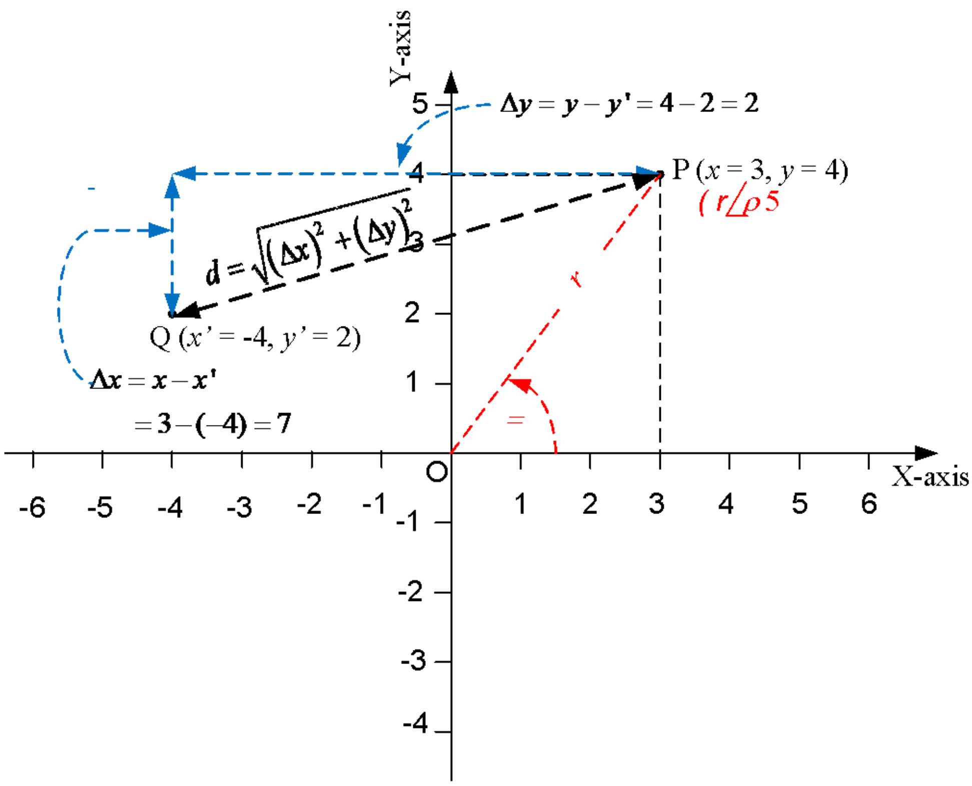
CHAPTER III: FOUNDATION MATHEMATICS

Basics of mathematics are like defining basic tools for carving an onward path. But, structure of mathematics is built on a foundation which is constituted by higher mathematics, complex number, vector and coordinate geometry using tools defined in Chapter-II, basics of mathematics. Interleaving of these constituents is done with the help of algebra and trigonometry. Use of basic identity in algebra is extended to define binomial theorem which under special condition defines e, e^x, $e^{i\theta}$, $\sin\theta$, other trigonometric identities. Thus a Universal set of numbers is constituted by complex number. Moreover, in a system nothing is static, everything at every moment is changing, and difference could be either in rate of change of the parameters being considered variable; this analytical part of mathematics is called differential calculus.. Likewise, present is either a starting point of an event or a result of variations in past; accordingly it is a study of cumulative variations and is called Integral Calculus. Study of all phenomenon realized in real world, is a subject matter of physics. The study of physics with its how and why is substantiated by foundation mathematics, and continues to create an increasing dependence on mathematics as journey of nature continues.

Coordinate Geometry: It is branch of mathematics which converts graphical representation of a point, line, shape in algebraic form. It is extension of number line system starts with defining a point in either plane or space. Since plane has length and breadth, position of a point is defined by two parameters called ***abscissa*** (X-coordinate) and ***ordinate*** (Y-coordinate). This system of representation of position of a point is called **Cartesian Coordinate System** as shown in the figure. Using Pythagoras Theorem it is represented **Polar Coordinate System** with the help of ***radius vector*** (r) and ***vectorial angle*** (θ) with reference point O as origin and X-axis as reference direction. These two systems of representation are shown in the figure.

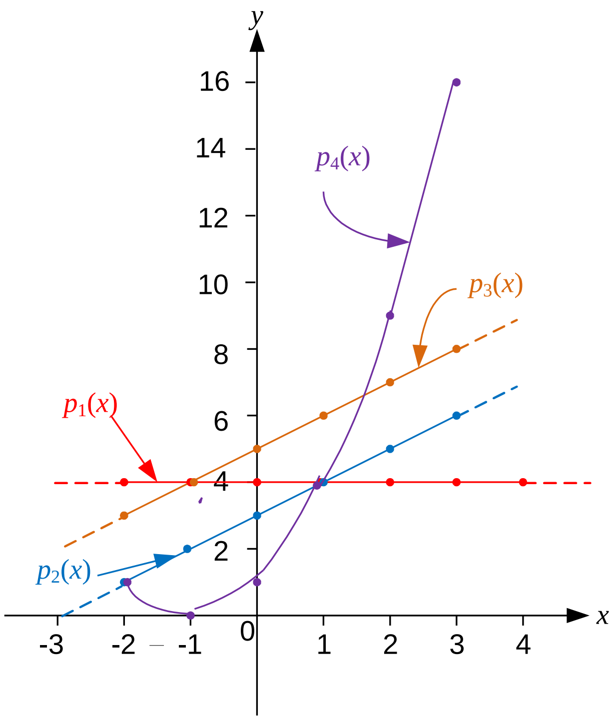


Equivalence of the two system of coordinates is $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

Distance between any two points whose coordinates (x_1, y_1) and (x_2, y_2) are known is $d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. These coordinates are also used to depict variation of a polynomials, known as graph, and is shown in the figure below using Four typical polynomials $p_1(x) = 4$, $p_2(x) = x + 3$,

