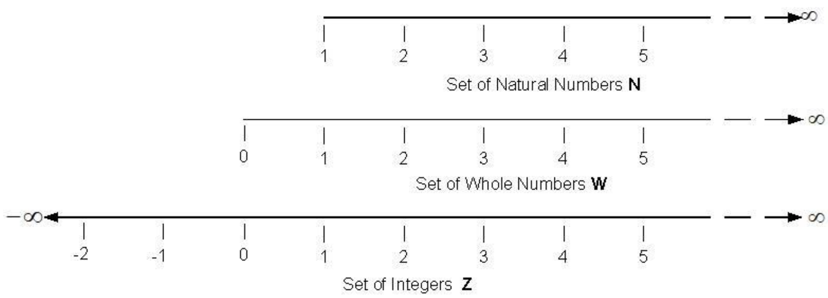
CHAPTER II: BASICS OF MATHEMATICS

Basics of mathematics in arithmetic, algebra and geometry, that are studied upto class Xth, form building blocks of whole world of mathematics, science, and engineering. Understanding of mathematics started with a feel of quantity smaller or bigger, taller or shorter, give or take, response to rhythm of sound; all of them grow naturally, without any training. This has not remained confined to natural responses, but gave rise to number system which later evolved into arithmetic as an independent branch of mathematics. Every problem in arithmetic handles a definite set of numbers unique to it, and tries to quantify every physical reality. But, algebra is all about translating any specific problem into a set of variables and thus evolve a general statement. This general statement with specific set of numbers, called values of variables, provides specific solution to the problem. Thus algebra is all about generalization of a problem. Whereas, geometry handles shapes which are attributes of every physical entity. All higher branches of mathematics are derived from addition, subtraction, multiplication and division of the three basic mathematical entities. A deeper insight into mathematics reduces at elemental level to ***(a)*** *ratio-proportion,* ***(b)*** *theory of indices and* ***(c)*** *Pythagoras theorem. The journey into mathematics in this chapter is made to deal this in a hybrid manner rather than dealing with each branch discretely.Accordingly, it will be observed that sequence of various concepts is based on need of how and why to understand the concepts that follows. An important massage in this journey is that mathematics is just not about knowing and using formulae, with necessary calculations. It is rather about using them in a manner which smartly simplifies calculations with a logical thought process.*

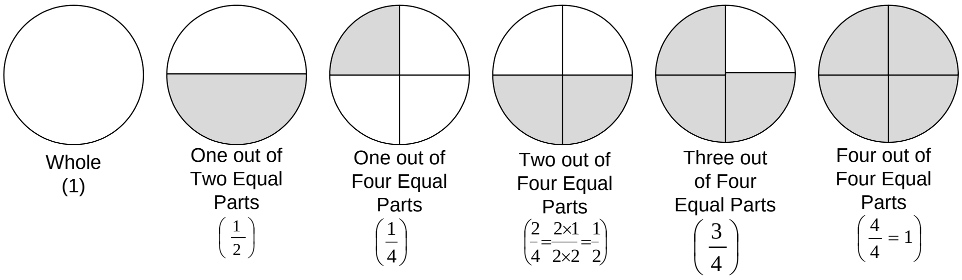
1. **Number System [1,2,3,4]:** History of number system, for that matter of mathematics, may be of interest to students of its evolutionary process while, most are interested in various types of numbers, their relevance and uses, a subject matter of arithmetic. It started with **Natural Numbers** for counting of objects, which could not be partitioned say man, cattle etc., from 1 to the highest number that could be counted. It leads to evolution of **infinity** a number much larger than any number that can be perceived. Representation of numbers is also pre-historical in the form of numerals. The group of integers has also been represented as a set N. Here, a word **Set** is being added to the vocabulary of mathematics which is nothing but collection of distinct objects, each of which is considered as an object in its own right; more of it would be dealt with separately. Comparison of numbers becomes more explicit when they are represented on the number line. A number on the right of another number is larger than the one on the left of it.

Experience of having absence of an object, under consideration, lead to evolution of **Zero**, is a reality and this set of Integers together with Zero was called **Whole Numbers W**. The experience of height vis-a-vis depth location of a place in one line with reference to a specific point, give and take transactions, etc. lead to evolution of complement of **N** as negative numbers and was named as **Integers Z**. Representation of these numbers on a number line is shown in the figure.



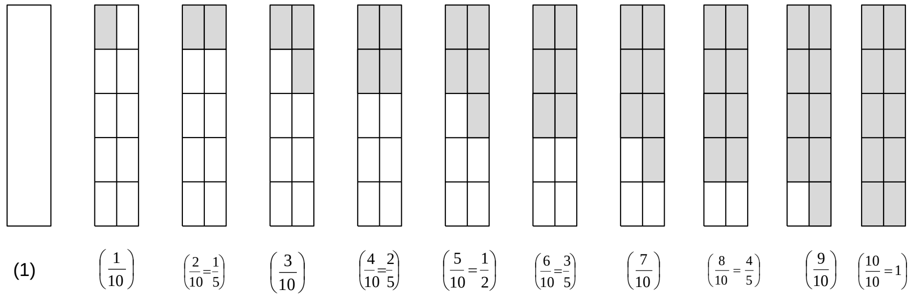
There are objects which can be partitioned viz. bread, paper etc. and part of a whole object does not find place in above set of numbers defined above. This lead to evolution of fractions and fractions together with whole objects were called **Real Numbers represented as set R.** Physical realization of fractions is as under–

In representation of factors, say one out of two as $\frac{1}{2}$, One (1) is called numerator and Two (2) is called denominator. Mathematical concept of fractions would be dealt with separately a little later. In language of mathematics, in English, fraction is represented with an integer written above the line is called **Numerator** and the number below the line is called **Denominator**. While in Hindi say $\frac{3}{4}$ it is called 4 हर काअंश 3; it means make Four Parts of the given object of which each part (हर भाग) is equal and out of it take only (अंश) 3 parts (भाग). It is considered to be most generic and meaningful

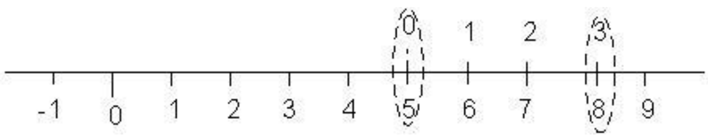


definition of fraction, within the colloquial vocabulary of the same language. On similar lines decimal fractions, using Zero, are elaborated as under –

Decimal system is useful not only in representing fractions over a wide range, in a common format using scientific notations that shall be addressed shortly. Before that, basics of geometry needs to be recalled, together and algebra, which shall be dealt with intermittently, as and when essential, to maintain the need of correlating basic concepts and satisfy inquisitiveness of students.

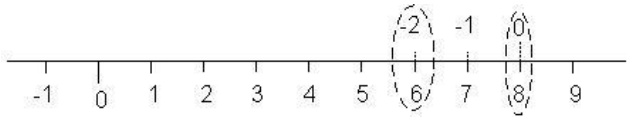


Basic Mathematical Operations [1,2,3,4]: There are four basic mathematical operations – *addition, subtraction, multiplication and division*. In **Addition** of two numbers, primarily (+)ve, First number is called **Augend** while the number to be added to this is called **Addend** and the result is **Sum.** *This operation is similar to increasing augend by addend.* A classical method of addition of two numbers is to count First addend beyond *augend*, such that augend is incremented by one every time till the addend is reached. This is a coarse method which does not go with the basic aim of solving problems speedily, effectively and accurately. An example of addition of Two single digits numbers $5+3=8$ is graphically elaborated.



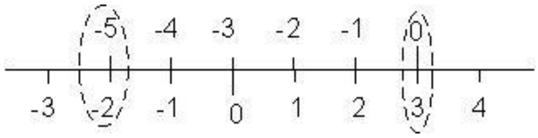
Another method is based on decimal representation widely in use. An example is given below. This method calls for adding each place value and if sum of each place value is greater than or equal to ten the next higher place value is incremented by one, called carryover. The reason for this method is that – a) at any place value digit cannot be greater than Nine i.e. $0 \leq z \leq 9$, and b) sum of two digits cannot be greater than incrementing eighteen (18), c) Carryover of addition of Two digits cannot be greater than One. This is extrapolated to addition of ***n*** digits, where carryover cannot be greater than (***n*-1**). This becomes more explicit with the addition operation in expanded form. Eventually, representation of numbers in expanded form is also a result of addition of digits with their respective place values. This method of addition, mathematical concepts, is elaborated in the example.

	Th	H	T	U	Representation of Numbers in Expanded Form
Carry over	1	1	0		1000 100 00
Augend	2	9	7	5	2000 + 900 + 70 + 5
Addend +		3	8	4	+ 0000 + 300+ 80 + 4
Sum	3	3	5	9	= 3000 + 300+ 50 + 9

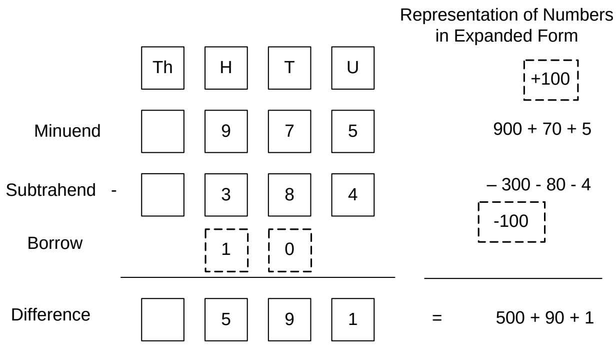


In **Subtraction** of two numbers there are two operands, primarily (+)ve One of them is. **Minuend** and the other is **Subtrahend** and the result is called **Difference**. *This operation* *is of taking away subtrahend from minuend*. Graphically subtraction of 2 (subtrahend) from 8 Minuend involving single digit numbers i.e. $8-2=6$ is as under.

In another case of single digit numbers where subtrahend is greater than minuend i.e. $3-5=-2$ this is also graphically elaborated here.



But, in general practice subtraction involves lager number having more than single digit. In such cases, when at any place, number of subtrahend is greater than that in the corresponding place in minuend then one is borrowed from next higher place, on the left, and simultaneously Ten is added to the minuend at the place under consideration. *It is important to note that a digit at any particular place is a number with the corresponding place value.* The borrowed number is written below the subtrahend; this corresponds to number in subtrahend of the borrowed place is added with Ten. This corresponds to logical manipulation of digits with corresponding place value.



Accordingly subtraction is carried out for each place. This situation can be compared with value of onions and potatoes being bought at the same rate. In case of Two Kg of each of the two, taking out onions and replace it with potatoes, as desired by the buyer, makes no difference to either of seller or buyer in the value of transaction. In case subtrahend is greater than minuend, the result is negative.

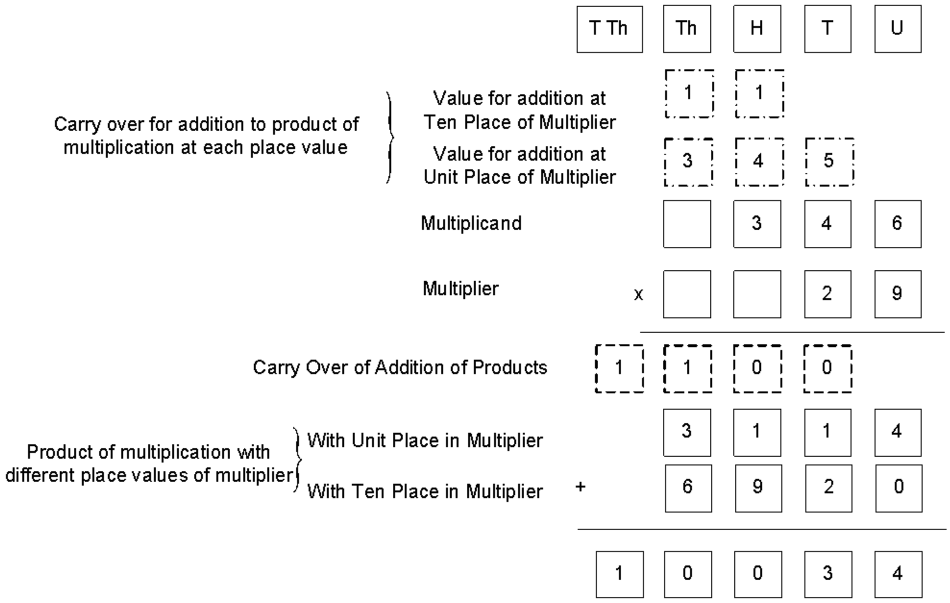
It can be seen that in subtraction sum of subtrahend and the difference is minuend; while in addition difference of sum and addend is augend. Thus it is concluded that in subtraction is reverse of addition.

There are situations when sum of multiple numbers some with (+) sign and (-)ve sign are encountered , all positive numbers are added together and taken as final minuend; likewise, all (-)ve numbers are added together and taken as final subtrahend. At the end, subtraction is carried out on the final minuend and subtrahend. Alternatively, addition and subtraction operations can be split in parts while maintaining the basic mathematical logic.

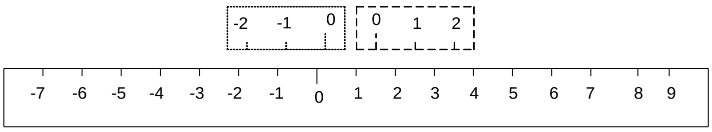
Result of **Multiplication** *operation is a value which is obtained when one of the operand is added to itself repetitively, as many times as the other operand*. The first operand is called **Multiplicand** while the other operand is **Multiplier** and the result is **Product**. The two operands are also called **factors** of the result; and each of the operand is also called a **Complementary Factor** of the other. Multiplication tables are simplest example of the operation. These tables are used as a tool in multiplication of two numbers together with addition operation. It is illustrated in the following example of tables of 2, 20 and 200 upto 10–

Table of 2			**Table of 20**			**Table of 200**		
2*1	= 2	= 2	20*1	= 20	= 20	200*1	= 200	= 200
2*2	= 2*1+2	= 4	20*2	= 20*1+20	= 40	200*2	= 200*1+200	= 400
2*3	= 2*2+2	= 6	20*3	= 20*2+20	= 60	200*3	= 200*2+200	= 600
2*4	= 2*3+2	= 8	20*4	= 20*3+20	= 80	200*4	= 200*3+200	= 800
2*5	= 2*4+2	= 10	20*5	= 20*4+20	= 100	200*5	= 200*4+200	= 1000
2*6	= 2*5+2	= 12	20*6	= 20*5+20	= 120	200*6	= 200*5+200	= 1200
2*7	= 2*6+2	= 14	20*7	= 20*6+20	= 140	200*7	= 200*6+200	= 1400
2*8	= 2*7+2	= 16	20*8	= 20*7+20	= 160	200*8	= 200*7+200	= 1600
2*9	= 2*8+2	= 18	20*9	= 20*8+20	= 180	200*9	= 200*8+200	= 1800
2*10	= 2*9+2	= 20	20*10	= 20*9+20	= 200	200*10	= 200*9+200	= 2000

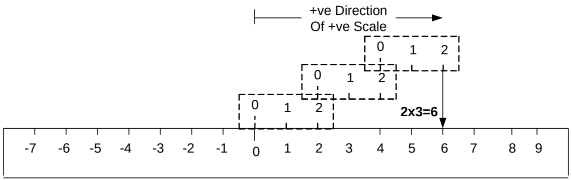
It can be observed that for every multiplication of 20, with any multiplier, product with the multiplication of 2, with the same multiplier, all values are shifted to left i.e. next higher place with unit place value as zero (0). This clearly signifies that in this case product becomes Ten (10) times. Likewise, for every multiplication of 200, value of each place value im multiplication of 20, is shifted to next higher place on left with the unit place value Zero (0). Using this principle multiplication process of any two discrete numbers is brought out below. Each of the operand is taken to be (+)ve.

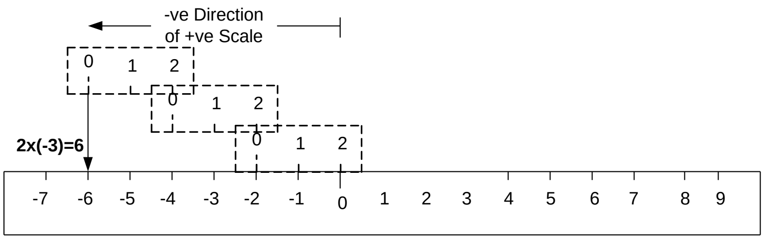


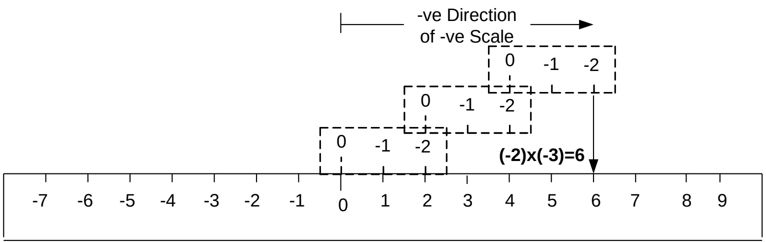
It needs care in handling carryover of multiplication and addition of products with individual place value of the multiplier.