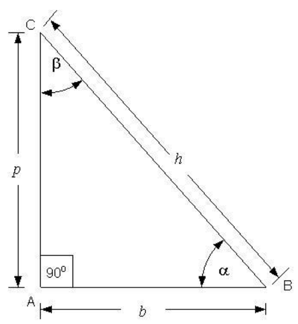
$p_3(x) = x+5$ and $p_4(x) = x^2+2x+1$, tabulated here under . In the event of a polynomial equated to a variable

x	$p_1(x)=4$	$p_2(x)=x+3$	$p_3(x)=x+5$	$p_4(x)=x^2+2x+1$
-2	4	$-2+3=1$	$-2+5=3$	$(-2)^2+2(-2)+1=1$
-1	4	$-1+3=2$	$-1+5=4$	$(-1)^2+2(-1)+1=0$
0	4	$0+3=3$	$0+5=5$	$(-0)^2+2(-0)+1=1$
1	4	$1+3=4$	$1+5=6$	$(1)^2+2(1)+1=4$
2	4	$2+3=5$	$2+5=7$	$(2)^2+2(2)+1=9$
3	4	$3+3=6$	$3+5=8$	$(3)^2+2(3)+1=16$



$y = p(x)$ becomes an equation. It will be seen that that $p_1(x)$ is invariant of x. While, $p_2(x)$ and $p_3(x)$ are First order polynomial with same coefficient of x with different constants and are therefore parallel straight lines; this is the reason that they are called linear polynomial. Whereas, $p_2(x)$ is second order polynomial. It has a curved plot. Coordinates of point of intersection of two plots satisfy both the polynomials. This concept is used in graphical method of solving simultaneous linear equation. But, the equations being unsolvable with same ratio of coefficients of variable, but dissimilar constants, is that they lead to parallel lines as shown in the figure. While, linear equations with identical ratio of coefficients of variables and also that of constants lead to coincident or collinear lines. This shall become clear a little later while straight lines are elaborated in continuation of Coordinate Geometry.

Trigonometry: Trigonometry, built on Pythagorean property of right angle triangle, defines ratios of sides of the $\perp\Delta$, known as ***Trigonometric Ratios*** is as under –



In the Δ if $\angle ABC=\alpha$, other than $\perp$ of the Δ is under consideration then, side opposite the right angle is called Hypotenuse (h), the side opposite the angle α is called Perpendicular (p) , and side forming the angle α with the Hypotenuse is called Base (b). Using this definition, primary ratios defined in trigonometry , and the mathematical convention is brought out in the table below –

Nomenclature	Convention	Trigonometric Ratio	Relationship	Inverse Circular Function
Sine of α	sin α	$=\frac{p}{h}$	$=\frac{1}{\text{cosec}\alpha}=\frac{1}{\csc\alpha}$	$\alpha=\sin^{-1}\frac{p}{h}$
Cosine of α	cosα	$=\frac{b}{h}$	$=\frac{1}{\sec\alpha}$	$\alpha=\cos^{-1}\frac{b}{h}$
Tangent of α	tan α	$=\frac{p}{b}$	$=\frac{\sin\alpha}{\cos\alpha}=\frac{1}{\cot\alpha}$	$\alpha=\tan^{-1}\frac{p}{b}$
Cotangent of α	cot α	$=\frac{b}{p}$	$=\frac{\cos\alpha}{\sin\alpha}=\frac{1}{\tan\alpha}$	$\alpha=\cot^{-1}\frac{b}{p}$
Cosecant of α	cosec α	$=\frac{h}{p}$	$=\frac{1}{\sin\alpha}$	$\alpha=\text{cosec}^{-1}\frac{h}{p}$
Secant of α	sec α	$=\frac{h}{b}$	$=\frac{1}{\sec\alpha}$	$\alpha=\sec^{-1}\frac{h}{b}$

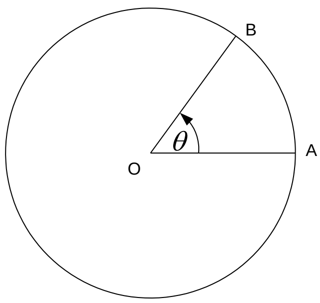
There are another two identities Versed Sine and Coversed Sine as under referred to in literature. They are –

$$1-\cos\theta=\text{vers}\,\theta,\text{ and }\ 1-\sin\theta=\text{covers}\,\theta.$$

Using property of Pythagoras Theorem : $p^2+b^2=h^2$; $\frac{p^2+b^2}{h^2}=\frac{h^2}{h^2}$; $\frac{p^2}{h^2}+\frac{b^2}{h^2}=1$; $\left(\frac{p}{h}\right)^2+\left(\frac{b}{h}\right)^2=1$. . In language of mathematics it is written as: $(\sin\alpha)^2+(\cos\alpha)^2=1\rightarrow\sin^2\alpha+\cos^2\alpha=1$. Expressions $(\sin\alpha)^2$ and $\sin^2\alpha$ are synonymous in mathematics and thus $\sin^2\alpha+\cos^2\alpha=1$is the basic identity of trigonometry from which other identities that are derived are as under –

$\sin^2\alpha+\cos^2\alpha=1$	$\sin^2\alpha=1-\cos^2\alpha$
	$\cos^2\alpha=1-\sin^2\alpha$
$\frac{\sin^2\alpha+\cos^2\alpha}{\cos^2\alpha}=\frac{1}{\cos^2\alpha}$	$\tan^2\alpha+1=\sec^2\alpha$
	$\tan^2\alpha=\sec^2\alpha-1$
$\frac{\sin^2\alpha+\cos^2\alpha}{\sin^2\alpha}=\frac{1}{\sin^2\alpha}$	$1+\cot^2\alpha=\text{cosec}^2\,\alpha$
	$\cot^2\alpha=\text{cosec}^2\,\alpha-1$

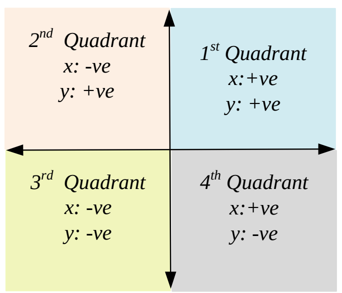
In trigonometry angles are invariably defined in radians and not degrees. In chapter while defining perimeter of a circle π was defined, and accordingly π radians were related to 180^0. Now discretely any angle θ is being defined in radians, with the help of following diagram. In a circle of radius r, an angle θ in radian formed by two radials OA and OB is : $\theta=\frac{\text{Length of arc AB}}{r}$ radians, here a care is required endure that unit of length of arc and radius must be same. Accordingly, value of trigonometric ratios most often used are shown in the table below.