Multiplication of Signed values, by repetitive addition, is elaborated in the figure below. Repetitive addition in (+)ve direction represents (+)ve multiplier, and in (–)ve direction represents (-)ve multiplier.



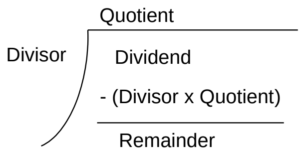




The summary of signed multiplications is as under-

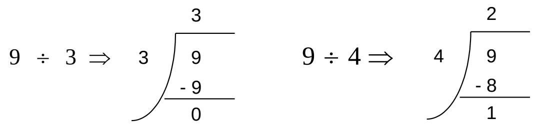
Signed Multiplication		
Multiplicand	**Multiplier**	**Product**
(+)	(+)	(+)
(+)	(-)	(-)
(-)	(+)	(-)
(-)	(-)	(+)

Division *is an operation of repetitive subtraction of one operand called* **Divisor** *from the other called* **Dividend**, *till difference after each subtraction is either Zero (0) or less than the divisor*. Number of times divisor is subtracted is called **Quotient**, and non-zero difference is called **Remainder**. The division operation is depicted as under –

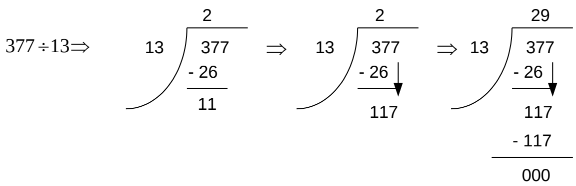


Altenately, **Dividend = Divisor x Quotient + Remainder**

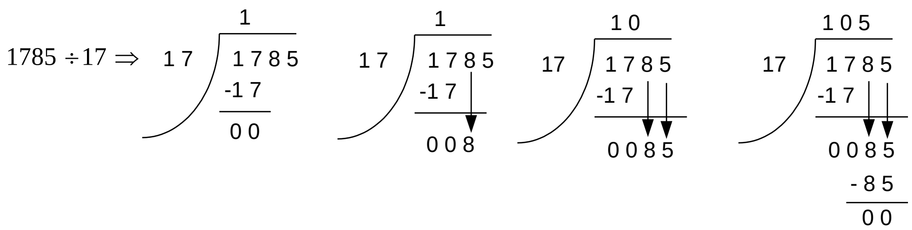
Euclid, an ancient mathematician recoded the above inference which is known as **Euclid's Lemma** as : given two integers ***a*** and ***b***, there exists unique integers ***q*** and ***r*** satisfying $a= bq + r, 0\leq r <b$. Two examples of Single digit division are as under; the table of the divisor is continued till it just crosses the dividend. At that point taking a step back in the table the multiplier is written in place of the quotient, and product is subtracted from the dividend. In first example the remainder is Zero (0) while in second example remainder is One (1).



In multiple digit division, Two examples, both of two digit divisors, are given below. In first example highest place two digits of dividend (37 – at Hundreds and Tens place) are greater than divisor (13); this gives Two (2) as highest place digit of the quotient and a remainder Eleven (11). Taking down the third digit at Units place (7- next lower place of the dividend), the next lower digit of the quotient is Nine (9) with Zero (0) remainder. Thus the dividend is completely divisible with the divisor.



In second example, with Two digit Divisor (17) , The Two digits at the highest place of the dividend are same as the divisor O(17). Therefore, the digit at highest place of the quotient is (1) and the remainder at this stage is Zero. But, on taking down next lower place digit of the dividend (8), which is smaller than the divisor, it is indivisible and therefore Zero (0) is placed at the next lower place of the quotient. The division is continued, by taking down next lower place digit of the dividend (5). At this stage next lower place digit in the quotient comes to 5 leaving a remainder Zero (0). Accordingly the quotient in this example is 105 and the dividend is completely divisible by the divisor.



In the division operation there are two important observations. *First, divisor cannot be Zero (0). Second, the division of integers to have a non-zero quotient (≠0) requires dividend greater than divisor*. Further, it can be verified that product of the divisor and quotient PLUS remainder gives dividend as per Euclid's Lemma. Accordingly division *operation is reverse of the multiplication operation. The only difference is that multiplication is started with lowest place value, while division is started with highest place value.*

BODMAS Rule: In real life rarely simple mathematical operations are performed on two operands, most of the problems involve compounding of mathematical operation and in view of this consistency and correctness of calculations is formalized. The BODMAS Rule stipulates order of precedence of operation is brought out in a table.

Operand	Remarks	Priority
Bracket	Solution proceeds from innermost to outermost bracket.	1st Priority
Exp**O**nent	Exponent is solving Power or Root of operand	2nd Priority
Divide and **M**ultiply	These two operations are equally ranked and carried out sequentially from left to right	3rd Priority
Addition and **S**ubtraction	These two operations are equally ranked and carried out sequentially from left to right	4th Priority

Modulation is an operation in mathematics which is similar to division, but in this significance of quotient which is ignored. *This operation is exercised on natural numbers* N. In modulation divisor is called **Modulus** while identity of Dividend is maintained, but remainder is called **Residue**. This operation is expressed as **a%b = c**; here **a** is dividend, **b** is modulus and **c** is the residue. It is also expressed as $\boldsymbol{a \equiv c\ (mod b)}$ and stated "***a** and **c** are congruent (mod **b**)."* Modulus concept is used in many applications like programming, intelligent codification. *In this operation Quotient is ignored.*

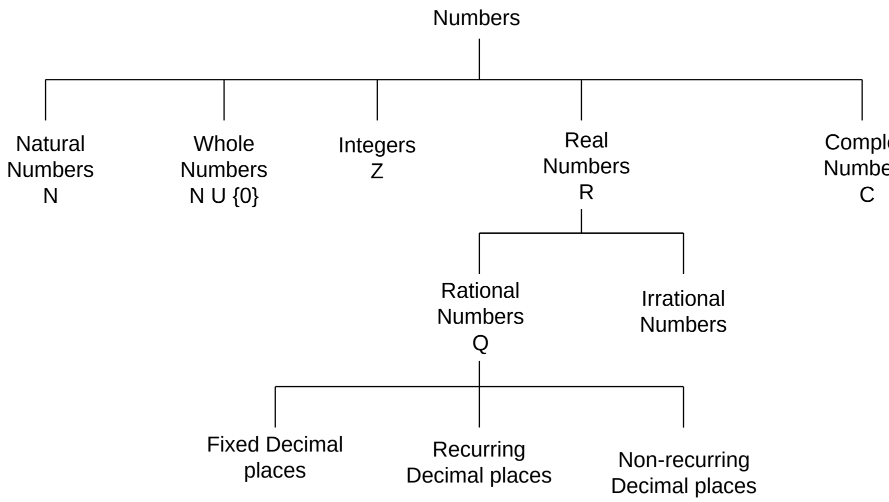
Numbers are further classified based on their specific nature with respect to its factors. *Even, Odd, Prime Numbers* are among Integers which include sets of *Natural Number, Whole Numbers and Integers* themselves. Another set of real Numbers are either Rational or Irrational. An **Even Number** is a number which is divisible by (2), while **Odd Numbers** are among those indivisible by Two (2). **Prime Numbers (P)** *is divisible by only One (1) or the number itself.* Thus, one of the factors of even number is essentially Two (2), but factor of odd number cannot be Two (2). One (1) is since a factor of every number it is called a **Special Number.** One (1) is taken in the category of odd numbers since it is less than Two (2). Any attempt to divide One (1) by any other integer would not result into a integer quotient. There are numbers which have prime numbers as its factors and are called **Composite Numbers** viz. 21 (=7x3). While, ***Co-prime Numbers*** *are a set of prime numbers which have its* ***greatest common divisor (g.c.d.) One (****1) and is represented (a,b)=1, where a and b are two integers which are relatively prime.*

Fractions were introduced earlier. They are classified as pure fractions and mixed fractions. **Pure fractions** are those where numerator is less than the denominator e.g.$\left.\frac{a}{b}\right|_{a<b}$. Whereas, **improper fractions** are those where,$\left.\frac{a}{b}\right|_{a>b}$. Improper factions are usually expressed as **mixed fraction** e.g. $q\frac{r}{b} = \left.\frac{a}{b}\right|_{a>b}$; here, $a = q \times b + r$, as per Euclid's Lemma. *Fraction* $\left(\frac{a}{b}\right)$*in its simplest form has no common factor in numerator and denominator and accordingly these are relatively prime i.e. (a,b)=1.* Further a fraction $\frac{x}{y}$ using *Theory of Indices* (brought out later in the chapter) can be expressed as xy^{-1}. There is a typical situation when y=0, then $-\frac{x}{y} = \frac{-x}{y} = -\infty$ (Infinity). *Infinity is a mathematical concept representing a number which much larger than any number under consideration, and hence, it is called* **Indeterminate**. Taking a fraction $\frac{x}{-y}$, where denominator $y = 0$. Mathematically, *Zero is absence of a quantity, as defined earlier and, therefore, it in neither (+)ve nor (-)ve.* Accordingly, $\frac{x}{-y} = \frac{x}{-0} \rightarrow \frac{x}{0} = \infty$. *This leads to an anomaly* $\frac{-x}{y} \neq \frac{x}{-y}$*. This anomaly is resolved by using a rationalizing factor (-1),* which is multiplied to both numerator and denominator, so as to convert denominator into a (+)ve number as $\frac{x}{-y} = \frac{x\times(-1)}{(-y)\times(-1)} = \frac{-x}{y} = (-)\frac{x}{y}$. This is consistent with the logic of mathematics.

As journey into mathematics proceeds into limits and continuities one would encounter 0^-, 0 and 0^+ (pronounced as Zero MINUS, Zero and Zero Plus, respectively) have a different meaning and this is based on direction in which Zero is approached. Nevertheless, it is a different application of mathematics and does not affect the rationalizing logic used above.

There is another relative operation between fractions i.e. greater (>), smaller (<) and equal. When two fractions being compared are +ve, then on making their denominator equal, the fraction having greater numerator is greater, and smaller fraction is the one having smaller numerator than the other. But, when the two fractions being compared are –ve then, then again on making their denominator equal, fraction having smaller numerator is greater, while the one with greater numerator is smaller than the other. This becomes explicit when fractions are represented on a number line; this shall be elaborated after a introduction to Geometry.

Representation of fractions as real number leads to classification in decimal form as **Rational Numbers** and **Irrational Numbers**. Rational numbers are further classified as **a)** ***Fixed Decimal Rational Number,*** e.g. 0.5, 1.7 etc. In this decimal numbers terminate. **b)** ***Recurring Decimal Rational Number*** *e*.g. 0.333..., 2.565656... etc. These numbers with recurring decimal numbers are represented as $0.\dot{3}$, $2.\dot{5}\dot{6}$. While, *irrational numbers* have **Non-recurring and non terminating decimal numbers,** which continue for infinitely large number of decimal places without repetitive patterns. Typical examples of irrational numbers are $\sqrt{2}, \sqrt{3}, \sqrt{5}$ (i.e. roots of all prime numbers or numbers having it is a factor), and π (Pie) and e (Euler's Number also called Napier's Constant). For all practical purposes Pie is equated to 22/7 or 3.14, while e is equated to 2.303; these two real numbers are most commonly used in analysis of natural phenomenon. Reasons of why π (Pie) and e are irrational numbers would be elaborated in next chapter **Foundation Mathematics**.



Classification of various types of numbers has been summarized graphically. *In this **Complex Numbers** are yet to be discussed and shall be taken together with Quadratic Equation, to build the concept, a little later while elaborating basics of algebra.*

Conversion of fractions into decimal form has been discussed earlier, while real number with recurring decimal places in the form of fractions requires a mathematical trick and is shown in this example: $x = 1.\dot{3}$; $10x = 13.\dot{3}$; $9x = 12$; $x = \frac{4}{3}$. Another example is : $x = 2.3\dot{5}\dot{6}$; $100x = 235.6\dot{5}\dot{6}$-> $99x = 232.3$; $x = \frac{2323}{990}$

Here, it is to be noted that multiplier of x is 10 if there is one recurring decimal, 100 $(= 10 \times 10 = 10^2)$ if there are two recurring decimals; *summarily the multiplier is* 10^n *where is the number recurring places which will become more explicit as theory of indices is covered in basics of algebra, a little later.*

There are some basic properties viz. *Closure, Commutative and Associative* properties of numbers for various mathematical operations. These are very nicely explained in **Mathematics, Textbook for Class VIII, NCERT** and are summarized in the table below. In ***Closure property*** *implies that any of the* $+, -, \times$ *and* $\div$ *mathematical operations on a set of a type of number results in a number of the same type*. Each of the operation on each type of number needs to be done to carried out to appreciate the closure property of different types of numbers as summarized in the table below –

Type of Number	Closure Property (Type of number in result remains unchanged) [a *op* b]			
	Addition	**Subtraction**	**Multiplication**	**Division**
Natural	Yes	No	Yes	No
Whole	Yes	No	Yes	No
Integer	Yes	Yes	Yes	No
Rational (Real)	Yes	Yes	Yes	No (if Divisor has numerator '0')

Likewise, in ***Commutative property*** *change interchange of operands does not change the result of the four mathematical operations*. Each of the operation on each type of number needs to be done to verify this property as summarized in the table below -

Type of Number	Commutative Property (Results remain unchanged) [a *op* b vis-à-vis b *op* a]			
	Addition	**Subtraction**	**Multiplication**	**Division**
Natural	Yes	No	Yes	No
Whole	Yes	No	Yes	No
Integer	Yes	No	Yes	No
Rational (Real)	Yes	No	Yes	No

Whereas, in ***Associative property*** *change of association and/or of the operands, inside a bracket, does not change the result of the four mathematical operations*. Each of the operation on each type of number needs to be done to verify this property as summarized in the table below –

Type of Number	Associative Property (Results remain unchanged) [a *op* (b *op* c) vis-à-vis (a *op* b) *op* c]			
	Addition	**Subtraction**	**Multiplication**	**Division**
Natural	Yes	No	Yes	No
Whole	Yes	No	Yes	No
Integer	Yes	No	Yes	No
Rational (Real)	Yes	No	Yes	No

Representation of fraction on a number line, as referred to earlier, requires understanding of basics of geometry and is being integrated into the elaboration of basic mathematics, and not treaded in exclusion.

Identities: There are two types of identities **- (a)** ***Additive Identity*** is that number when added to any number its value does not change and that is **Zero (0),** viz. a+0=a, **(b)** ***Multiplicative Identity*** is that number when multiplied to any number its value remains unchanged and that is **One (1),** viz. $a \times 1$=a**.**

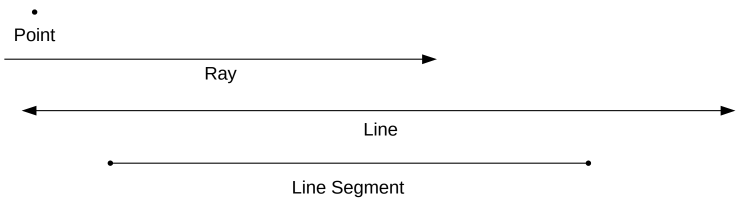
Inverse: There are two types of inverse in respect of numbers – **(a)** ***Additive Inverse*** is that number when added to a number result is **Zero (0),** viz. a + (-a) =0, conversely *additive inverse is negative of the number*. (b) ***Multiplicative Inverse*** is that number which when multiplied to a number result is **One (1)**, viz. $a \times b = 1 \rightarrow (a \times b) \times \frac{1}{a} = \frac{1}{a} \rightarrow b = \frac{1}{a}$.

Thus, *multiplicative inverse of a is* $\frac{1}{a}$.

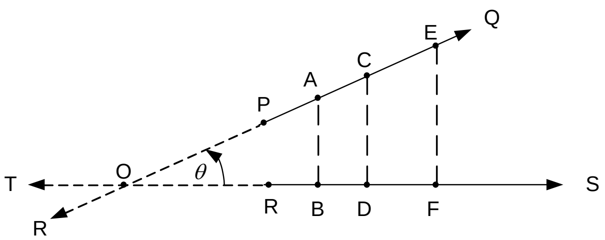
BASICS of GEOMETRY [1,2,3,4,5,9]: It starts with the definition *of a **Point** whose measure of dimensions is infinitesimally small i.e. tending to be Zero expressed as* $x \rightarrow 0$ *. It is a mathematical concept which implies that x is much smaller than any smallest number under consideration, but greater than Zero*. The moment $x = 0$, x ceases to exit so would be the Point. Inclusion of (0) in the set of natural numbers (N) gives a new set of Whole numbers (W

)elaborated earlier. This Zero connects Positive Integers (Z^+) and Negative integers (Z^-) to give a whole set of Integers (Z). And so also a set of real numbers (R).