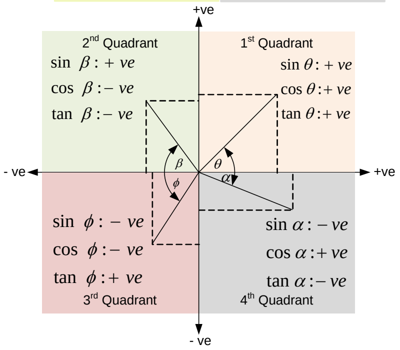


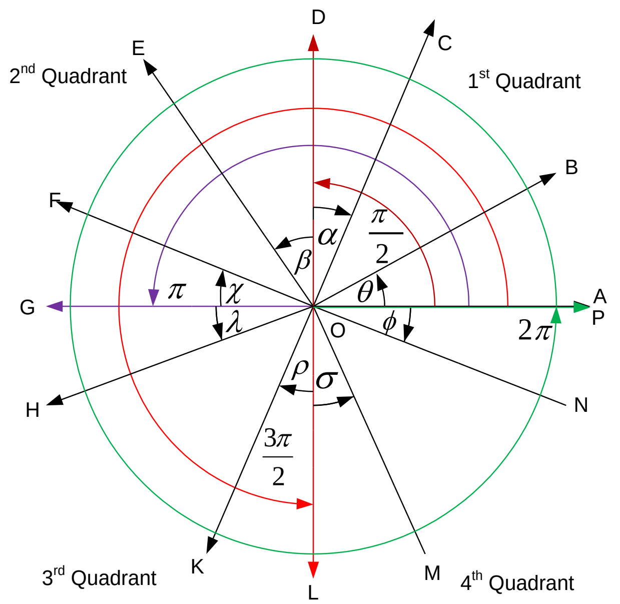
Trigonometric Ratios	θ in degrees and radians				
	0^0 0 rad	30^0 $\frac{\pi}{6}$ rad	45^0 $\frac{\pi}{4}$ rad	60^0 $\frac{\pi}{3}$ rad	90^0 $\frac{\pi}{2}$ rad
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Once this is done various situations may arise when trigonometric ratios are required to be determined in four quadrants as defined in the diagrams below. On the left-side diagram sign of trigonometric ratios are indicated with signed values of projections of hypotenuse on horizontal axis (Base) and vertical axis (Perpendicular). The diagram on the right-side depicts angles which are used in generic convention in the table shown below.

Using these diagrams angles used in determining trigonometric ratios are brought out in the table below, and +ve value of the angle is always in anti-clock-wise direction.

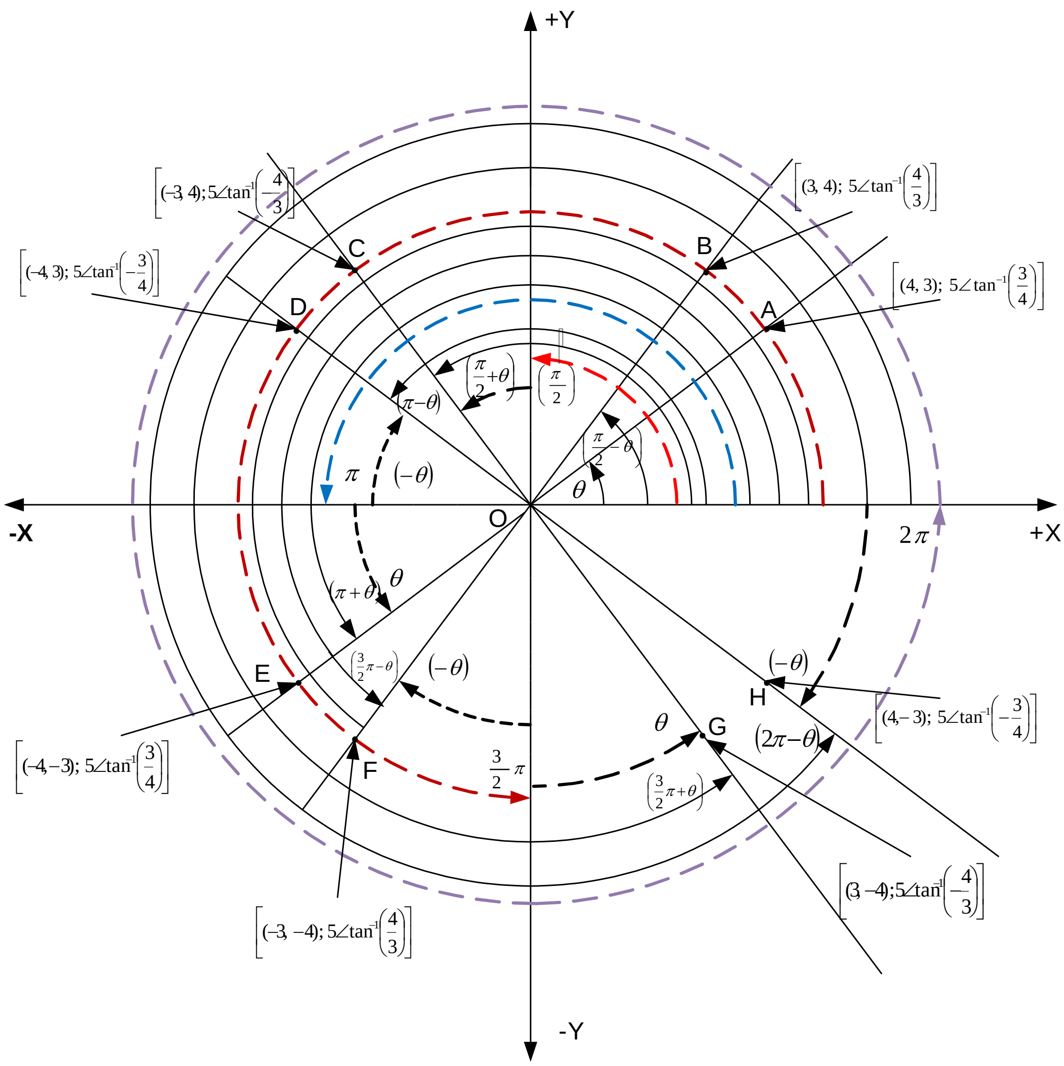




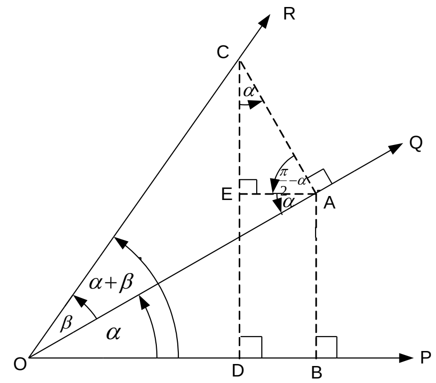


Angle	Value of angle for Trigonometric ratios	Angle	Value of angle for Trigonometric ratios
$\angle$AOB	θ	$\angle$AOH	$\pi+\lambda$
$\angle$AOC	$\frac{\pi}{2}-\alpha$	$\angle$AOK	$\frac{3\pi}{2}-\rho$
$\angle$AOD	$\frac{\pi}{2}$	$\angle$AOL	$\frac{3\pi}{2}$
$\angle$AOE	$\frac{\pi}{2}+\beta$	$\angle$AOM	$\frac{3\pi}{2}+\sigma$
$\angle$AOF	$\pi-\chi$	$\angle$AON	$2\pi-\phi$ or $-\phi$
$\angle$AOG	π	$\angle$AOP	2π

This is elaborated with greater details in a diagram below taking points A, B, C, ... H on perimeter of a circle of radius 5 with its centre at O. Coordinates of these points are taken in permutation and combination of ±3 and ±4 and their corresponding angles. Coordinates of the points are shown in both Cartesian and polar coordinates.



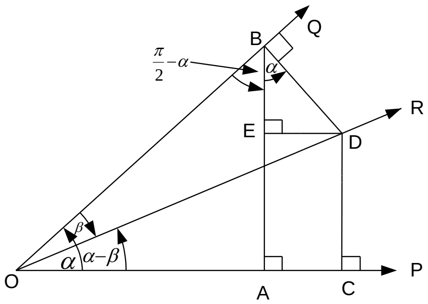
Defining angles as shown in the table requires determining trigonometric ratios as sum or difference of angles, and is as under. Initially identity for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ is created by taking rays OP, OQ and OR and angle α and β as shown in the figure below. A point C is taken on OR, forming an angle $(\alpha+\beta)$ with ray OP. From point C $\perp$s are drawn on Rays OQ and OP, meeting them on points A and D respectively. From point A $\perp$s are drawn on line CD and OP. This geometry leads to $\angle ACE = \alpha$. Accordingly in ΔODC -



In ΔODC; $\sin(\alpha+\beta)=\frac{DC}{OC}=\frac{DE+EC}{OC}=\frac{AB}{OC}\times\frac{OA}{OA}+\frac{EC}{OC}\times\frac{AC}{AC}$. Multipliers AC and OA are considered such that they together with the sides in multiplicand fraction lead to sine and cosine of constituent angles α and β.

Accordingly, $\sin(\alpha+\beta)\ =\frac{AB}{OA}\times\frac{OA}{OC}+\frac{EC}{AC}\times\frac{AC}{OC}$,

or $\sin(\alpha+\beta)\ = \cos\alpha\sin\beta+\sin\alpha\cos\beta$



On similar lines, $\cos(\alpha+\beta)=\frac{OD}{OC}=\frac{OB-DB}{OC}=\frac{OB}{OC}-\frac{EA}{OC}=\frac{OB}{OC}\times\frac{OA}{OA}-\frac{EA}{OC}\times\frac{AC}{AC}=\frac{OB}{OA}\times\frac{OA}{OC}-\frac{EA}{AC}\times\frac{AC}{OC}$, or $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

Extending the identity for sum angles, to the difference of angles can be done geometrically using figure as under.

A simpler method could be to replace β= -ϕ, *in* identities $\sin(\alpha+\beta)$ and $\cos(\alpha-\beta)$, such that: $\sin(-\emptyset)=-\sin\emptyset$ and $\cos(-\emptyset)=\cos\emptyset$. This leads to : $\sin(\alpha-\emptyset)=\sin\alpha\cos\emptyset-\cos\alpha\sin\emptyset$, and $\cos(\alpha-\emptyset)=\cos\alpha\cos\emptyset-\sin\alpha\sin\emptyset$

Based on these formulations of sine and cosine of sum and difference of angles, similar formulations for tangent and cotangent can be derived as under –

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}=\frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\to\tan(\alpha+\beta)\ =\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

Likewise, it can be proved that – $\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$; $\cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cos\beta+\cot\alpha}$; $\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cos\beta-\cot\alpha}$

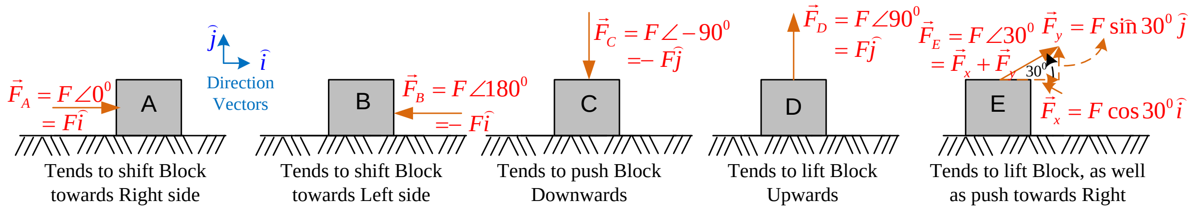
Further, a set of identities, derived from those basic identities, are as under –

$\sin(\alpha+\beta)+\sin(\alpha-\beta)\ =2\sin\alpha\cos\beta$	$\sin A\cos B\ =\frac{1}{2}\left(\sin\left(\frac{A+B}{2}\right)+\sin\left(\frac{A-B}{2}\right)\right)$
$\cos(\alpha+\beta)+\cos(\alpha-\beta)\ =2\cos\alpha\cos\beta$	$\cos A\cos\ B\ =\ \frac{1}{2}\left(\cos\left(\frac{A+B}{2}\right)+\cos\left(\frac{A-B}{2}\right)\right)$
$\cos(\alpha-\beta)-\cos(\alpha+\beta)\ =2\sin\alpha\sin\beta$	$\sin A\sin\ B\ =\ \frac{1}{2}\left(\cos\left(\frac{A+B}{2}\right)-\cos\left(\frac{A-B}{2}\right)\right)$

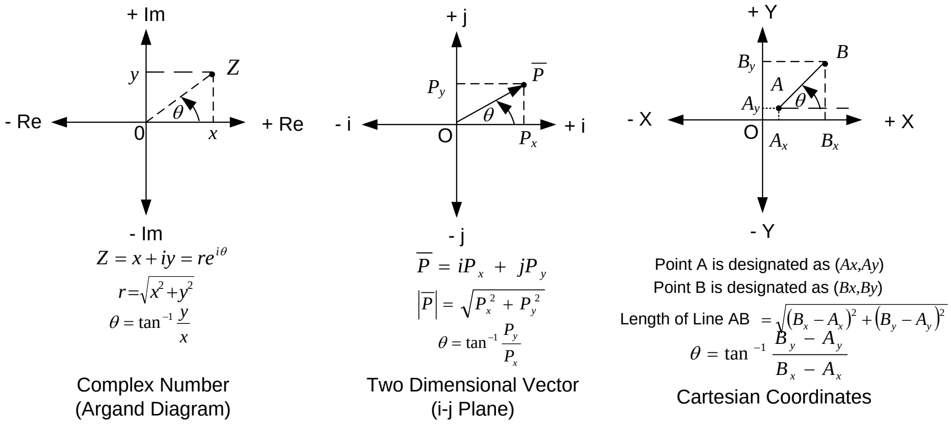
These set of identities, with minor variance and extension, find extensive application both in mathematics and physics.