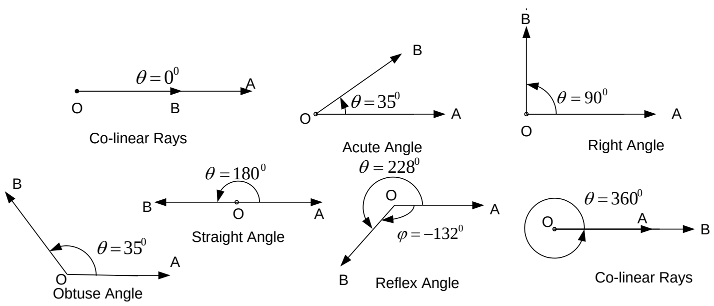
Graphically, for the purpose of **Geometry**, point is also explained in the form of shape having its size infinitesimal. It is visualized from taking following numbers in increasing order of denominator, viz 1/10, 1/100, 1/1000, 1/10000 .... Thus, Point can be defined but its conceptualization on paper, shall have to some dimensions to be visible. A ***Line*** is nothing but a trace of a point when moved. Accordingly, *line shall have a length but not width*. But, for conceptualizing line on paper some width is essential to be visible. If length of a line between two points, in a plane or space, is shortest it is called a **Straight Line** and a line between two points longer than the straight line is a **Curve**. *Straight lines* are classified as Ray, Line and Line Segment. ***Ray*** *is a line starting from a point, but extending to infinity*, while ***Line*** *starts from infinity on one side and extends to infinity on the other side*. But, *a line between two discretely defined points is a* ***Line Segment***. In line Segment end points are shown of dimension larger than thickness of line, only for conceptualization, and should not be confused to be so. These classifications are shown below.



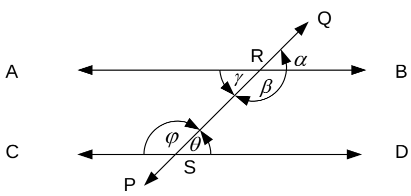
Lines can be grouped in parallel lines and angular lines. If *two lines have shortest distance between them at any point is uniform then they are called* ***Parallel Lines***. Concept of ***shortest distance*** would be elaborated a little later, after properties of a triangle. If, shortest *distance of any point, on one line, from the other line is varying then they are* ***Angular Lines,*** *also called* ***inclined lines or non-parallel lines***. In case of angular lines separation between angular lines increases on one side of a point on one line, then on the opposite side separation between lines would decrease and finally they converge at a point. *Measure of deviation between these non-parallel lines is called* ***Angle and represented with a its nomenclature preceded by a symbol*** $\angle$. In the figure below an angle *$\angle QOS$ or simply $\angle O$, where O is point of intersection of lines, it is also called* ***vertex***.



In the figure above, a point C on ray PQ has a separation CD from ray RS. A Point E on the right of C and point A on the left of C have separations, from ray RS, EF and AB respectively. The separation on the right side is visibly increasing while on the left side are gradually decreasing. Projection of the rays PQ and RS on left, the direction of decreasing separation, and they converge at point O. *This change in separation at different points of one line with the other line is called deviation and is expressed in geometry as angular separation or Angle, denoted as either of $\angle POR$, $\angle AOB$, $\angle COD$, $\angle EOF$ or $\angle QOS$, in the instant case, as per convenience. Measure of angles is in Degrees varying from 0^0 to 360^0*. Accordingly, angles are classified as **Acute Angle** ($>0^0$ and $<90^0$), **Right Angle** ($=90^0$) ∟,**Obtuse Angle** ($>90^0$ and $<180^0$), **Straight Angle** ($=180^0$) and **Reflex Angle** ($>180^0$ and $<360^0$), shown below. *It will be seen that angles are measured in anti-clockwise and are of positive magnitude*. Further, a line after one complete rotation (i.e. deviation 360^0) becomes ***Co-Linear.*** *Reflex angles if measured in clockwise, i.e. in a direction opposite to the convention, their measure is of negative magnitude.* Further, this logic of deviation, between parallel lines is 0^0 since they never meet. It will become more explicit when concepts of trigonometry is introduced



A pair of **Adjoining Angles** (having one side common) whose sum is 180^0 are called **Supplementary Angles**, while such angles **Complementary Angles** when their sum is 90^0. Right angle has wide application in geometry and mathematics and is expressed as ⊥. Application of the concept of angle in parallel lines is next step into geometry through figure below.



AB and CD are parallel lines (expressed as ||), having a transversal PQ which intersects these || lines at points R and S. It is a starting point draw some important conclusions and for this various angles are given **notation in Greek Alphabets** $\alpha, \beta, \lambda, \theta$ etc. There, is no restriction in naming the angles, and one can choose any non-numeral as per convenience be it alphabets or alphanumeric. These angles can have any value specific to a problem under consideration. These notations, forms a beginning of **algebra** to be elaborated soon after, shall be used to derive important conclusion in respect of parallel lines, here under -as under –

a) Since $\angle SRQ$ and $\angle ARB$ are straight angles, hence $\alpha + \beta = 180^0$ and $\gamma + \beta = 180^0$. Here, β is common in both the straight angles and, therefore, it leads to α = γ, i.e. ***opposite angles*** *formed by intersection of two lines are equal.* (**Conclusion 1**).

b) Angle between lines AB and CD is zero; being parallel lines, and if line CD is moved until it overlaps line AB, it would lead to ∠QSD overlap ∠QRB or $\theta = \alpha$. These are the corresponding angles formed by transversal on same side of the parallel lines . This leads to another conclusion ***corresponding angles*** *of a transversal on parallel lines are equal* (**Conclusion 2**).

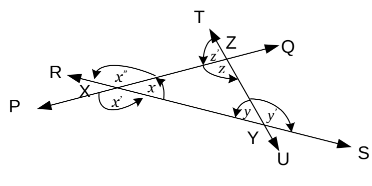
c) Applying conclusion 2 by adding angle β to the equality it leads to $\theta + \beta = \alpha + \beta = 180^0$. Accordingly, $\theta + \beta = 180^0$; *sum of* ***interior angles*** *formed by transversal of parallel lines is 180^0*. **(Conclusion 3).**

d) Combining the identities evolved during conclusion (a) and (c) Since, $\beta + \gamma = \theta + \beta = 180^0$. Here, taking away β from both the sides, it leads to ,$\theta = \gamma$; i.e. ***alternate angles*** *formed by transversal of a parallel lines are equal.* **(Conclusion 4).**

These Four conclusions are extremely helpful in moving forward in the geometry.

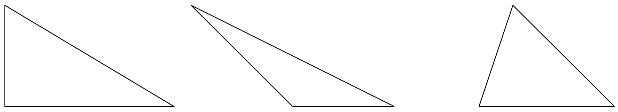
Basic of Algebra: *It is that branch of mathematics which converts a specific observation, involving some definite set of quantities into a* ***general statement****, which can be used in solving any other specific problems with the related quantities, is called* ***Algebra****.* Algebra involves **variables** and **constants**. While deriving conclusions, above, some ***equations*** were used involving variable α, β, γ, θ and φ, while constant in these statements is 180^0. *Equations are like weighing balance represented by sign* =. It has a left hand side and right hand side. An equation remains balanced when same quantity is added, subtracted, multiplied or divided on both sides of the equation, uniformly. This can be illustrated with any equation, say $x = y$, and performing these basic mathematical operations uniformly on both sides. Taking a step backwards into geometry is required to make a fast-forward move in the mathematics.

Basics of Geometry (Contd. 1): Trace of a line form a **surface** *which has length, width but not thickness. If the trace of a line is also along a line, not curve, then it is called* ***Plane. Width*** *of plane is virtually length of trace of the line, but its* ***zero thickness*** *comes from the concept of line which has no width.* Three non-co-linear, non-parallel lines, but co-plainer lines form a shape called ***triangle*** enclosed between three lines, having three points of intersection called **vertices**, as shown below; *line segments joining the vertices are line segments but in this shape are called* ***sides*** *of triangle, angles formed by sides on the vertices are* ***angles of triangle****.*

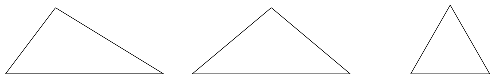


In the figure below, lines PQ, ES and TU are intersecting at points X, Y, and Z and accordingly a shape XYZ is formed called ***Triangle****, having three sides and three* ***Interior Angles x, y and z****, formed by the sides XY, YZ and ZX of the triangle*. A triangle is expressed as Δ suffixed with its vertices, in the instant case it is ΔXYZ. Angle formed by any side with any other extended side is called ***Exterior Angle***, viz. *x*' is an exterior angle at vertex X w.r.t. side XZ , Likewise, *y*' and *z*' are exterior angles at vertices Y and Z, w.r.t. sides XY and YZ respectively, as shown in the figure. Exterior angle of an angle at the vertex of the triangle can be on either side of the angle under consideration and depends upon the side of the triangle under consideration. Thus, in the figure *x*' and *x*" are supplementary angle at vertex X with respect to sides XZ and XY of the triangle, respectively. It is to be noted that *x*'= *x*" being vertically opposite angles at X. *Essential condition for a triangle to be formed is that the lines forming the triangle must be coplanar.*

Triangles can be classified based on their interior angles and sides as under. It is seen that in a ***Right Angle Triangle***, *only one interior angle is* 90^0(expressed as ∟) and *other two interior angles are acute angles*. This is attributed to First Property of Triangles which requires the other angles two angles together with ∟ is 180^0. This satisfies condition of the Two angles being complementary angles, i.e. each of the two angles is less than 90^0 . Thus, the two remaining angles have to be acute angle, i.e. <90^0. In a ∟Δ, side opposite to ∟ is called ***hypotenuse***, and it is the longest side in the Δ, being side opposite to largest ∠. This correspondence between length of side and angle of the triangle opposite to the side, as a general case, shall be discussed little later. In ***Obtuse Angle Triangle***, *only one interior angle is obtuse*, and remaining two interior angles are acute by logic similar to that in right angle triangles. In ***Acute Angle Triangle*** *all angles are acute*, and can satisfy the condition of sum of interior angles.

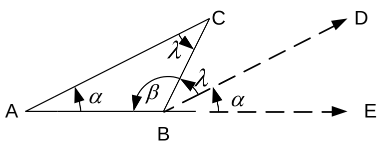

Right Angle Triangle Obtuse Angle Triangle Acute Angle Triangle

The other classification Δs is based on length of sides. In ***Scalene Triangle*** *all the three sides are unequal*. While in ***Isosceles Triangle*** *only Two sides are equal*. Triangles of these types may or may not have an obtuse angle. In ***Equilateral Triangle*** *all sides are equal*, and their angles are acute i.e. $60^0 = \frac{180^0}{3}$.


Scalene Triangle Isosceles Triangle Equilateral Triangle

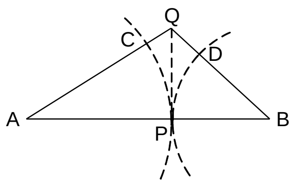
There are **Four important properties of a triangle**, First three of this are discussed below, while Fourth property shall be discussed soonafter elaboration of types Δs

a. In the ΔABC, in the adjoining figure, above, AB is extended towards E and a ray parallel to line AC is drawn from B towards D. Here, ∠CAB = ∠DBE=□, being corresponding angles formed by transversal AE to the parallel lines AC and BD. Likewise, ∠ACB = ∠CBD=□, being alternate angles formed by transversal CB to the two parallel lines. ∠ABC + ∠CBD + ∠DBE = $\alpha+\beta+\lambda$ = ∠ABE =180^0 ; ∠ABE is a straight angle. Algebraically, this equation is the *sum of interior angles of a triangle is equal to* 180^0; (***First Property***).



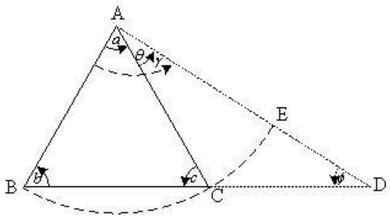
b. Further, exterior angle ∠CBE = ∠CBD + ∠DBE = $\alpha+\lambda$, and ∠ACB + ∠CAB= $\alpha+\lambda$; This leads to a conclusion *exterior angle of a triangle is equal to sum two opposite interior angles (****Second Property****).*

c. In a Δ ABQ a perpendicular is drawn from vertex Q on side AB which intersects the side at P, as shown in the figure, such that AB_{Length} = AP_{Length}+ PB_{Length}. Taking A as centre an arc of radius AP is drawn which intersects side AQ at point C such that AQ_{Length} = AC_{Length}+ CQ_{Length} . It implies that AQ_{Length} > AC_{Length} → AQ_{Length} > AP_{Length} , since AC_{Length} = AP_{Length} both being radials. Likewise, for arc of radius BO intersecting BQ at point D, it woud lead to BQ_{Length} > BD_{Length} → BQ_{Length} > PB_{Length} , since BD_{Length} = DP_{Length} . Accordingly, adding the two inequalities (AQ_{Length} + BQ_{Length})> (AP_{Length} + PB_{Length}). It finally leads to a conclusion (AQ_{Length} + BQ_{Length})> (AB_{Length}) since points A,P and B are collinear and consecutive. Conversely



(AB_{Length}) <(AQ_{Length} + BQ_{Length}). Thus, it is stated that *in a triangle length of any side of a triangle is shorter than sum of the length of other two sides of the triangle (**Third Property**).*

d. An equilateral ΔABC where AB_{Length} = BC_{Length} = CD_{Length} is converted into a scalene ΔABD by extending line BC upto D and joining AD, It implies that BD_{Length} = BC_{Length} + $CD_{Length} \rightarrow$ BD_{Length} = AB_{Length} . An arc of radius AB with Centre A is drawn so as to intercept side AD at E, such that AB_{Length}=AE_{Length} and AD_{Length} = AE_{Length} + ED_{Length} $\rightarrow$ AD_{Length} > AB_{Length} .

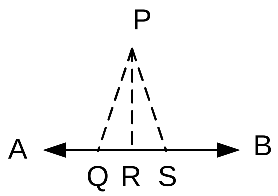


Thus, in the scalene ΔABD, side AB is shortest. Using third property AC_{Length} + CD_{Length} > AD_{Length} , or BC_{ength} + CD_{Length} > AD_{Length} $\rightarrow$ BD_{Length} > AD_{Length} . Accordingly, AB_{Length} <AD_{Length} < BD_{Length}.

In the equilateral ΔABC angles $a = b = c$. Further, as per Second property of the Δs in ΔACD, external angle $c = \theta = \phi$, therefore $\phi < c$ and by construction $\gamma = \alpha + \theta$, therefore, $\gamma > b$.

Accordingly, arranging angles of ΔABD in ascending order $\phi < b < \gamma$. It leads to an important general conclusion that ***magnitude of angles opposite to the sides of a Δ are in order of their lengths. It implies that angle opposite to the longest side of a Δ is largest and conversely angle opposite shortest side of a Δ is shortest*** **(Fourth Property).**

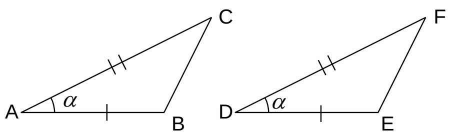
This third property of a triangle is used to prove that *length of a perpendicular drawn from an exterior point on a line is shortest distance of the point from the line also called distance of the point from the line*. This property has been used earlier to define parallel lines and inclined lines. Take a line AB and a point P external to it. Draw perpendicular PR from P on line AB. Take two pints Q and S on either side of the base R of the perpendicular and join them by line PQ and PS. This gives two ∟Δs PRQ and PRS where PQ_{Length}>PR_{Length} and PS_{Length}>PR_{Length}. As Q and S points approach R, QR_{Length} $\rightarrow 0$ and RS $_{Length}$ $\rightarrow 0$, in turn $PQ_{Length} \rightarrow PR_{Length}$ and PS_{Length} $\rightarrow$ PR_{Length} . Accordingly, ***the perpendicular distance of an external point from a line is the shortest distance***.



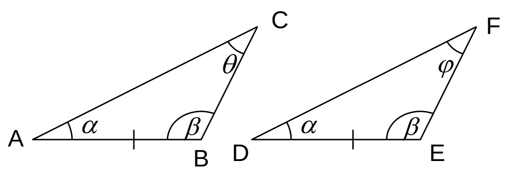
Congruence of triangles, and conditions thereof, are important conclusions in journey of mathematics. *General conditions of congruency are **a)** Side-Angle-Side (SAS), **b)** Angle-Side-Angle (ASA), **c)** Right_Angle-Hyptenuse-Side (RHS) and **d)** Side-Side-Side (SSS)* and are being proved.

a. SAS Condition: In figure shown below, Δs ABC and DEF, sides given that, AB=DE, AC=DF and $\angle$CAB =$\angle$FDE.