Introduction to Vectors: Until now quantities that were referred to had only magnitude and such quantities are called scalar. Nevertheless, unknowingly we were taken into set of integers and real numbers represented on a number line. Each number on the right of another number on the number line was greater; conversely a number on the left of another number was smaller. Taking this logic Zero on number line is taken to be a reference and all numbers (quantities) on right of Zero are +ve and on the left of Zero are –ve. Thus in essence sign attached to a number is directional attribute of the quantity, and is an introduction to Vector Quantity. This will become more evident when sum or difference of collinear vectors is determined from generic formulation of addition of vectors, to be discussed later in the chapter.

Better visualization of vectors is possible in 2D as shown in the figure below; while most generic visualization comes in 3D space, Need of a vector quantity is due to effect that the direction attribute of it brings in and is elaborated in the figure below. Effect of Force is easy to visualize and hence Force $\left(\vec{F}\right)$ as a vector has been used. Since, each of the force on the block, despite equal magnitude but different direction, produces different effect. Therefore, each of the vector is different from the other and are identified with magnitude and corresponding angle. Here, $\hat{i}$ and $\hat{j}$ are directional unit vectors along Two perpendicular axes.



Integration of Vectors, Complex Numbers, and Coordinate Geometry with Trigonometry: Using this knowledge of trigonometry a journey into mathematics and physics can be started to correlate complex numbers, vectors and coordinate geometry. Accordingly, in the diagram below it has been elaborated as to how trigonometry is helpful in *correlation*.



Particulars	Complex Numbers	2D-Vectors	2D-Coordinate Geometry
Space	*Argend Diagram*	*i-j plane*	*Cartesian Plane*
Representation	$Z = x + iy$	$\bar{P} = iP_x + jP_y$	Point A as (x, y)
Special	x and y are real while $i = \sqrt{-1}$	P_x and P_y are real while i and j are orthogonal directions	X (Ordinate) and Y(Abscissa) are orthogonal Axes
Magnitude	Modulus $z = \sqrt{x^2 + y^2}$	Magnitude $\lvert P\rvert = \sqrt{P_x^2 + P_y^2}$	Distance of Point A from Origin $r = \sqrt{A_x^2 + A_y^2}$
Angle (anticlockwise direction)	Amplitude $\theta = \tan^{-1}\frac{y}{x}$ $-\pi < \theta \le \pi$	Angle with i direction $\theta = \tan^{-1}\frac{P_y}{P_x}, 0 \le \theta \le 2\pi$	Angle with X axis $\theta = \tan^{-1}\frac{A_y}{A_x}, 0 \le \theta \le 2\pi$
Orthogonal Representation	$x = \lvert z\rvert\cos\theta$ $iy = i\lvert z\rvert\sin\theta$	$P_x = \lvert P\rvert\cos\theta$ $P_y = \lvert P\rvert\sin\theta$	$A_x = r\cos\theta$ $A_y = r\sin\theta$
Polar Representation	$ze^{i\theta}$	$P\angle\theta$	(r, θ)
Space	Any hypothetical plane (2D)	This 2D representation can be extrapolated in 3D using cylindrical or spherical unit vectors (directional).	This 2D representation can be extrapolated in 3D using cylindrical or spherical coordinates

Permutations and Combinations: This is a journey to understand possible ways of arranging a given set of objects. It further leads *Binomial Theorem* and defining *Napier's constant*(e), one of the most generic irrational number. This journey is started with taking products of successive number and their mathematical representation called **factorial n** and in mathematical connotation it is represented as $\lfloor n$ or n!. It is as under –

1!=1	2!=2x1=2x1!	3!=3x2x1=3x2!	4!=4x3x2x1=4x3!	....	$n! = n(n-1)(n-2)\dots 2\times 1$ $= n\lfloor(n-1);$ or $n(n-1)!$

Physical significance of $\lfloor n$ or n! Is that the distinct manner in which given objects are arranged in different possible ways. Taking example of a couple Husband (on the left) and Wife (on the right side) , and vice-versa are two distinct arrangements and called ***Permutations*** of given objects, in the instant case they are Two. But, since distinct husband and with his wife form a distinct couple, whichever ways they are arranged, it is a distinct ***Combinations***. Mathematical representation of ***permutations*** starting from One to Four objects, taking objects in increasing order starting from 1, is brought out here under, and is then generalized in algebraic form using *Factorial notation*.

Distinct objects	**No of Alphabets [n]**	**No of possible arrangements (Permutations) of Four Alphabets [r]**			
		One at a time (r=1)	**Two at a time (r=2)**	**Three at a time (r=3)**	**Taking Four at a time (r=4)**
a	1	a **(1 No)**	-	-	-
a, b	2	a;b **(2 No)**	ab;ba **(2 No)**	-	-
a, b, c	3	a; b; c **(3 No)**	ab; ba; ca; ac; bc; cb **(6 No)**	abc; bca; cab; acb; bac; cba **(6 No)**	-
a, b, c, d	4	a ; b; c; d **(4 No)**	ab; bc; cd; da; ac; bd; ca; db; ad; ba; cb; dc **(12 No)**	abc; bcd; cda; dab; abd; bca; cdb; dac; acd; bda; cab; dba; acb; bdb; cad; dbc; adb; bac; cba; dca; adc; bad; cbd;dcb **(24 No)**	abcd; bcda; cdab; dabc; abdc; bcda; cdba; dacb; acdb; bdac; cabd; dbca; acbd; bdca; cadb; dbac; adbc; bacd; dabc; cdab; adcb; badc; dacb; cdba **(24 No)**
Algebraic generalization of Permutations for a set of n objects, with r objects taken at a time such that $r \le n$ is evolved from the above examples , and is as under					
$^n p_r = \frac{n!}{(n-r)!}$		$^4p_1 = \frac{4!}{(4-1)!}$	$^4p_2 = \frac{4!}{(4-2)!}$	$^4p_3 = \frac{4!}{(4-3)!}$	$^4p_4 = \frac{4!}{(4-4)!}$
		$^4p_1 = \frac{4!}{3!}$	$^4p_2 = \frac{4!}{2!}$	$^4p_3 = \frac{4!}{1!}$	$^4p_4 = \frac{4!}{0!}$
		$\frac{4\times 3!}{3!} = 4$	$\frac{4!}{2!} = \frac{4\times 3\times 2!}{2!} = 12$	$\frac{4!}{1!} = \frac{4\times 3\times 2\times 1!}{1!} = 24$	$^4p_4 = \frac{4\times 3\times 2\times 1}{0!} = \frac{24}{1} = 24$