Proof: Place vertex D over A and orient ΔDEF such that DE aligns with AB, vertex E will coincide with vertex B; given AB=DE. The Line DF would align with line AC; given $\angle$CAB =$\angle$FDE. The vertex F would coincide with vertex C; AC=DF. Since, all the three vertices of ΔDEF coincide with those of ΔABC, the two triangles are congruent (***congruence proved).***



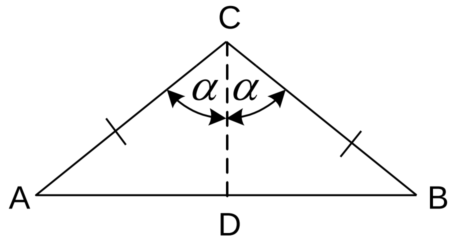
b. ASA Condition: In figure given below, Δs ABC and DEF, sides given that, $\angle$CAB =$\angle$FED, AB=DE, and $\angle$CBA =$\angle$FED, or any side with adjoining two angles are equal e then Δs are congruent.



Proof: Place vertex D over A and orient ΔDEF such that DE aligns with AB, vertex E will coincide with vertex B; given AB=DE. Line DE would coincide with AC. Given ∠CAB = ∠FED = □, line ED would coincide with line AC; Likewise, with the given ∠CBA = ∠FED = □, the line BC would lie on line EF. The point of intersection of two coinciding lines, starting from identical set of points would also coincide, and thus Vertex F would coincide with C. Moreover, using First property of Δs in ΔABC, θ =180 - $\alpha + \beta$, and in ΔABC, ϕ =180 - $\alpha + \beta$, thus, with R.H.S. being equal $\theta = \phi$. Since, all the three vertices of ΔDEF with those of ΔABC, the two triangles are congruent (***congruence proved***).

Other Two theorems of congruence would be proved after proving properties is isosceles triangles, using the above Two theorems of congruence, which would be found useful in the derivations.

Isosceles Triangle: *Isosceles* Δs *have* ***three properties*** – **(i)** *Angles opposite equal sides are equal,* (ii) *bisector of angle, formed by two equal sides, bisects the third side,* ***(iii)*** *bisector of the angle formed by equal sides is perpendicular to third side.*



i. **Proof of Property (i) :** In the figure below, sides AB=AC and hence it is isosceles Δ. Draw a `bisector of ∠ACB such that it meets side AB at point D. Thus twoΔs ADC and CDB are formed. In these two triangles since, AC=BC, ∠ACD=∠DCB = □; by construction, and CD is common side. Therefore, Δs ADC and CDB are congruent by SAS condition. Hence, ∠CAD=∠CBD, being corresponding angles of congruent Δs . Thus ***property* (i) *is proved***.

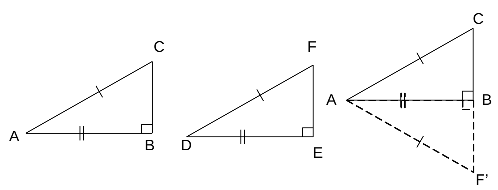
ii. **Proof of Property (ii)** : Further, from congruency ∠CDA=∠CDB. Further, corresponding sides AD and DB of the two congruent Δs are equal, hence, point D bisects line AB. Thus point bisector of the angle formed by equal sides of Δ bisects the third side, i.e. the side of the triangle opposite the vertex. Thus ***property* (ii) *is proved***.

iii. **Prrof of Propert (iii)** : Moreover, Further, from congruency ∠CDA=∠CDB. Where AD and DB are collinear. Hence, ∠CDA + ∠CDB = ∠ADB = 180^0; straight angle by construction. Therefore, ∠CDA=∠CDB=90^0, the bisector of the angle formed by equal sides of Δ is ⊥ to third side. Thus ***property* (iii) *is proved***.

Derivations of Congruent Δs is continued here –

c. ***RHS Condition:*** In two Δs ABC and DEF, sides given that, ∠ABC =∠DEF =90^0 i.e. they are right angle triangles (∟Δs). In this, hypotenuse, opposite, AB=DE, and ∠CBA =∠FED, and hypotenuse AC=DF.

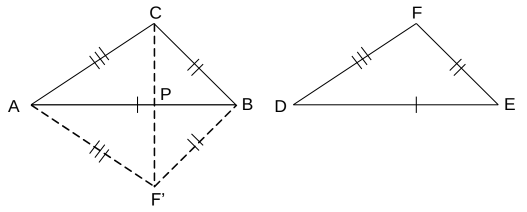
Proof: Place vertex D over A and reorient ΔDEF such that the line DE to coincide AB, the vertex E would coincide with B; given sides AD=DE. Now, rotate Δ DEF through 180^0 along line DE such that vertex F fall at F' opposite to the vertex C and both Δ ABC (original) and ΔDEF (after rotation) are in the same plane.. Since ∠CBF' = ∠CBA +∠ABC = 90^0 + 90^0 =180^0, given. Hence, ∠CBF' is a straight ∠ or CF' is a straight line.



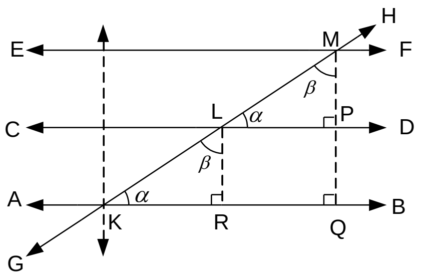
Now, in ΔACF' so formed, AC=AF', and hence ∠ACB =∠AF'B = ∠DFE as per property (i) of isosceles Δ and BC=BF', point B being mid point of CF' as per property (ii) of isosceles Δ . Thus in the Δs ABC and ABF', (a) side AD is common, (b) ∠ACB =∠AF'B , being ∟, and (c) BF'=BC. Accordingly by SAS condition Two corresponding angles of Δs ABC and ADF' are congruent. Therefore, of Δs ABC and DEF are also congruent. Thus, ***RHS condition of congruency is proved***.

d. **SSS Condition:** in figure given below two Δs ABC and DEF have sides AB=DE, BC=EF and CA=FD.

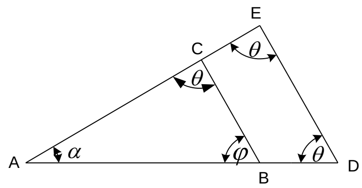
Proof: Place vertex D over E and lay side DE on AB. Vertex E would coincide with B since lengths AB=DE. Now, turn Δ DEF through 180^0 about side AB such that vertex F lies as F' on side of AB opposite to that of vertex C and both the Δs ABC and ABF' are on the same plane. Join vertices C and F', crossing line AB at point P. Thus two isosceles Δs AF'C, and CF'B are formed where ∠ACP =∠AF'P and ∠BCP =∠BF'P; angles opposite to equal sides are equal. Adding the two equalities, ∠ACP' +∠PCB= ∠AF'P +∠BF'P; ∠ACB =∠AF'C =∠AFC. Thus, it satisfies SAS conditions of congruency for the Two Δs the Δs ABC and ABF'. Accordingly, Δs ABC and DEF thus ***SSS condition of congruency of Δs is proved***.

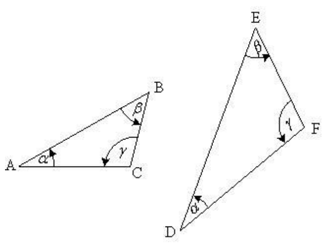


Next stage is examining ***property of Similar Triangles*** *which have three interiors angles mutually equal*. Conceptual clarity in this respect would need a revisit to || lines together with congruency of Δs. Lines, AB, CD and EF are equidistant parallel lines having a transversal GH intersecting the || lines at points K, L and M respectively. Draw ⊥ LR, to lone AB from point L and ⊥ MP on line CD and extend it to Q on line AB. In Δs KLR and MLP, ∠KLR =∠LMP, being corresponding angles, Sides LR=MP being ⊥ distance between equidistant || lines, and ∠LRK = ∠MPL= ʟs. Thus, ASA conditions of congruency of the Two Δs are satisfied and hence sides KL = LM and KR = LP. Now in Δs KLR and KMQ, ∠LKR is common, while ∠KLR =∠KMQ being corresponding ∠s , and ∠LRK=∠MQK being ʟs . Thus, these two Δs KRL and KQM have all three ∠s equal, and hence by definition they are similar Δs. Further, LR = PQ = MP. Hence, in Δs KRL and KQM, MQ= 2xLR; likewise MK = 2xKL, and KQ = 2xKR. Accordingly, in the similar Δs under consideration their corresponding sides are $\frac{KR}{KQ} = \frac{LR}{MQ} = \frac{KL}{KM}$is equal to a constant; in the instant case this constant is $\frac{1}{2}$.



Thus, it proves the ***property of similar Δs wherein ratio of length of corresponding sides is equal, or algebraically of the same proportion***. It can be generalized as under for figure of similar Δs ABC and AED shown in the figure below. It is for convenience that vertex A is taken to be common among the two Δs and sides AC and AB are taken to be collinear with sides AE and AD, respectively. Here, ∠A = α is common, while ∠C=∠E =θ ,∠B=∠D = φ, and therefore theΔs are similar. With the property of similar Δs: $\frac{AB}{AD} = \frac{AC}{AE} = \frac{BC}{DE} = K$, where K is constant or *length of sides of similar Δs are of same proportion*. It is not necessary for the similar Δs to have either one common vertex and over laying side as shown in the figure above. This property of similar proportion of corresponding is valid for any set of Δs whose three angles are equal to the other as shown in the figure, irrespective of their orientations. In this corresponding sides are those which joins two vertices Δs which have equal angles. For illustration side AB in ΔABC is between vertices A and B having their corresponding angles □ and □, while, in ΔDEF, side DF is between vertices D and E, and therefore these two are corresponding sides.





This makes a case to introduce ***Ratio –Proportions*** through ***Algebra***.

Basics of Algebra (contd.) [1,2,3,4,6,7,8]: It is a method of expressing a specific case in a generalized form, which can be used in any specific case with a set of values related to it. The ***expression*** is *represented with* ***algebraic constants***, *having fixed values,* ***variables*** *which can assume or be assigned a value based on specific case or condition and numeric* *constants*. An algebraic expression, typically shown below, has x, y and z as *variables*, while a, b, c, d as *algebraic constants* and 4 is a *numeric constant*.

$$x + axy - by \div cz + dz + 4$$

In the *algebraic expression, arithmetic operators* $+, -, \times, \div$ *and* $=$ *retain the same meaning as elaborated in BODMAS rule*. In case coefficient of any variable is 1, it is omitted in the expression as seen in the first term of expression. Likewise, in second term ✕ is omitted between coefficient ***a***, and, variables ***x*** and ***y***. Likewise, ✕ is omitted in second and third term. The above expression is open ended since it does not have sign $=$ for equating it to either set of variables or constant, and is called ***polynomial***. Use of operator $=$ equates two polynomial to create an expression called ***equation.*** In equation only specific values of the variables would satisfy the conditions of equality. In absence of operator $=$ expression is called polynomial and value of the polynomial would depend upon the values of variables under consideration.

A *polynomial is further classified in terms of variables with a single term* is called *Monomial*, with Two terms *Binomial*, with three terms *Trinomial* and so on. Accordingly, a polynomial having only a constant term is called a *Constant polynomial*. In case this constant is Zero then it is called *Zero polynomial*. Further, if polynomial involves only one variable then it is called polynomial of single variable, if two terms then polynomial of Two variables, and so on.

Another classification of polynomial is based on index of variables called **degree or order, which is always an integer**. A polynomial having only one or more constants $a + b + c = ax^0 + bx^0 + cx^0$ has variable terms with Zero exponent and hence it is called ***polynomial of Zero Degree***. Polynomials involving terms having single variable with or without a coefficient is of first degree (called exponent or power) viz. *x+ay+by* etc. are called *polynomial of first order or degree, it is also called linear polynomial*. In case of even a single variable occurring with exponent 2 or two variable, with or without coefficient, as factors of any term in a polynomial are called *polynomial of second order or degree or quadratic polynomial* viz. x^2, a + bx + cx^2, a + bx + cy + dx^2 + ey^2, etc. are polynomials of second order or degree. Accordingly, based on highest sum of exponents of the variables degree of a polynomial is determined; degree and order of polynomial are use interchangeable being synonyms. Here, **it is important to note that degree of variable in each terms of the polynomial has to be as (+) ve integer, while coefficients are real numbers**. As an example $ax^2 + bx + c$ is trinomial of Second order, while $ax + b + \frac{c}{x}$ and $ax + bx^{\frac{1}{2}} + c$ are not polynomials. Further elaboration of polynomials creates a necessity to understand ***Theory of Indices,*** *which shall be elaborated a little later*.

It is essential to dovetail understanding of **equality and inequality**. *A polynomial when equated to a constant or another polynomial is called* **Equation**, *and two sides of the equation like a weighing balance are called Left Hand Side (LHS) and Right Hand Side (RHS), e.g.*$f(x) = 4$. Any of the basic mathematical operation (+, - , ×, and ÷) when identically performed simultaneously on both sides, the equation remains unchanged as under –

$$f(x) + 3 = 4 + 3\ ;\ f(x) - 5 = 4 - 5\ ;\ f(x) \times 2 = 4 \times 2\ ;\ \frac{f(x)}{7} = \frac{4}{7}$$

It is to noted that identical mathematical operation means that addend, subtrahend, multiplier and divisor on both sides of the equation are necessarily identical.

Necessary ***condition for equality of polynomials*** is that *coefficients of variables with identical power must be individually equal*. This property finds an extensive application in mathematics. This is illustrated below –

$$A_1x^3 + A_2x^2 + A_3x^1 + A_4xy + A_5y^2 + A_6 = B_1x^3 + B_2x^2 + B_3x^1 + B_4xy + B_5y^2 + B_6$$

Here, $A_1 = B_1;\ A_2 = B_2;\ A_3 = B_3;\ A_4 = B_4;\ A_5 = B_5;\ A_6 = B_6;$

Linear Equations: Polynomial of First order in single variable has been defined above as linear polynomial. Likewise *an equation of first order in a single variable is called* ***Linear Equation***. A typical example to solve linear equation is given.

$$4x-7=21 \rightarrow 4x=21+7 \rightarrow 4x=28 \rightarrow x=\frac{28}{4}=7$$

Solution of linear equation requires, keeping its variable term on one side of the equation and all constants on the other side, and leading to find value of variable. But, in case of a linear equation with variables more than One, keeping one variable on LHS and other variable(s) on the RHS together with constants. Thus, the value of variable on LHS, obtained in this manner shall contain remaining variable(s) and constants. This is an incomplete solution. Solution of linear equations involving multiple variable require – **(a)** *number of **independent linear equation**, involving one or more variables, under consideration, must be equal to number of variables involved*. This form a set of equations called **simultaneous linear equations (SLE)**, For example in case of Two, Three, Fours.... variables under consideration the set of SLE shall have Two, Three, Four ..... equations, respectively. And, **(b)** these simultaneous linear equations are converted into linear equation of each variable for arriving at the solution. Here, at (a) ***independent linear equations*** are elaborated with an example of SLE of Two variables, $a_1x+b_1y+c_1=0$, and $a_2x+b_2y+c_1=0$. Necessary condition for this is $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$. But, it has two possibilities – **(i)** if $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$ then equation represent ***parallel lines***, and **(ii)** if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ then they represent ***coincident or collinear lines***. This aspect shall be discussed in next chapter on **Foundation Mathematics** while elaborating Coordinate Geometry.

For simplicity SLE are taken in two variables viz. $7x-15y=2$ and $x+2y=3$. There are three algebraic methods of solving SLE, and in essence they all are based on the same principle at (b) cited in previous para. Each of the method is being elaborated using the same set of SLE shown in example cited above.

(i) **Substitution Method**:

Step 1: Solve 2nd equation for variable $x \rightarrow x=-2y+3$, for simplification.

Step 2: Eliminate variable x in 1st equation by substituting its value obtained in Step1. It leads to –

$7(-2y+3)-15y=2 \rightarrow -14y-15y+21=2 \rightarrow -29y=-19 \rightarrow y=\frac{19}{29}$, thus value of variable y has been obtained.

Step 3: Substitute value of variable y in equation obtained in Step 1. It leads to –

$x=-2\left(\frac{19}{29}\right)+3 \rightarrow x=-\frac{38}{29}+\frac{3}{1} \rightarrow x=\frac{-38+29\times3}{29}=\frac{-38+87}{29}=\frac{49}{29}$, thus value of x has also been obtained.

(ii) Elimination Method:

Step 1: A pair of Two equations are taken and coefficients of one variable are equated in the Two equations in pair-

$$(7x-15y)\times1=2\times1 \rightarrow 7x-15y=2$$

$$(x+2y)\times7=3\times7 \rightarrow 7x+14y=21$$

Step 2: Subtract one of the above equation from other, in the instant case 2nd equation is subtracted from the 1st-

$$(7x-15y)\times(-1)=2\times(-1) \rightarrow -7x+15y=-2$$

$$7x+14y=21$$

$0\times x+29y=19 \rightarrow y=\frac{19}{29}$, thus value of variable y is obtained.

Step 3: Substitute value of in step 2 in any equation, in the instant case it is taken to be First –

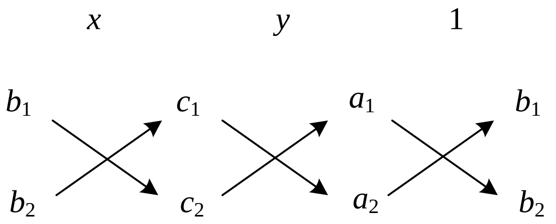
$$7x-15\left(\frac{19}{29}\right)=2\rightarrow 7x=2+\frac{15\times 19}{29}=\frac{2\times 29+285}{29}=\frac{343}{29}\rightarrow x=\frac{343}{29\times 7}=\frac{49}{29}$$, this value of x

(iii) **Cross-multiplication Method:** This method is an improved form of Elimination Method, involving concept of Determinants and Matrix. These two topics are elaborated in a separate Chapter in Mathematics Section and is more appropriate for computer application.

Step 1: Transform SLE with all terms on LHS, such that each of them have RHE equal to Zero, as under -

$$x+2y=3\rightarrow \quad x+2y-3=0\equiv a_2x+b_2y+c_2=0$$
$$7x-15y=2\rightarrow 7x-15y-2=0\equiv a_1x+b_1y+c_1=0$$

Step 2: Arrange the coefficients in the manner shown below



$$\equiv \frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

Step 3: Values of the variables are – $x=\dfrac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$; $y=\dfrac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$

Conclusion: (i) Each of the three methods give same values of the variables and are therefore mathematically equivalent. Therefore, choice of a method of solving SLE depends upon convenience.

(ii) Solving SLE involving more than Two variables is an extension of these methods to reduce them into linear equations of each of the variable as brought out at requirement (b) of solving them.

Inequalities of two expressions is expressed with notation > when LHS is greater than RHS, while notation < is used when LHS is smaller than RHS. *Similar to equation, inequality remains unchanged when identical mathematical addition or subtraction is performed on both sides of an equality*. But, <u>only</u> *during multiplication and division operations if multiplier and/or divisor is (+)ve then inequality remains unchanged*. However, *in case the multiplier/divisor is (-)ve the inequality reverses*. This can be understood with the concept of number line; the negative multiplier changes the position of the value of an expression on number-line from left-to-right and vice-versa and, therefore, a value on right and this is basic reason of reversal of inequality.

Theory of Indices [4,5,7,8]: It starts with understanding of x multiplied to itself and is represented, in language of mathematics as $x\times x=x^2$. In arithmetic if x = 3, then 3^2=9. Algebraically x multiplied to itself ***n*** times is $x\times x$ ... n times $=x^n$. In this expression x is called **base**, while n is called ***index, exponent*** or ***power*** of x. These expressions are compact form of an expression containing multipliers called *factors*. It is similar to that of prime factors in arithmetic. Here, each factor in isolation has index One (1) which is not written explicitly similar to that in algebra when coefficient is One (1).

Multiplication of two algebraic exponentials, with a common base and different exponents. It is explained with : $x^a\times x^b=x^{a+b}$, known as ***First Law of Theory of Indices***. This becomes explicit with an example, using only integer bases and indices for convenience, as under –

$$x^2\times x^3=(x\times x)\times(x\times x\times x)=x\times x\times x\times x\times x=x^5$$

This logic is extended to exponent of a term which in itself is an exponential term viz. $(x^a)^b$. It is similar to multiplication of an exponential term to itself (***b-1***) number of times: $(x^a)^b = x^{ab}$, known as ***Second Law of Theory of Indices***. Here, ***ab*** is not a separate variable but it is equivalent to $a \times b$. This is explained with an example $(x^4)^2 = (x^4) \times (x^4) = x^{4\times2} = x^8$ where net exponent is product of two exponents. In another situation having different bases, but identical indices is expressed as $x^a \times y^a = (xy)^a$, known as ***Third Law of Theory of Indices***. It can be illustrated through an example, using associative property of multiplication, as under –

$$5^3 \times 7^3 = (5 \times 5 \times 5) \times (7 \times 7 \times 7) = (5 \times 7) \times (5 \times 7) \times (5 \times 7) = (5 \times 7)^3$$

Using these laws it can be proved that: $\frac{x^a}{x^b} = x^{a-b}$, and $x^0 = 1$, where *a*=*b*. Decimal number in scientific notations uses theory of indices e.g. 1567.79=1.56779x10^3, while 0.000156=1.56x10^{-4}. This simplifies multiplication and division, with desired degree of precision, and would be brought out in **logarithm** and introduction to **Physics**, that would be discussed in separately in another chapter.

In case exponent (n) is fractional, say a^n, here $n = \frac{p}{q}$ **it leads to two possible representation and each of them has different connotations. Let,** Mathematically, in the expression if $a = 2$, $n = \frac{p}{q}$, where $p = 1$ and $q = 2$, then $4^{\frac{1}{2}} = \pm 2$ and can be verified square of the two values, as per theory of indices, but when a is written under radical sign as $\sqrt{4} = 2$, which only a (+) ve value and is identified in mathematics as ***Surd***. Thus, in *surds fractional index is represented with radical sign involving, while x is a real and* (+)*ve number*. Thus, two expressions $x^{\frac{a}{b}} = (x^a)^{\frac{1}{b}} \to (x^a)^{\frac{1}{b}}$, while $\sqrt[b]{x^a} = \sqrt[b]{x^a} \to \left(\sqrt[b]{x}\right)^a$. follow different mathematical treatment. Surds are used to represent those arithmetical numbers which cannot be determined exactly irrational number and the radical sign is also called root. It is to be noted that there are two possibilities of root of a rational number – **(a)** it can be rational viz. $\sqrt[3]{27} = \sqrt[3]{3^3} = 3^{\frac{3}{3}} = 3$ or **(b)** it could be irrational viz. $\sqrt[2]{8} = \sqrt[2]{2^3} = 2^{\frac{3}{2}} = 2 \times 2^{\frac{1}{2}} = 2 \times \sqrt{2}$ and this is because of occurrence of $\sqrt{2}$ in the final result. Nevertheless, use of radical sign makes it surd.

Typically, a rational number represented within a radical sign viz. $\sqrt{4}$ is called ***quadratic surd*** of 4, is equal to +2. Surds possess *associative property*, where$\sqrt{6} = \sqrt{3 \times 2} = \sqrt{3} \times \sqrt{2}$ and $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{2^2} \times \sqrt{2} = 2\sqrt{2}$. This shows that surds follow laws of multiplication. This is extended to division also viz. for fraction of surds $\frac{\sqrt{5}}{\sqrt{3}} = \sqrt{5} \times \frac{1}{\sqrt{3}}$, and determining value of this expression is made with division operation using approximate values of $\sqrt{5}$ and $\sqrt{3}$. This type of conversion can be simplified ***rationalizing denominator*** of the denominator-surd into a rational-number as under –

$$\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{\sqrt{15}}{3}$$

Here, $\sqrt{2}$ or $\sqrt{15}$ are called ***pure surds***, while $2\sqrt{2}$ and $\frac{\sqrt{15}}{3}$ are called ***mixed surds***. Addition or subtraction of two or more dissimilar surds is called ***Compound Surd***. A compound surd having two terms is called ***Binomial Surd*** and so on similar to the nomenclature of *polynomials*. Two binomial compound surds say $\sqrt{x} + \sqrt{y}$ and $\sqrt{x} - \sqrt{y}$, where both x and y are real numbers, are called ***Conjugate or Complementary Surds***. Products of conjugate surds $\left(\sqrt{x} + \sqrt{y}\right) \times \left(\sqrt{x} - \sqrt{y}\right) = x - y$ is a rational number provided both *x and y are real numbers which be either rational or irrational*. Conversion of a compound surd using its conjugate surd is also called rationalization of compound surd. Here conjugate surd is called ***rationalizing factor*** of the compound surd; this is mutually true for both the compound and conjugate surds.

This product of compound surd with its conjugate surd would be (+)ve or (-)ve based on relative values of x and y. Likewise, division of surds by another binomial surd is carried out by its, *rationalization* the denominator; this is done by multiplying numerator and denominator with a surd conjugate of the denominator surd e.g. –

$$\frac{\sqrt{4}+\sqrt{5}}{\sqrt{3}+\sqrt{2}} = \left(\frac{\sqrt{4}+\sqrt{5}}{\sqrt{3}+\sqrt{2}}\right) \times \left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right) = \frac{\sqrt{12}+\sqrt{15}-\sqrt{8}-\sqrt{10}}{3-2} = \sqrt{15}+\sqrt{12}-\sqrt{10}-\sqrt{8}$$

In the above example, as per definition of surd, all numbers within radical sign were taken to be +ve. If it were a –ve number say -4, then -1 within radical sign is dissociated, using associative property, and written as $\sqrt{-4} = \sqrt{(-1)\times 4} = \sqrt{-1}\times\sqrt{4} = 2\sqrt{-1}$. *This typical situation with present knowledge of multiplication of signed numbers cannot be explained and shall be illustrated little later as a special case of **quadratic equation**.* This gave rise to new identity $\sqrt{-1} = i$ (called **iota** a Greek alphabet) or simply an unsigned imaginary ***i*** Accordingly, $\sqrt{-4} = 2i$. Despite ***i*** defined as imaginary has an important role defining vibrations and oscillations a reality of every system that is experienced in life. Accordingly, *x+iy, where both x and y are real, but presence of i makes second terms, with a coefficient ı is called a **complex number**. In complex number x and iy are called real component and imaginary component, respectively*. Thus roots of quadratic equation, under typical condition, are ***complex conjugates*** i.e. $x \pm iy$. *A fraction of complex numbers is resolved into a rational denominator by multiplying the numerator and the denominator of the fraction with rationalizing factor which is a complex conjugate of the denominator*. A complex root i.e. either $x+iy$ or $x-iy$, does not exist in isolation and are the basic cause of oscillatory behaviour which would be **studied later in physics**. Following identities are extremely helpful in handling complex numbers – $\sqrt{-1} = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ …….

This has been be summarized here under –

k	$i^{(4k+1)}$	$i^{(4k+2)}$	$i^{(4k+3)}$	$i^{(4k+4)}$
0				
1				
2	i	-1	-i	1
3				
4				
…				

Based on these discussions, following inferences are derived –

1. Value of a Surd is always a +ve ,
2. Pure surd is an irrational number,
3. Base of surds can be multiplied if their fractional indices are of same order viz. $\sqrt[n]{a}\times\sqrt[n]{b} = \sqrt[n]{(a\times b)}$,
4. Multiplication of two pure surds is an irrational rational number viz. given that $\sqrt[p]{a}$ and $\sqrt[q]{b}$ both are irrational numbers as per (2) above and therefore their product $\sqrt[p]{a}\times\sqrt[q]{b}$ is also irrational number, as per closure property of irrational numbers discussed in number systems.
5. Multiplication of a surd $\sqrt[n]{b}$ and a rational number a is named ***Mixed Surd***, viz.
6. Two mixed surd $a+\sqrt[p]{b}$ and $\sqrt[p]{b} = \sqrt[q]{d}$ can be equal iff rational and irrational parts of both the surds are mutually equal viz. if $a+\sqrt[p]{b} = c+\sqrt[q]{d}$ then $a=b$ and $\sqrt[p]{b} = \sqrt[q]{d}$.

There is another typical inference in respect of imaginary irrational numbers viz. $\sqrt{-3} = (i\sqrt{3})$ and $\sqrt{-5} = (i\sqrt{5})$. In this case product of irrational numbers is irrational, as per its closure property. But, imaginary characters of the result is regulated by product of iota product as is evident from example $\left(i\sqrt{3}\right)\times\left(i\sqrt{5}\right)=i^2\left(\sqrt{3}\times\sqrt{5}\right)=-\left(\sqrt{3}\times\sqrt{5}\right)$. It violates closure property during multiplication of imaginary irrational numbers.

Ratio and Proportions*[1]:** This concept of ratio wherein two different quantities are compared it is mathematically expressed as ratios of same proportion, e.g. ratio of number of boys (20) and girls (15) in a class is mathematically expressed as $20:15$ or$4:3=\frac{4}{3}$; ratio can also be expressed as a simple fraction. In a ratio, say $a:b$ first term (a) is called ***antecedent, while the second term (b) is called ***consequent***. Ratio is used to define quantitative comparison of two distinct quantities say average percentage number of marks obtained in an examination by boys and girls are 88% and 66%, then ratio of these percentage marks is 88%:66% $=4:3=\frac{4}{3}$. It is having same ratio as that of number of students, despite nature of the number of students and percentage marks being different. Such ratios are mathematically stated to be ***proportion.*** In algebraic form say for two set of quantities a and b, and c and d be such that $a:b=\frac{a}{b}=k$, and $c:d=\frac{c}{d}=k$, then mathematically $a:b :: c:d \rightarrow \frac{a}{b}=\frac{c}{d}=k$; here k is called ***proportionality constant*** *and mathematically it is equal to value of the fractions, a real +ve number*. Once ratio is defined it is possible to determine individual quantity out of summative quantity of the constituents. As an example a concrete mixture is 1:3:5 of cement, metal and sand by volume, then for 5 units (by volume) of cement the mixture would contain 15 units of sand and 25 units of sand. Likewise, for a 150 Cu-M concrete requirement of cement, metal and sand would be $\frac{1}{1+3+5}\times 150=\frac{150}{9}=16\frac{6}{9}=16\frac{2}{3}$ Cu-M, $\frac{3}{1+3+5}\times 150=\frac{450}{9}=50$ Cu-M, $\frac{5}{1+3+5}\times 150=\frac{750}{9}=\frac{250}{9}=27\frac{7}{9}$ Cu-M respectively. This can be applied to any quantity of relevance, and it is seen that fractions, used to define part of a quantity becomes intrinsic to ratios when considered in totality.

If there are three discrete quantities such that $a:b=b:c$, then $\frac{a}{b}=\frac{b}{c}=k$. It leads to mathematical induction $b^2=ac$ or $b=\sqrt{ac}$. Such proportionality is expressed as $a:b:c:d\ldots=\cdots$ is equivalent to $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\ldots.=k$ and is called ***continued proportion***.

The algebraic properties of ratio-proportion are. ***alternando, invertrendo, Componendo, Dividendo and Componendo-Dividendo***. These properties find application in algebraic manipulation of a mathematical expression for simplification of generalized statement. These are explained with another example of a school where numbers of students in each class are different, but the ratio of boys and girls is same viz. in class 9^{th} and 10^{th} is of same proportion, and is mathematically stated as $B_9:G_9::B_{10}:G_{10}\rightarrow a:b::c:d\rightarrow\frac{a}{b}=\frac{c}{d}$. Each of the property is mathematically is derived below-

Invertendo: Given that $\frac{a}{b}=\frac{c}{d}=k\rightarrow\frac{1}{k}=\frac{1}{\frac{a}{b}}=\frac{1\times\frac{b}{a}}{\frac{a}{b}\times\frac{b}{a}}=\frac{\frac{b}{a}}{1}=\frac{b}{a}$. Likewise, $\frac{1}{k}=\frac{d}{c}$; , therefore $\frac{d}{c}=\frac{b}{a}$, is another form of the ratio-proportion called ***invertendo***.

Alternando: Taking $\frac{a}{b}=\frac{c}{d}$, on cross multiplication, $a\times d=b\times c$. On dividing both sides of equation by $c\times d$, it leads to $\frac{a\times d}{c\times d}=\frac{b\times c}{c\times d}\rightarrow\frac{a}{c}=\frac{b}{d}$ is a new form of the ratio-proportion called a***lternando***.

Componendo: Taking $\frac{a}{b}=\frac{c}{d}$ and add One (1) to both sides of the ratio-proportion, it leads to $\frac{a}{b}+1=\frac{c}{d}+1$. Solving the new equation $\frac{a+b}{b}=\frac{c+d}{d}$ is a new form called ***componendo.***

Dividendo: It is a modified form of componendo where instead of adding One (1) is subtracted from both sides of the given ratio-proportion, it leads to $\frac{a}{b}-1=\frac{c}{d}-1$. Solving the new equation $\frac{a-b}{b}=\frac{c-d}{d}$ is another form called ***dividendo.***

Componendo-Dividendo: This is a combination of the independent equation of componendo divided by equation derived from dividend. It leads to $\frac{\frac{a+b}{b}=\frac{c+d}{d}}{\frac{a-b}{b}=\frac{c-d}{d}}$.This simplifies into $\left(\frac{a+b}{b}\right)\times\left(\frac{b}{a-b}\right)=\left(\frac{c+d}{d}\right)\times\left(\frac{d}{c-d}\right)\rightarrow\frac{a+b}{a-b}=\frac{c+d}{c-d}$, is a newer form of ratio-proportion called ***componendo-dividendo.***

Relationship between two quantities is also classified as ***Direct Proportion*** and ***Inverse Proportion;*** *it is algebraic representation of arithmetic variation in quantities handled by* ***Unitary method,*** brought out at the end of this chapter. However, this concept is being integrated with an example. Say, price of a notebook is fixed say it is Rs *X per notebook*. If number of notebooks to be purchased is N, then total cost *C* (in rupees), of the notebooks, would be $C = NX$. In the language of algebra it is expressed as $\frac{C}{N} = X$. Another way of *expressing this in the language of algebra (mathematical notations) is* $C \propto N$. *Where, X- rate per notebook, is the constant of proportionality, and it stated in language of mathematics as C is in* ***Direct Proportion*** *to X, or simply Proportional to number of copies.*

Another example is of distance, speed and time. Distance between home and school is say *D* meters. If *S* is the speed, in meters per minute, at which one walks to the school and time taken to reach the school is *T*. These three quantities can be related as $T = \frac{D}{S} = D \times \frac{1}{s}$. In this example distance *D* being fixed, this relationship is expressed as $T \propto \frac{1}{S}$, i.e. *time of travel is* ***Inversely Proportional*** *to speed of travel.* In this relationship if all the three are taken as variables, the expression becomes $D = T \times S$. If T, time of travel, is taken as constant then distance of travel can be expressed as $D \propto S$. In day-to-day experiences parameters being considered are generally variables. Accordingly, ***Direct or Inverse Proportion*** *is a matter of consideration concerning the relationship of variables involved.* Nevertheless, *further journey into physics, in chapters to follow, would involve some universal constants to make the Direct and Inverse proportions discrete*.