It will be seen that and $1!=1$, it is shown in the table above for One Object. But, an obvious curiosity is how can be $0!=1$. At this point it essential recall concept of Zero, which implies nothing, and there can be one only one way to have nothing i.e. discard everything. This is the reason as to why $0!=1$. Thus in a generic sense *permutations, i.e. number of possible arrangements, of n things for all values of* $r \le n$*, out of the set is mathematically expressed as* $^n p_r = \frac{\lfloor n}{\lfloor(n-r)}$.

In an arrangements of more than one object, out of the given set, arrangements of a typical set of objects is treated as a single combination. Taking a set $\{a,b,c\}$ its permutation $^3P_3 = \frac{3!}{(3-3)!} = \frac{3!}{0!} = 3! = 1\times 2\times 3$, viz. (a,b,c), (a,c,b), (b,c,a),

(b,a,c), (c,a,b) and (c,b,a) form One combination. Accordingly, an arrangements of a typical set of objects is treated as single combination. Thus, a generalized mathematical concept and expression of ***combinations***, is $\left[\ {}^{n}C_{r}=\frac{\lfloor n}{\lfloor r \lfloor (n-r)}\right]$, is arrived at using the concept of *permutations*. Here, in denominator $r!$ represents no of possible arrangements of ***r*** objects, taken together, as single combination. Therefore, number of combinations in a possible arrangements of a given set of of objects brought out above, is brought out in the Table below. For convenience of identification, repetitive arrangements of each subset of objects shown in different colour are ~~struck out~~ (*This just for representation*).

Distinct objects	**No of Alphabets (n)**	**r No of possible arrangements of Four Alphabets (r)**			
		One at a time (r=1)	**Two at a time (r=2)**	**Three at a time (r=3)**	**Taking Four at a time (r=4)**
a	1	a (1 No)	-	-	-
a, b	2	a; b (2 No)	ab; ~~ba~~ (1 No)	-	-
a, b, c	3	a; b; c (3 No)	ab; ~~ba;~~ ca;~~ac;~~ bc; ~~cb~~ (3 No)	abc; ~~bca; cab;~~ ~~acb; bac; cba~~ (1 No)	-
a, b, c, d	4	a ; b; c; d (4 Number of un-struck elements)	ab; ~~ba;~~ ac; ~~ca;~~ ad; ~~da;~~ bc; ~~cb;~~ bd; ~~db;~~ cd; ~~dc~~ (6 = Number of un-struck pairs)	abc; bcd; cda; dab; ~~acb; bdc; cad; dbc;~~ ~~abd;bca~~; ~~cdb~~; ~~dac~~; ~~adb; bac; cba; dca;~~ ~~acd~~; ~~bda~~; ~~cab~~; ~~dba~~; ~~adc; bad; cbd; dcb~~ (4= Number of un-struck sets)	abcd; ~~bcda; cdab; dabc;~~ ~~abdc;bcda; cdba; dacb;~~ ~~acdb;bdac; cabd; dbca;~~ ~~acbd; bdca; cadb; dbac;~~ ~~adbc; bacd; dabc; cdab;~~ ~~adcb; badc; dacb; cdba~~ (1= Number of un-struck sets))
Algebraic generalization of Combinations for a set of n objects, with r objects taken at a time such that $r \le n$ is evolved from the above examples , and is as under					
${}^{n}C_{r}=\dfrac{n!}{r!\times(n-r)!}$		${}^{4}C_{1}=\dfrac{4!}{1!\,(4-1)!}$	${}^{4}C_{2}=\dfrac{4!}{2!\,(4-2)!}$	${}^{4}C_{3}=\dfrac{4!}{3!\,(4-3)!}$	${}^{4}C_{4}=\dfrac{4!}{4!\,(4-4)!}$
		${}^{4}C_{1}=\dfrac{4!}{3!}$	${}^{4}C_{2}=\dfrac{4!}{2!\times 2!}$	${}^{4}C_{3}=\dfrac{4!}{3!\times 1!}$	${}^{4}C_{4}=\dfrac{4!}{4!\times 0!}$
		$\dfrac{4\times 3!}{3!}=4$	$\dfrac{4\times 3\times 2!}{2\times 1\times 2!}=6$	$\dfrac{4\times 3!}{3!\times 1!}=4$	$\dfrac{4!}{4!}=1$

Having developed concept of ***combination*** and *its mathematical expression* it is applied to identities having binomial factors upto four and it is extended to ***n*** order by mathematical induction -

Identities of Factors		**Identities of an Indexed Factor**	
$(x+a)$	$=x+a$	$(x+a)$	$=x+a$
$(x+a)(x+b)$	$=x^2+(a+b)x+ab$	$(x+a)^2\vert_{a=b}$	$=x^2+2ax+a^2$ $={}^{2}C_0x^2a^0+{}^{2}C_1x^{2-1}a+{}^{2}C_2x^{2-2}a^2$
$(x+a)(x+b)(x+c)$	$=x^3+(a+b+c)x^2$ $+(ab+bc+ca)x$ $+abc$	$(x+a)^3\vert_{a=b=c}$	$=x^3+3ax^2+3a^2x+x^3$ $={}^{3}C_0x^3a^0+{}^{3}C_1x^{3-1}a+{}^{3}C_2x^{3-2}a^2+{}^{3}C_3x^{3-3}a^3$
$(x+a)(x+b)(x+c)(x+d)$	$=x^4+(a+b+c+d)x^3$ $+(ab+ac+ad+bc+bd+cd)x^2+$ $(abc+abd+bcd+cda)x$ $+abcd$	$(x+a)^4\vert_{a=b=c=d}$	$=x^4+4x^3a+6x^2a^2+4xa^3+x^4$ $={}^{4}C_0x^4a^0+{}^{4}C_1x^{4-1}a+{}^{4}C_2x^{4-2}a^2+{}^{4}C_3x^{4-3}a^3+{}^{4}C_4x^{4-4}a^4$

Thus with the increasing degree of a binomial polynomial, the series developed in the above polynomial it is possible to deduce by mathematical induction a binomial of order n which can be expressed as –

$$(x+a)^n={}^{n}C_0x^na^0+{}^{n}C_1x^{n-1}a+{}^{n}C_2x^{n-2}a^2\ \dots{}^{n}C_rx^{n-r}a^r\ \dots+{}^{n}C_{n-2}x^{n-\overline{n-2}}a^{\overline{n-2}}+{}^{n}C_{n-1}x^{n-\overline{n-1}}a^{n-1}+{}^{n}C_nx^{n-n}a^n$$

$$= x^n + nx^{n-1}a + n(n-1)x^{n-2}a^2 \dots {}^nC_r x^{n-r}a^r \dots + n(n-1)x^2a^{n-2} + nxa^{n-1} + a^n$$

This is called ***Binomial Theorem***, it leads to various cases for fractional or (–)ve order index of the *Binomial Theorem*. A typical case of Binomial Theorem is –

$$\left(1+\frac{1}{n}\right)^n\Bigg|_{n\to\infty} = 1 + \frac{n}{1}\times\frac{1}{n} + \frac{n(n-1)}{2!}\times\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\times\frac{1}{n^3} + \cdots\Bigg|_{n\to\infty}$$

In this expression as $n \to \infty$ each of the term $\frac{n(n-1)}{n^2} = \frac{n(n-1)(n-1)}{n^3} = \cdots = 1$, accordingly, the expression reduces to –

$$\left(1+\frac{1}{n}\right)^n\Bigg|_{n\to\infty} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\frac{1}{r!}\Bigg|_{r\to\infty} = e$$

This ***e*** is called ***Napier's Constant*** or ***Euler's Number;*** it is an irrational number having an approximate value 2.718. On similar lines expression for e^x is –

$$e^x = \left(1+\frac{1}{n}\right)^{nx}\Bigg|_{n\to\infty} = 1 + \frac{nx}{1}\times\frac{1}{n} + \frac{nx(nx-1)}{2!}\times\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{3!}\times\frac{1}{n^3} + \cdots\Bigg|_{n\to\infty}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\frac{x^r}{r!}\Bigg|_{r\to\infty}$$

This expression of e^x is logically extended to $e^{i\theta}$ and $e^{-i\theta}$ as under –

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\ e^{-i\theta} &= 1 - i\theta + \frac{i^2\theta^2}{2!} - \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} - \frac{i^5\theta^5}{5!} + \cdots \\ &= 1 - i\theta - \frac{\theta^2}{2!} + \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} - \frac{i\theta^5}{5!} + \cdots \end{aligned}$$

Thus both $e^{i\theta}$ and $e^{-i\theta}$ are complex expressions. At this referring back to resolution and representation of Complex Numbers earlier in the chapter –

$\bar{Z} = ze^{i\theta} = z(\cos\theta + i\sin\theta)$ and if $\bar{Z} = ze^{-i\theta} = z(\cos\theta - i\sin\theta)$, here θ is in radians.

This further resolves to, $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$. It leads to an identity $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, likewise, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. Thus using above exponential series –

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} \dots \text{and } \sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7} \dots$$

It is to be noted that $\cos\theta$ and $\sin\theta$ derived above are real numbers, as also derived at trigonometry being ratios of real numbers. It is interesting to reveal that Trigonometry is stated to have been discovered by Aryabhatt about 6th Century, while Napier's Constant was discovered around 1614 AD; both the discoveries in gap of 1000 years are so interdependent yet interrelated. This leads to another interesting formulation suggested by de Moivre's according to which –

$$(\cos\theta + \text{i}\sin\theta)^n = (\cos n\theta + \text{i}\sin n\theta) \text{ and } (\cos\theta - \text{i}\sin\theta)^n = (\cos n\theta - \text{i}\sin n\theta)$$

This is a very simple inference based on representation of complex number in polar form as under –

$$\left(e^{i\theta}\right)^n = e^{i(n\theta)} = \cos n\theta + \text{i}\sin n\theta, \text{ and } \left(e^{i\theta}\right)^{-n} = e^{i(-n\theta)} = \cos n\theta - \text{i}\sin n\theta$$

This natural journey into mathematics has taken strides into higher mathematics, where any number can be represented as a subset of complex number and it is possible to determine its any root. Complex number and it's roots as members of Universal set is shown in the table below (next page).

Hyperbolic Functions: Unlike circular trigonometric functions (derived from a unit Circle) hyperbolic functions are derived from Unit Hyperbola. Hyperbolic functions were independently introduced by Vincenzo Riccati and Johann Heinrich Lambert , in the second half of 18th century. Riccati used a bit different convention, while convention introduced by Lambert is continuing; it is used here-