Application of Proportionality and Inverse Proportionality in arithmetic simplifies problem solving, rather than constructing long and tedious ***unitary statements***.

Inequality in ratio: It can be created by adding or subtracting a number to both antecedent and consequent. Taking a ratio $\frac{a}{b}=k$ in which *x* is added to both the terms of the ratio then. $\frac{a+x}{b+x}=k'\neq k$. This inequality is greater of lesser would depend upon value of $k>1$ or $k<1$.

a. **If** $k > 1$: Applying ***dividendo*** to original ratio and modified ratio $\frac{a-b}{b} = k - 1$ and $\frac{(a+x)-(b+x)}{b+x} = \frac{a-b}{b+x} = k'-1$. Dividing LHS with LHS and corresponding RHS with RHS, $\frac{\frac{a-b}{b}}{\frac{a-b}{b+x}} = \frac{k-1}{k'-1} \to \frac{b+x}{b} = \frac{k-1}{k'-1}$. It leads to another two possibilities –

- **i.** If x is (+)ve $b+x > b$ and, therefore, the equation reduces $k-1 > k'-1 \to k > k'$, accordingly $\frac{a}{b} > \frac{a+x}{b+x}$.
- **ii.** If x is (-)ve $b+x < b$ and, therefore, the equation reduces $k-1 < k'-1 \to k < k'$, accordingly $\frac{a}{b} < \frac{a+x}{b+x}$.

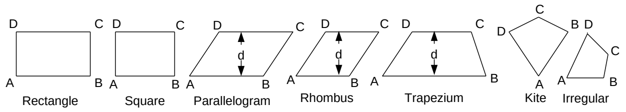
b. **If** $k < 1$: Applying ***dividendo*** to original ratio and modified ratio $\frac{a+b}{b} = k + 1$ and $\frac{a-b}{b+x} = k'-1$. Dividing LHS with LHS and corresponding RHS with RHS, $\frac{\frac{a-b}{b}}{\frac{a-b}{b+x}} = \frac{k-1}{k'-1} \to \frac{b+x}{b} = \frac{k-1}{k'-1}$. It leads to another two possibilities –

- **i.** If x is (+)ve $b+x > b$ and, therefore, the equation reduces $k-1 > k'-1$. As per premise $k-1 < 0$ and, therefore, it is (-)ve. Taking (-)ve number on a number-line, absolute value of a larger (-)ve number is smaller than the absolute value of smaller (-)ve number, therefore, $k < k'$ and accordingly, $\frac{a}{b} < \frac{a+x}{b+x}$.
- **ii.** If x is (-)ve $b+x < b$ and, therefore, the equation reduces $k-1 < k'-1 \to k < k'$, accordingly, $\frac{a}{b} < \frac{a+x}{b+x}$.

Equality-Inequality of Two Ratio: Let $\frac{a}{b}$ and $\frac{c}{d}$ are two ratios. It leads to three cases as under –

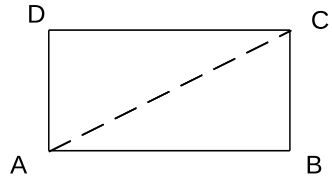
- **a. Equality:** If $\frac{a}{b} = \frac{c}{d} \to ad = bc \to ad - bc = 0$
- b. **Greater Inequality:** If $\frac{a}{b} > \frac{c}{d} \to ad > bc \to ad - bc > 0$
- c. **Lesser Inequality:** $\frac{a}{b} < \frac{c}{d} \to ad < bc \to ad - bc < 0$

Basics of Geometry (Contd.):***Quadrilateral*** *is a shape which is enclosed by four sides, and thus it has four vertices. Based on sides classification of quadrilaterals is*- **a)** *Rectangle,* **b)** *Square,* **c)** *Parallelogram,* **d)** *Rhombus,* **e)** *Trapezium,* **f)** *Kite and* **g)** *Irregular Quadrilateral,* as shown in the figure.

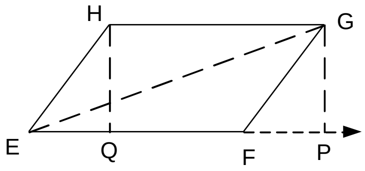


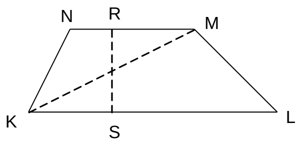
It would be seen that Square is a special case of Rectangle, while rhombus is a special case of parallelogram. Here, an important derivation of the relationship between area of a triangle and rectangle, a parallelogram and a trapezium shall be made which is useful in derivation of Pythagoras Theorem, an important inference mathematics.

In figure given below ABCD is a rectangle. Properties of parallel lines suggest that BC or AD are perpendicular distance between AB || DC, likewise AB and DC are perpendicular distance between BC||AD. The rectangle ABCD is formed by either traversing line AB or DC over distance BC and *area thus covered is expressed in mathematics as* $AB_{Length} \times BC_{Length}$. Constructing a diagonal AC, it bisects the rectangle in two Δs ABC and ACD as they satisfy *RHS condition of congruency*. Therefore, areas $ABCD_{ar}=\Delta ABC_{ar}+ \Delta ACD_{ar}=2x\Delta ABC_{ar}$. Accordingly, ΔABC_{ar} = ½ $ABCD_{ar}$=½(ABxBC)=½((Length) x (Height)).



Likewise, in parallelogram EFGH, shown below, diagonal EG is constructed and a ⊥GP is drawn on extended line segment EF. Here, PG is ⊥distance between EP || GH. Two Δs EFG and EGH are congruent as they satisfy either of SAS and ASA conditions and hence their areas are equal. Further, Δs EHQ and EGH, can also be easily proved congruent by RHS theorem and shall have same area, Thus area of parallelogram EFGH when added Δ FPG and subtracted with Δ EHQ, it will result in a rectangle HQPG of an equal area. Here it is to be noted that base of the rectangle HQPG is $QP_{Length}= EF_{Length} + FP_{Length} - EQ_{Length} \rightarrow QP_{Length}= EF_{Length}$. Moreover, both the paralleogram EFGH and rectangle HQPG are between the same pair of lines EP || GH . Thus three conclusions are arrived at – **(a)** Areas of a rectangle and a parallelogram with equal base and same pair of || lines are equal, **(b)** Area of a parallelogram is equal to product of length of its base and distance between the base and line opposite to it (in the example elaborated $EFGH_{ar}=EF \times GH$) and **(c)** Diagonal of a parallelogram bisect it in Two congruent Δs.





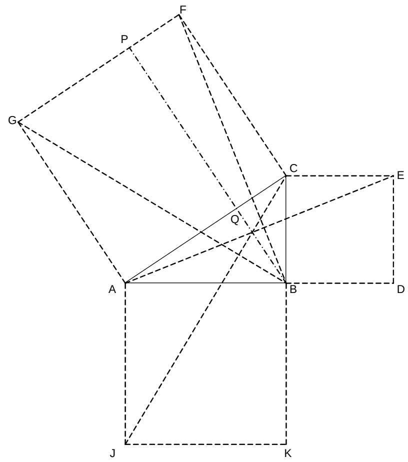
In case of Trapezium KLMN, shown here, diagonal KM divides the shape in Two unequal scalene Δs KLM and KMN. But, they have equal heights being between ||KL || MN. Therefore, $KLMN_{ar} = \Delta KLM_{ar} + \Delta KMN_{ar}$= ½ ((KLxRS) + (MNxRS) = ½ ((KL + MN)x(RS)). It leads to : $KLMN_{ar}$ = ½ (Sum of length of || lines)x(Distance between ||lines)

Properties of Quadrilateral are summarized in the table below –

Shapes	**Sides**	**Angles**	**Diagonals**	**Area**	**Perimeter**
Rectangular	AB=CD; BC=DA	∠A=∠B=∠C=∠D=∟	Equal and bisect	ABxBC	2 (AB+BC)
Square	AB=BC=CD=DA	∠A=∠B=∠C=∠D=∟	Equal, bisect and ⊥	$(AB)^2$	4xAB
Parallelogram	AB = & || CD ; BC = & || DA	∠A=∠C; ∠B=∠D	Unequal and bisect	(ABxd) *	2 (AB+BC)
Square	AB=BC=CD=DA; Opposite sides ||	∠A=∠C; ∠B=∠D	Unequal, bisect and ⊥	(ABxd) *	4xAB
Trapezium	AB≠BC≠CD≠DA; AB || CD	Opposite angles may not be equal	Unequal and may not bisect	½ *(AB+CD)xd*	(AB+BC+CS+ DA)
Kite	AB=CD; BC=CA	∠A ≠∠C; ∠B =∠D	Unequal, and ⊥	½ (ACxBD)	2(AB+BC)
Irregular Quadrilateral	AB≠BC≠CD≠DA	∠A ≠∠C ≠∠B ≠∠D	Unequal	Based on actual sides and angles	(AB+BC+CD+ DA)

Note: * - d is the ⊥ distance between || sides of quadrilateral.

Pythagoras Theorem: It is a property of ∟Δs which stipulates $(\text{Hypotenuse})^2 = (\text{Base})^2 + (\text{Height})^2$. In ∟ΔABC, shown below, horizontal line (AB) is called Base, while vertical line (BC) is called Height and the third side, opposite to the ∟ of the Δ is called ***Hypotenuse*** and is longest side of Triangle. *These definitions of Base and Perpendicular would be assigned an additional attribute when Trigonometry is introduced in later chapter*. On the given ΔABC constructions are made as shown by the dotted lines. This also creates two rectangles AQPG and QCFP out of square ACFG on hypotenuse.



Further, Δs ABG and AJC are congruent by SAS conditions. Further, ΔAJC and square AJKB are between KC||AJ, and have common base AJC therefore, ΔAJC_{ar} = ½ $(\text{Square AJKB})_{ar}$. With the congruency this equality can be extended to $(\text{Rectangle AQPG})_{ar} = (\text{Square AJKB})_{ar}$. Likewise, Δs AEC and BCF are congruent. Moreover, Δ ECA and square BDEC are between EC|| AD, and have common base EC therefore, ΔEAC_{ar} = ½ $(\text{Square BDEC})_{ar}$. With the congruencies of the Δs, equality can be extended to $(\text{Rectangle QCFP})_{ar} = (\text{Square BDEC})_{ar}$. With these two identities, it leads to to Two geometrical equations:

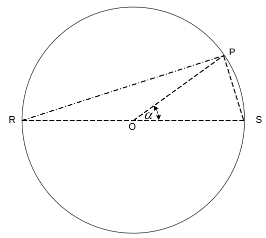
$(\text{Rectangle AQPG})_{ar} + (\text{Rectangle QCFP})_{ar} = (\text{Square AJKB})_{ar} + (\text{Square BDECD})_{ar}$

$$(\text{Square ACFG})_{ar} = (\text{Square AJKB})_{ar} + (\text{Square BDECD})_{ar}$$

This can be algebraically written as $AC^2 = AB^2 + BC^2$, ***a proof of Pythagoras Theorem***.

This gives rise to ***Pythagorean Triplet***, a part of number theory, derived with help of algebra and shall be taken a little later in this chapter.

Circle: *Is a closed shaped traced by a point maintaining a constant distance from a fixed point called* ***Centre, this arc is called Circle.*** Since it is a closed shape *it has neither a beginning nor an end. Distance of any point on the Circle from its centre is called* ***Radius*** *(designated as* ***r).*** Two radii OP and OS shall form an angle α at the centre, and **Arc Segment** PS. *The line segment PS joining ends of the arc PS, is called* ***Chord*** *of the arc. While, shape enclosed by lines PO, OS and arc PS is called* ***Sector***. *Perimeter of circle is* $2\pi r$. Here,∠ SOR=180^0; a straight angle and it is expressed in scale of Radians as π ($\frac{22}{7} = 3.14$). So also ∠ ROS is π radians. Accordingly, an angle traversed in tracing a circle is $360^0 = 2\pi$ Radians.

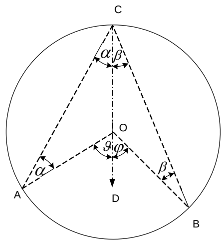


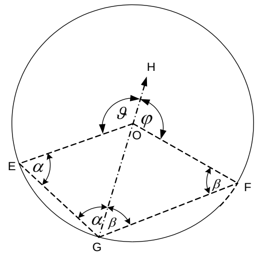
There are Five properties of Circle as – a) ***Perimeter*** *of circle is 2πr,* **b)** ***Area*** *of a circle is* πr^2, **c)** ***Diameter*** *is the longest Chord of a circle,* **d)** *Angle formed by an arc of a circle on its Centre is twice the Angle formed by the ends on any point on the complementary arc of the circle, and* **e)** *In a quadrilateral with a circumscribing circle (A circle passing through vertices) sum of pair of opposite angles of quadrilateral is π Radians.*

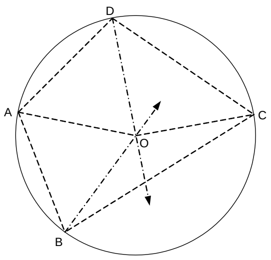
Perimeter of a circle is $2\pi r$ *(**Property a**) and Area of a circle is* $\pi r^2 (=\pi \frac{D^2}{4})$ (***Property b***). First property gives rise to defining irrational number Pi $\left(\pi \approx \frac{22}{7}\right)$, Pi is a Greek alphabet. Using this definition of Pi a ratio of circumference and diameter of circle is an irrational constant. Second property of the circle, for the present, is taken as such, while its proof shall be taken in Chapter III, when **Calculus** is introduced.

In the above figure take any point P anywhere on the perimeter of the circle. Three Chords RS, SR and PR form a Δ RSP, where RS is ***Diameter,*** it passes through Centre O, *with its length is Expressed as* ***d.*** The diameter intercepts circle at two points R and SP. As per property of Δ, relating length of sides, $RS_{Length} < (RO_{Length} + OP_{Length}) \rightarrow RP_{Length} < 2r$;); $RP_{Length} < d$; here *diameter, d=2r* . As point P approaches S, RP → RS = D, and PS→ 0 and points R, O and S become collinear. Hence, *Diameter is the longest chord of Circle [**Property c:Concluded**]*

In figure, on the left below, points A and B on the circle of radius ***r*** are below centre O and form two arcs. First is ***smaller arc or minor arc***, which starts at A and ends at B, traversing along the circle, but it is below the centre O of the circle. Whereas, the ***larger arc or major arc*** starts and ends from the same two points A and B of the circle but traverses along the circle through point C on the circle, above the centre. From point C, a ray CD is drawn passing through the centre of the circle. In the circle, three radii OA, OB and OC are identified. The ΔAOC, is isosceles (OC = OA = r) and together with this ∠AOD=∠OAC + ∠OCA = 2α; another property of Δ relating exterior angle to the opposite interior angles. Likewise, ∠BOD=∠OBC + ∠OCB = 2β. Adding these two equalities, ∠BOA= 2(α + β) = 2∠ACB. i.e. angle formed by an arc on the centre is equal to twice the angle formed by the an arc on any point on the remaining arc of the circle. This property is also valid for figure, on the right, below. *[**Property d: Concluded**]. This property is equally valid for the point if it is on the minor arc also, as shown in the figure.*

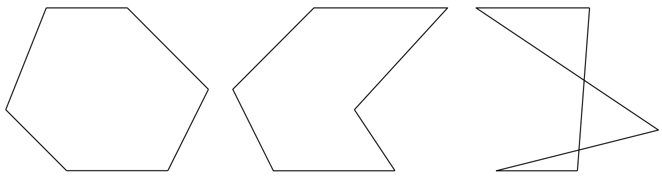






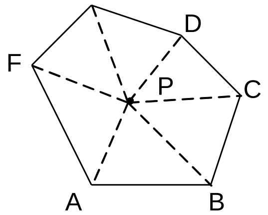
In a figure below a quadrilateral is drawn whose all four vertices are on a circle. With the *property (d)* above it can stated that $\angle AOC_{Obtuse} = 2\angle ABC$; likewise, $\angle AOC_{Reflex} = 2\angle ADC$. Adding the two equalities: $\angle AOC_{Obtuse} + \angle AOC_{Reflex} = 360^0 = 2\angle ABC + 2\angle ADC$; it leads to a conclusion that opposite angles $\angle ABC + \angle ADC = 180^0$. On similar lines it can be proved that $\angle DAB + \angle DCB = 180^0$. Thus it can be generalized that in a quadrilateral having a circumscribing circle, opposite angles are supplementary angles (**Property e: concluded**)

Polygon: *Any closed shaped since has three or more sides it is **Polygon**. Classification based on sides is **Regular Polygon** (having all sides equal); equilateral triangle and square also belong to this classification. While, **Irregular Polygon** having unequal sides.* Another classification of polygon is based on interior angles; ***Convex Polygon, Concave Polygon*** and ***Complex Polygon***, are shown in the figure below.

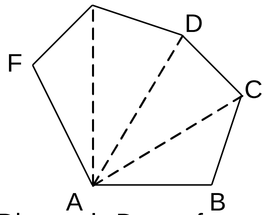


Convex Polygon Concave Polygon Complex Polygon

There are Two properties of convex polygons. Taking a point **P** inside a polygon, shown in the figure below on the left, and joining it vertices of the polygon would create ***n*** triangles. Sum of interior angles of ***n*** triangles is $2\pi n$ Radians. Since, sum of angles formed by vertices of ***n*** triangles on the internal point is 2π; subtracting, this from *sum of all interior angles of triangles gives sum of interior angles of a polygon = 2π(n-2) Radians*, (***First Property : Concluded***).



Internal Angles of Polygon



Diagonals Drawn from a Vertex of a Polygon

*In a polygon any line from a vertex which joins another vertex, not forming side of a polygon (non-co-linear point) is called **Diagonal**.* Taking case of a *Hexahon* where diagonals are drawn from any vertex (say A), as shown in the figure, it has three (*n*-3) Diagonals. The count is reduced by 3 due to the vertex under consideration and two adjoining vertices on either side do not form diagonal. In a polygon each of the remaining vertex forms Diagonal with vertex under consideration and vice-versa. Thus total number of Diagonal is reduced to half. Accordingly, *number of Diagonals shall in polygon is*= $\frac{n}{2}(n-3)$. ***(Second Property: Concluded).***

Basic of Algebra (Contd.): A polynomial in single variable, $7x + 4x^3 + 2$, in mathematical operations, for convenience, it is written in the order of descending indices of the variable as $4x^3 + 0x^2 + 7x + 2$. In the given polynomial term containing x^2 is missing yet, it is retained in the revised form with a coefficient Zero (0) , *a mathematical equivalent*. During ***addition and subtraction of polynomial*** they are so written that variables of same index are below another. Finally, coefficients are added or subtracted as per desired operation.

Case of multiplication is explained with an example from arithmetic in which multiplicand (56732) and multiplier (23) are written in expanded form as under, where variable *x* represents 10 (in decimal system) -

(Multiplicand) $56732 = 50000 + 6000 + 700 + 30 + 2$

$= 5 \times 10^4 + 6 \times 10^3 + 7 \times 10^2 + 3 \times 10^1 + 2 \times 10^0$

$= 5x^4 + 6x^3 + 7x^2 + 3x + 2$

(Multiplier) $23 = 20 + 3$

$= 2 \times 10^1 + 3 \times 10^0$

$= 2x + 3$

The multiplication of the two expressions would be as under –

$$5x^4 + 6x^3 + 7x^2 + 3x^1 + 2x^0$$
$$\times \quad 2x^1 + 3x^0$$

------ ---

$(15x^4 + 18x^3 + 21x^2 + 9x^1 + 6x^0) \times 2x^1 \rightarrow \quad 10x^5 + 12x^4 + 14x^3 + 6x^2 + 4x^1 + 0x^0$

$(15x^4 + 18x^3 + 21x^2 + 9x^1 + 6x^0) \times 3x^0 \rightarrow \quad 15x^4 + 18x^3 + 21x^2 + 9x^1 + 6x^0$

$$10x^5 + 27x^4 + 32x^3 + 27x^2 + 13x^1 + 6x^0$$

In this expression of product by replacing value of x in product we get –

$$10 \times 10^5 + 27 \times 10^4 + 32 \times 10^3 + 27 \times 10^2 + 13 \times 10^1 + 6 \times 10^0$$

= 1000000 + 270000 + 32000 +2700+130+6

=1304836

This elaboration is only to convey the ***basic concept of algebraic multiplication***, which in practice is done intuitively. *In **Algebraic Multiplication** each term of multiplicand is multiplied to each terms of multiplier, using theory of indices, and process in principle is similar to that in arithmetic multiplication.*

*Similarly, **Algebraic Division** is like arithmetic division, and is left to be attempted, while mentoring, for which a clue can be taken from the above multiplication. During division also dividend and divisor are arranged in descending order of the index of the variable. Consequently quotients are determined by matching highest index of divisor to the highest index of the dividend at each stage of division, progressively until a remainder is either zero or non-zero as per Euclid's Lemma.*

*There are some typical multiplications results which are widely used in algebra, and at times for simplification of arithmetic calculation, known as **Identities.*** These are brought out below for demonstrating their verification, durig mentoring and shall be found useful, very often–

1.$(x+p)(x+q) = x^2 + (p+q)x + pq \rightarrow (x+a)^2 = x^2 + 2ax + a^2|_{p=q=a}$
2. $(x-p)(x-q) = x^2 - (p+q)x + pq \rightarrow (x-a)^2 = x^2 - 2ax + a^2|_{p=q=a}$
3. $(x+p)(x-q) = x^2 + (p-q)x - pq \rightarrow (x+a)(x-a) = x^2 - a^2|_{p=q=a}$
4. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + 3ab(a+b) + b^3$
5. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - 3ab(a-b) - b^3$
6. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
7. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
8. $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 \dots + x^2y^{n-3} + xy^{n-2} + y^{n-1})\Big|_{n:\,even\ or\ odd\ integer}$
9. $x^n - y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 \dots + x^2y^{n-3} - xy^{n-2} + y^{n-1})|_{n:an\ even\ integer}$
9.$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 \dots - xy^{n-2} + y^{n-1})|_{n:an\ odd\ integer}$

When a polynomial on the RHS is equated to either another polynomial or Zero, it becomes an ***Equation***. The *left hand side* (**LHS**) of the First three identities, listed above, are factors of the polynomial on *right hand side* (**RHS**). This is in accordance with the definition of multiplication operation or prime factors of any non-prime number. If the polynomial is equated to Zero (0), then either of the factor can be equated to zero. In third identity, equating first Factor to Zero, we get $x = -p$ while equating second factor to Zero we get $x = q$. *Thus, -p and q are called* ***Roots of the Equation,*** $x^2 + (p-q)x - pq = 0$. *This equation is of order Two (2), and is called* ***Quadratic Equation***.

A polynomial $f(x)$on n^th order, as shown, below has n roots $\propto_1, \propto_2, \dots \propto_{n-1}, \propto_n$:

$$f(x) = A_0x^n + A_1x^{n-1} + \cdots A_{n-1}x^0 + A_n = (x-\alpha_1)(x-\alpha_2)\dots.(x-\alpha_{n-1})(x-\alpha_n)$$

A polynomial of order n is completely divisible any of the factor, and taking value of x as any of the root viz. $\propto_1, \propto_2, \dots \propto_{n-1}, \propto_n$ *the polynomial gets equated to zero; this is called* ***Factor Theorem***. *In an n^th order polynomial there are* ***n number of roots.*** *This is explained by the fact that the polynomial in factored form taking any one of the roots say* $x = \alpha_1$ *then* $f(x) = (x-\alpha_1)|_{x=\alpha_1}[(x-\alpha_2)\dots.(x-\alpha_{n-1})(x-\alpha_n)]$. *This reframes the polynomial into* $f(x) = (\alpha_1 - \alpha_1)[(x-\alpha_2)\dots.(x-\alpha_{n-1})(x-\alpha_n)] = 0 \times [(x-\alpha_2)\dots.(x-\alpha_{n-1})(x-\alpha_n)] = 0$

If the polynomial $f(x)$ is divided by $(x-k)$, then extending Euclid's Lemma into algebra , it leads to $f(x) = Q(x)(x-k) + R$. Taking value of $x = k$, the term $Q(x)(x-k)$ in the Lemma reduces to Zero and accordingly, $f(x)|_{x=k} = R$, where R is remainder and is called ***Remainder Theorem***. *If remainder is Zero (0), then* $(x-k)$ *is a factor of the polynomial.*

Factorization of a Quadratic Polynomial: A polynomial of form $p(x) = ax^2 + bx + c$, It can be of two forms based on signed value of a and c, and in turn each of them can be classified in another Two forms based on signed value of b . It is elaborated with respective examples in table below.

$p(x) = ax^2 \pm bx + c$ (Here, sign of a and c are same)		$p(x) = ax^2 \pm bx - c$ (Here, sign of a and c are opposite	
$p(x) = ax^2 + bx + c$	$p(x) = ax^2 - bx + c$	$p(x) = ax^2 + bx - c$	$p(x) = ax^2 - bx - c$
$6x^2 + 17x + 5$ • Take result of product $a \times b = 6 \times 5 = 30$ • Find factors of the result	$6x^2 - 17x + 5$ • Take result of product $a \times b = 6 \times 5 = 30$ • Find factors of the result	$8x^2 + 2x - 1$ • Take result of product $a \times b = 8 \times 1 = 8$ • Find factors of the result	$8x^2 + 2x - 1$ • Take result of product $a \times b = 8 \times 1 = 8$ • Find factors of the result

whose sum is equal to absolute value of $b=17$ ○ $1\times 30=30, 1+30=31\neq$ (rejected) ○ $2\times 15=30,\ 2+15=17$ (selected), **Halt.** ○ Rewrite polynomial substituting value of $17=2+15$, and proceed - • $6x^2+(2+15)x+5$ ○ $=6x^2+2x+15x+5$ ○ $=(6x^2+2x)+(15x+5)$ ○ $=2x(3x+1)+5(3x+1)$ ○ $=(3x+1)(2x+5)$ • Thus factors are $(3x+1)$ and $(2x+5)$ with roots $x=-\frac{1}{3}$ and $x=-\frac{2}{5}$, respectively. This conclusion can be verified with factor theorem.	whose sum is equal to absolute value of $b=17$ ○ $1\times 30=30, 1+30=31\neq$ (rejected) ○ $2\times 15=30,\ 2+15=17$ (selected), **Halt.** ○ Rewrite polynomial substituting value of $17=2+15$, and proceed - • $6x^2-(2+15)x+5$ ○ $=6x^2-2x-15x+5$ ○ $=(6x^2-2x)+(15x-5)$ ○ $=2x(3x-1)+5(3x-1)$ ○ $=(3x-1)(2x-5)$ • Thus factors are $(3x-1)$ and $(2x-5)$ with roots $x=\frac{1}{3}$ and $x=\frac{2}{5}$, respectively. This conclusion can be verified with factor theorem.	whose sum is equal to absolute value of $b=8$ ○ $1\times 8=8, 8-1=7\neq 2$ (rejected) ○ $2\times 4=8,\ 4-2=2=2$ (selected), **Halt.** ○ Rewrite polynomial substituting value of $2=4-2$, and proceed - • $8x^2+(4-2)x-1$ ○ $=8x^2+4x-2x-1$ ○ $=(8x^2+4x)-(2x+1)$ ○ $=4x(2x+1)-1(2x+1)$ ○ $=(2x+1)(4x-1)$ • Thus factors are $(4x-1)$ and $(2x+1)$ with roots $x=\frac{1}{4}$ and $x=-\frac{1}{2}$, respectively. This conclusion can be verified with factor theorem.	whose sum is equal to absolute value of $b=8$ ○ $1\times 8=8, 8-1=7\neq 2$ (rejected) ○ $2\times 4=8,\ 4-2=2=2$ (selected), **Halt.** ○ Rewrite polynomial substituting value of $2=4-2$, and proceed - • $8x^2-(4-2)x-1$ ○ $=8x^2-4x+2x-1$ ○ $=(8x^2-4x)+(2x-1)$ ○ $=4x(2x-1)+1(2x-1)$ ○ $=(2x-1)(4x+1)$ • Thus factors are $(4x+1)$ and $(2x-1)$ with roots $x=-\frac{1}{4}$ and $x=\frac{1}{2}$, respectively. This conclusion can be verified with factor theorem.
The factorization process is identical in both the cases with a difference in signed value of b, during factorization, and thus it results into different factors and roots.		The factorization process is identical in both the cases with a difference in signed value of b, during factorization, and thus it results into different factors and roots.	

*A polynomial, in single variable, of second order is called **Quadratic Equation*** is of special interest is mathematics; it gave the concept of ***imaginary numbers***. Let the equation $ax^2+bx+c=0$ has roots $ax^2+bx+c=(x-\propto)(x-\beta)=0$. The equation can be written as $(\sqrt{a}x)^2+bx=-c$, Conversion of LHS on a perfect square both sides need to be added with $\frac{b^2}{4a}=\left(\frac{b}{2\sqrt{a}}\right)^2$. Accordingly, new form of the equation is : $(\sqrt{a}x)^2+bx+\left(\frac{b}{2\sqrt{a}}\right)^2=\left(\frac{b}{2\sqrt{a}}\right)^2-c$; $\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2=\frac{b^2-4ac}{4a}$. Alternatively, this leads to $\sqrt{a}x+\frac{b}{2\sqrt{a}}=\pm\frac{\sqrt{b^2-4ac}}{2\sqrt{a}}$, which further simplifies into $\sqrt{a}x=\pm\frac{\sqrt{b^2-4ac}}{2\sqrt{a}}-\frac{b}{2\sqrt{a}}\rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Accordingly, the two roots α and β, in terms of coefficients a, b and c of quadratic equation, as $\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$. There is a specific observation $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$. This can be corroborated from the initial expression of the quadratic equation with its roots and is most generalized form of its solution.

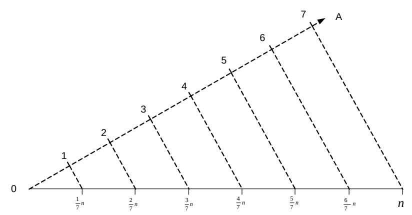
Square root of a +ve number $\left(n^{\frac{1}{2}}\right)$ is either +ve or –ve, and but $\sqrt{n}$ is a surd and is always +ve. If numerical values a, b and c are real but, such that $\mathrm{b}^2 - 4\mathrm{ac} < 0 \rightarrow \mathrm{b}^2 - 4\mathrm{ac} = -n$, then $\sqrt{\mathrm{b}^2 - 4\mathrm{ac}} = \sqrt{-\mathrm{n}} = \left(\sqrt{(-1)\mathrm{n}}\right) = \left(\sqrt{-1}\right) \times \sqrt{\mathrm{n}}$. Here, $\sqrt{-1} = i$, this is a new identity called ***iota*** represents an imaginary coefficient; when iota is multiplied to any real number it becomes an ***Imaginary Number***. When $\mathrm{b}^2 - 4\mathrm{ac} < 0$, the roots of the quadratic equation would be combination of real and imaginary numbers and it is called ***Complex Numbers***. *Complex roots is a reality and is basic characterstic of every oscillatory change.* It finds an extensive application in analysis of a system which will be a subject matter of higher mathematics and oscillations, to be taken up later.

Pythagorean Triplet: *Taking any number say m, a set of three other numbers derived from it which satisfies property of∟Δ as per Pythagoras theorem is called* ***Pythagorean Triplet***. Algebraically the set is $\{(m^2-1),(2m),(m^2+1)\}$; here (m^2+1) represents diagonal of ∟Δ. *The triplet can be verified as under –*

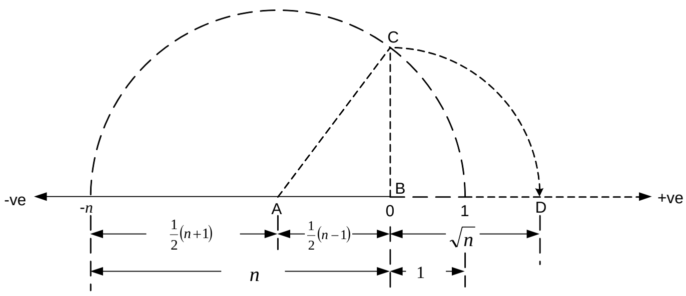
$$(m^2+1)^2 = (2m)^2 + (m^2-1)^2 \; ; \; m^4 + 2m^2 + 1 = 4m^2 + (m^4 - 2m^2 + 1).$$

Another ***Pythagorean Triplet*** with ***m***, directly as one side of a ∟Δ. *is* $\left\{\left(\frac{m^2}{4}-1\right), m, \left(\frac{m^2}{4}+1\right)\right\}$, *and it can be verified on the similar lines.*

Basic of Arithmetic (Contd.): In a figure below a fraction (say 3/7) of any number (n) is to be represented on a number line. It is difficult to find an exact and a measurable value of $\frac{3}{7}n$ and represent it on a number line. This has been done geometrically by using *property of similar triangle; corresponding sides are in same proportion*. Following steps are involved – **(a)** Take a number line and mark on it given number ***n*** on it, **(b)** At any angle, a ray OA and mark equidistant points (of any length) in number at least equal to denominator of the fraction, **(c)** starting from O, in the instant case it is Seven (7), join 7^{th} mark on the ray to the mark corresponding to ***n*** on number line, **(d)** draw, from mark corresponding to the numerator on the ray, line parallel to that drawn at (c) above intersecting the number line at (a) above. **(e)** *the distance between 0 and point of intersection of line drawn from mark* 3 *is the desired length* $\frac{3}{7}n$.



Another case is of representing an Irrational Number of the form . $\sqrt{2}, \sqrt{3}.....\sqrt{n}$, on a number liner here n can be any number which can be represented on a number line. It involves following step as shown in the figure below- (a) mark on the number line two points at 1 and (-n) , (b) a point A at $\left(-\frac{1}{2}(n-1)\right)$ on the number line, (c) draw a semicircle with a radius $\left(\frac{1}{2}(n+1)\right)$, (d) Point 0 on number line is marked B and a ⊥ is drawn on it which intercepts semicircle at point C, (e) taking a radius BC, draw an arc with centre 0, intercepting the number line on its +ve side at point D, (f) the line segment OD has length equal to $\sqrt{n}$.



This is proved using Pythagoras Theorem on ∟Δ. *ABC* such that $(\mathrm{BC})^2 = (\mathrm{AC})^2 - (\mathrm{AB})^2 \rightarrow (\mathrm{BC})^2 = \left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2 = \left(\frac{n+1}{2} + \frac{n-1}{2}\right)\left(\frac{n+1}{2} - \frac{n-1}{2}\right) = n \rightarrow BC = \sqrt{n}$.

These two cases are very good example of an *integrated approach in mathematics where Arithmetic, Algebra and Geometry complementing each other, and thus concepts of higher mathematics are built.* As one proceeds into higher topics, more of the integration of different topics of mathematics would be involved.

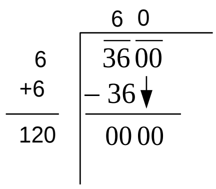
In real life problems numbers are generally encountered over a range of values, called ***Interval*** and are classified as under -.

1. If$a \leq x \leq b$ is called a ***Closed Interval*** and denoted as $[a,b] = \{x: a \leq x \leq b\}$.
2. If $a < x < b$ is called an ***Open Interval*** and denoted as $]a,b[= \{x: a < x < b\}$. It is also expressed as $(a,b) = \{x: a < x < b\}$
3. If $x < a \leq b$ is called ***Semi-closed or Semi-open Interval***$]a,b] = \{x: a < x \leq b\}$. It is also expressed as $(a,b] = \{x: a < x \leq b\}$
4.
5. If $x \leq< a < b$ is called ***Semi-closed or Semi-open Interval***$[a,b[= \{x: a \leq x < b\}$. It is also expressed as $[a,b) = \{x: a \leq x < b\}$

Square Root of Numbers: There are two methods, First by ***Fractions*** and second by ***Successive Division.*** The first method is common to arithmetic and algebra where given number or polynomial is expanded into product of smallest fractions and then fractions are paired into set of two identical fractions. Square Root is the product of taking one fraction from each pair of fractions, as long as all fractions are +ve real. It is illustrated is as under -

Algebra:$\sqrt{25x^4y^2} = \sqrt{(5x^2y)(5x^2y)} = 5x^2y$
Arithmetic: $\sqrt{3600} = \sqrt{(2 \times 2)(2 \times 2)(3 \times 3)(5 \times 5)} = 2 \times 2 \times 3 \times 5 = 60$

In case of variables having perfect square-root index of each variable are even, and in case of numbers it has even pairs of each of the prime-factors.



The second method of continuous division is shown in the figure. It is observed that highest result of single digit is in tens and, therefore, the number whose square-root is to be determined is taken in *sets of two digits starting from unit place towards left*. In case of real numbers the decimal component is taken in *sets of two digits starting from first decimal place towards right.* Practice of this method is left for mentor to elaborate to the students for clarity.

2	124, 80, 140,150
2	62, 40, 70, 75
2	31, 20, 35, 75
2	31, 10, 35, 75
3	31, 5, 35, 75
5	31, 5, 35, 25
5	31, 1, 7, 5
7	31, 1, 7, 1
31	31, 1, 1, 1
	1, 1, 1, 1

LCM and HCF: *With a given set of numbers or polynomial its* ***Lowest Common Multiple (LCM)*** *is the lowest number or polynomial of which each number or polynomial of the set is a fraction*. Algebraic and arithmetic examples are here under –

Algebraic Example: x^3, x^2y, xy^2, y^3 has LCM x^3y^3. It is observed that all the monomials in the set with their highest power constitute the LCM.

Arithmetic Example: With a set of numbers124,80, 140,150 can be done by fractions method, as done in square-root, or successive division method us shown in the figure and $LCM = 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 5 \times 7 \times 31 = 260400$. In the process to determine LCM the successive divisions of each of the number in the set is done with prime numbers till last quotient becomes One (1). The process is continued till all numbers are reduced to quotient One. Product of these divisor Prime Numbers is the LCM.

Highest Common Factor: *A factor which is common to all number or polynomials is called* ***HCF***. In the set of algebraic polynomials used above to illustrate LCM, the HCF is this typical case is$x^0y^0 = 1$, while in the above example of arithmetic set of numbers the HCF is 2. After first division by 2, in the example of LCM, next successive dividends do not give integer quotients, and hence ignored as factors in HCF.

Arithmetically, **Euclid's Algorithms of HCF**, *is based on Euclid's Lemma* $a = bq + r; 0 \leq b < r$ *brought in division operation on a set of two integers (a,b) say* $\left(\frac{a}{b}\right)$. This classical algorithm, brought out here as under, is considered to be quite tedious as compared to algebraic method.

Step 1: Apply *Euclid's Lemma* to ***a*** and ***b*** to determine ***q*** and ***r***.
Step 2: If r=0, stop and q is HCF, else apply the *Lemma* to q and r.
Step 3: The division process is continued till remainder is Zero (0) and the last divisor is HCF.

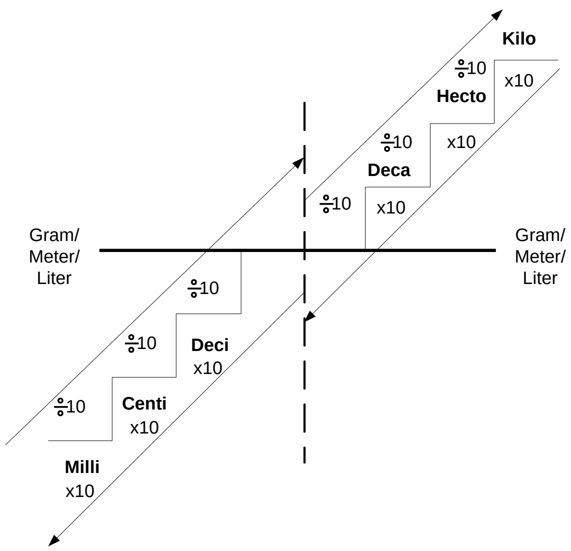
***It is to be noted that for a set of two numbers or polynomials then the product of their LCM and HCF is equal to the product of the two Numbers and polynomials*.**

Addition of Fractions: Addition of fractions having different denominators is done by converting the fractions into a form where each fraction has equal denominator, i.e. **LCM** of all the denominators. This is achieved by multiplying numerator and denominator of each fraction with the complementary factor of denominator in the *LCM*. Numerators of such updated fractions are just added, keeping their denominators as the LCM. Thus problem is reduced to simple adding fractions having common denominators. A typical example is shown in the figure where LCM of the denominators of the addends is 210 and the sum of the fractions is $\frac{491}{210}$. Same is true for subtraction of factors.

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} \rightarrow \frac{1}{2}\times\frac{\frac{210}{2}}{\frac{210}{2}}+\frac{2}{3}\times\frac{\frac{210}{3}}{\frac{210}{3}}+\frac{3}{5}\times\frac{\frac{210}{5}}{\frac{210}{5}}+\frac{4}{7}\times\frac{\frac{210}{7}}{\frac{210}{7}}$$

$$=\frac{1\times\frac{210}{2}+2\times\frac{210}{3}+3\times\frac{210}{5}+4\times\frac{210}{7}}{210} \rightarrow \frac{1\times105+2\times70+3\times42+4\times30}{210}$$

$$=\frac{105}{210}+\frac{140}{210}+\frac{126}{210}+\frac{120}{210} = \frac{491}{210}$$



Conversion of units: Basic units, generally used in day-to-day life, like length, weight and volume are expressed in range **milli to kilo** and a particular form is based on quantity under consideration. Buying grains is done in Kilogram, Liquid medicines are given in Millilitre and measurement of a size of copy is done in Centimetre. Conversion from ***Milli to Kilo and Kilo to Milli***, is in decimal system and is summarized in the figure.

Applications of Basics of Arithmetic [1,2,3,4,5,6,10]: Cases of arithmetic applications are encountered in day to day life and are generally commercial in nature. This can also be seen as algebraic problem and that make solution of problems generic suitable for use in specific calculations using appropriate values. Such uses of mathematics are called Commercial Mathematics.

Percentage: In Algebra, ***Percentage*** $x\% = \frac{X}{Q} \times 100$; Q is a whole quantity of which X is to determined as fraction of 100 (Cent), similar to century in cricket. Arithmetically this is done by unitary method, and this requires setting up unique unitary statement for each problem. *In algebraic method care is required in choosing correct values of variables in the formula.*

Average: If there is a set of values, say marks obtained by student of a class in examination, average marks of the class is sum of the marks obtained by each student divided by the number of students. This big statement can be mathematically represented as under –

$$\bar{X} = \frac{X_1 + X_2 \dots X_n}{n} = \frac{\sum_{k=1}^{n} X_k}{n}$$

Here, $X_1 + X_2 \dots X_n$ are the marks by each of the ***n*** student respectively, and $\bar{X}$ is the average mark. The crisp *algebraic representation of the* ***Average Value*** is $\frac{\sum_{k=1}^{n} X_k}{n}$.

Interest: Interest calculation involves principal (***P***) is the amount on which interest (***I***) is to be calculated at ***R***% rate per period of time, and number of periods (***T***) for which I is calculated. While, Amount (***A***) after the T, is sum of P and I. Thus the relationship of ***Simple Interest*** is expressed as –

$$I = \frac{P \times R \times T}{100}; A = P + I$$

Here, *consideration of units is important and any mistake in this would create error in results*. Units of P, A and I is same while unit of period T should be commensurate with that included in T, if not, the conversion of one into the scale of another is essential.

In **Compound Interest**, amount of each period of compounding is used as principal for the next period of compounding. Accordingly, it requires conversion of ***T*** and ***R*** commensurate with the period of compounding. It is algebraically expressed as –

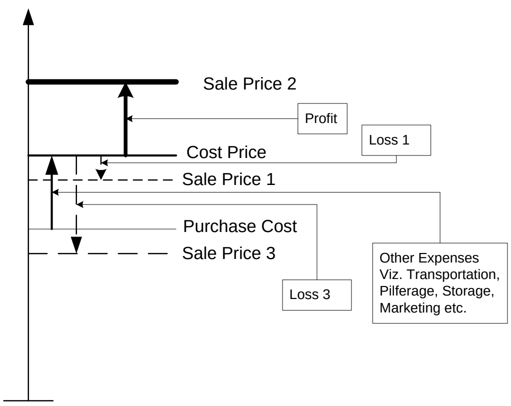
$$A = P\left(1 + \frac{R}{100}\right)\left(1 + \frac{R}{100}\right) \dots T\ times = P\left(1 + \frac{R}{100}\right)^{T};$$

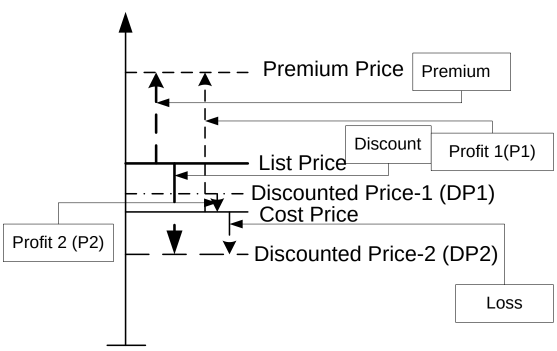
$$\text{Compound Interest } CI = A - P = P\left(\left(1 + \frac{R}{100}\right)^{T} - 1\right)$$

Profit and Loss &Discount and Premium: Calculation of Profit (**P**) or Loss (***L***) are always difference between sale price (**SP**) and cost price (**CP**); generally it is expressed as a percentage of **CP.** It is graphically illustrated below, and accordingly –

$$P\% = \frac{SP - CP}{CP} \times 100; \text{ and} L\% = \frac{CP - SP}{CP} \times 100$$

Likewise, in shopping one comes across List Price (LP), Discount (D%) and Premium (Pr%). In the context of Profit and Loss calculations on the cost price (CP), calculation of sale price (SP) is done on List Price, by loading discount or premium. This is illustrated in another figure.





Calculation of *D*% and *Pr*% :$D\% = \frac{LP - DP}{LP} \times 100$;

$Pr\% = \frac{Pr - LP}{LP} \times 100$;

Calculation of Profit with Premium, and Discount when DP>CP is :$P1\% = \frac{Pr - CP}{CP} \times 100$; $P2\% = \frac{DP - CP}{CP} \times 100$

While on case of Discount leading to DP<CP the loss is $L\% = \frac{CP - DP}{CP} \times 100$.

Speed, Time and Distance &Work, Time and Persons: Relationship of Speed (*S*), Time (*T*) and Distance (*D*) in algebraic form is $D = S \times T$. Likewise, in Work (*W*), Time taken to complete the work (*T*) and Persons doing work (P), assuming that rate of doing work is same for all persons engaged the relationship in algebraic form is $W = P \times T$. *These two types of problems appear to be of different types but mathematically they are identical. Their relationship can be treated as of Direct Proportion (distance vrs. time, at a constant speed) or Inverse Proportion (time vrs distance at a constant speed , and is based on the way parameters are defined.*

Area and Volume: These types of problems are application algebra into and with actual values of the variables they become arithmetic in nature. Three Dimensional Geometry involving volume is nothing but extension of Two Dimensional geometry with its Third Dimension involving space, which shall follow in later chapters.

Statistics and Probability: These two branches, though seen independent, are extension of basics of mathematics and shall be dealt with separately in chapters to follow on mathematics.

Trigonometry and Coordinate Geometry: These topics are barely covered in basic mathematics, as much of it is an inseparable part of ***Foundation Mathematics***, a chapter to follow. Accordingly, these topics together with integration is elaborated in following chapter.

Conclusion: Illustrations in this chapter will help to believe that mathematics is all about our observations of real world which can be quantified, correlated and analysed to draw useful inferences. In this chapter endeavour has been made to build the basics, as they come up, to be of natural consequence. Assimilation of these concepts, to enable one to use them adroitly, and swiftly, requires practising them through mental revision (which authors calls as meditation) and solving of problems. This we call making concepts intuitive. Accordingly, text books and reference books, listed below or any other, that are readily available should be advised by mentors to the students for practicing concepts. These books are time invariant and can become available as second hand book, an economical proposition to target students, as well as it is environment friendly. In real life nothing is encountered in a simplistic manner. Even best of the books and teachers cannot provide a readymade solution to all the problems or the ones that one would encounter in real life. Even it be so, it not possible for anyone to carry complete set of books. But, clarity of concepts is the only thing that helps in correlating observations, in correct mathematical form, and to get a right answer or solution.

Mentors can also promote group learning among students, by way complementing their colleagues in the group, in solving their problems. This will help each student, while gaining proficiency at learning, to become a good team player. This attribute of personality is a necessity. It it is not developed in school days, it costs heavily to every individual when it is required to put in place.

Here, importance of understanding of problem in a language in which it is encountered, defined or narrated also becomes crucial tool in its resolution. Every successful person has a good command over language and, therefore, every student of mathematics, science and engineering whom society and profession looks upon as problem solver, must not ignore proficiency in language, and general reading as a means to maintain and build it, else all this learning would remain confined to books with no utility in real life.

This chapter is part of common section, followed by Foundation Mathematics, that will go in understanding Mathematics, Physics and Chemistry upto class XIIth. There after separate section have been developed on Mathematics, by Prof S.B. Dhar, Physics by Dr Subhash Joshi and Dr Vibhu Mishra and on chemistry by Mrs Kumud Bala. All these resources are being selflessly made available as free web resource. Mentors and students are welcome to make their observations andor suggestion to make value addition and make it more purposeful, for the larger good.

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