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}} = -i\sin(ix)$$

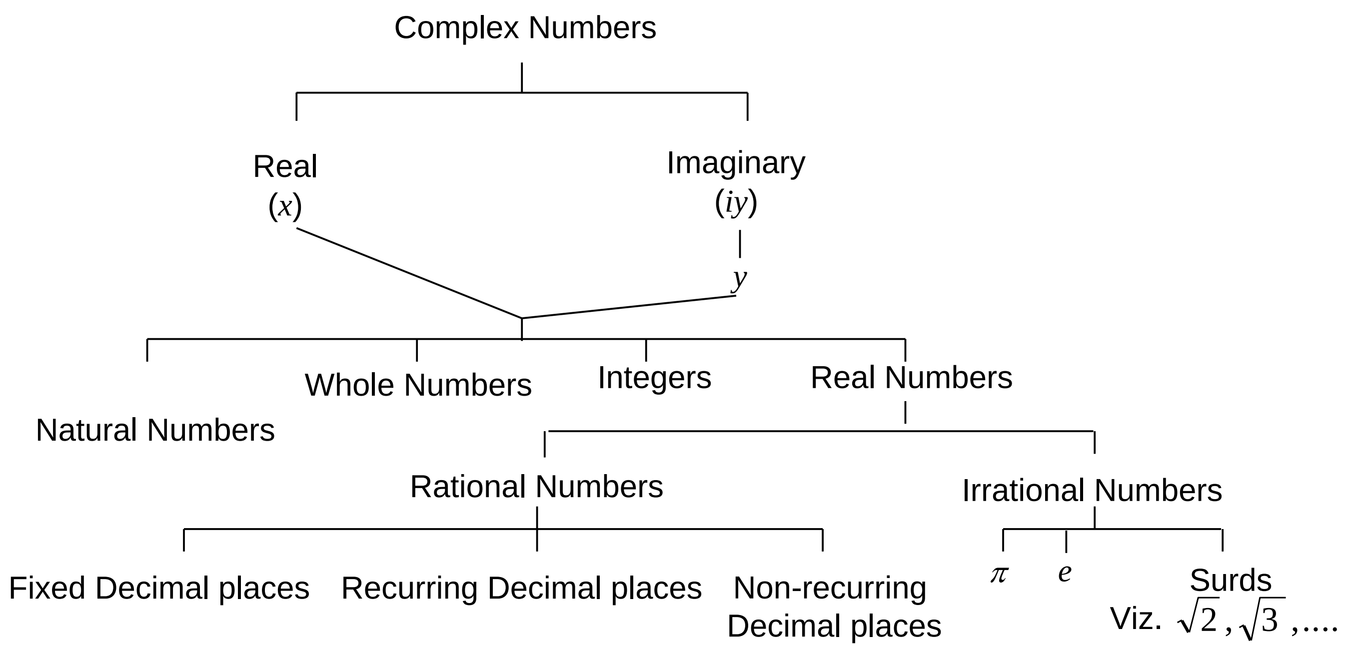
$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}} = \cos(ix)$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = -i\tan(ix) \quad ; \quad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = i\cot(ix)$$

$$\text{sech}\, x = \sec(ix) \; ; \; \text{cosech}\, x = i\,\text{cosec}(ix)$$

Hyperbolic functions are specific case of trigonometric functions having their independent variable restricted to imaginary axis. It finds lot of applications in physics and engineering as one pursues journey in these fields and is brought in the table below.

Representation of number system within ***Universal Set*** of Complex Numbers would be as under –



Particulars	**All possible forms of Complex Numbers $ze^{i\theta}$**							
Rectangular form	$x + i0;$	$x + iy;$	$0 + iy;$	$-x + iy;$	$-x + i0;$	$-x - iy;$	$0 - iy;$	$x - iy;$
Polar Form $0 < \alpha < \frac{\pi}{2}$	$xe^{i(2n\pi)}$	$ze^{i(2n+\alpha)\pi}$	$ye^{i\left(2n+\frac{1}{2}\right)\pi}$	$ze^{i\left(2n+\frac{1}{2}+\alpha\right)\pi}$	$xe^{i(2n+1)\pi}$	$ze^{i\left(2n\pi-\left(\frac{\pi}{2}+\alpha\right)\right)}$	$ye^{i\left(2n-\frac{1}{2}\right)\pi}$	$ze^{i(2n\pi-\alpha)}$
Occupies	+ve Re Axis	1st Quadrant	+ve Im Axis	2nd Quadrant	-ve Re Axis	3rd Quadrant	+ve Im Axis	4th Quadrant
General expression for fractional root $\left(\frac{p}{q}\right)$. Here, $0 \le n < q$	$\left(xe^{i(2n\pi)}\right)^{\frac{p}{q}}$	$\left(ze^{i(2n+\alpha)\pi}\right)^{\frac{p}{q}}$	$\left(ye^{i\left(2n+\frac{1}{2}\right)\pi}\right)^{\frac{p}{q}}$	$\left(ze^{i\left(2n+\frac{1}{2}+\alpha\right)\pi}\right)^{\frac{p}{q}}$	$\left(xe^{i(2n+1)\pi}\right)^{\frac{p}{q}}$	$\left(ze^{i\left(2n\pi-\left(\frac{\pi}{2}+\alpha\right)\right)}\right)^{\frac{p}{q}}$	$\left(ye^{i\left(2n-\frac{1}{2}\right)\pi}\right)^{\frac{p}{q}}$	$\left(ze^{i(2n\pi-\alpha)}\right)^{\frac{p}{q}}$
	$x^{\frac{p}{q}}e^{i(2n\pi)\frac{p}{q}}$	$z^{\frac{p}{q}}e^{i(2n+\alpha)\pi\frac{p}{q}}$	$y^{\frac{p}{q}}e^{i\left(2n+\frac{1}{2}\right)\pi\frac{p}{q}}$	$z^{\frac{p}{q}}e^{i\left(2n+\frac{1}{2}+\alpha\right)\pi\frac{p}{q}}$	$x^{\frac{p}{q}}e^{i(2n+1)\pi\frac{p}{q}}$	$z^{\frac{p}{q}}e^{i(2n+1+\alpha)\pi\frac{p}{q}}$	$y^{\frac{p}{q}}e^{i\left(2n-\frac{1}{2}\right)\pi\frac{p}{q}}$	$z^{\frac{p}{q}}e^{i(2n\pi-\alpha)\frac{p}{q}}$
	Based on real value of index of $\boldsymbol{e}$ of $\boldsymbol{\theta}$ in the complex number, the result would be anywhere on Re-Im plane							

Number of roots of a complex number is equal to q, i.e. denominator of the fractional root, and accordingly in general expression of complex number, in polar form, $0 \le n < q$ and at $n = q$ it starts repeating the sequence as that at $n = 0$. This can be tested for any number. In most of the books algebra roots of $x^3 = 1$ are $1, \omega, \omega^2$ such that $1 + \omega + \omega^2 = 0$, where, $\omega = \frac{-1+i\sqrt{3}}{2}$ and $\omega^2 = \frac{-1-i\sqrt{3}}{2}$. Basic equation for this is can be written in the form $x^3 - 1 = 0$. This can be factorized as $x^3 - 1 = (x-1)(x^2 + x + 1) = 0$ and serves as a clue. Representation of roots becomes much simpler in the complex form of a number (in Argend Diagram) in polar form and can be verified. Radical of a number called Surd, discussed in Chapter-II, should not be confused with root of a number, and therefore Surds are distinctly shown in the reclassification together with π and e.

Vectors (contd.): Representation of vector is done in Two Dimension and Three Dimension as under –

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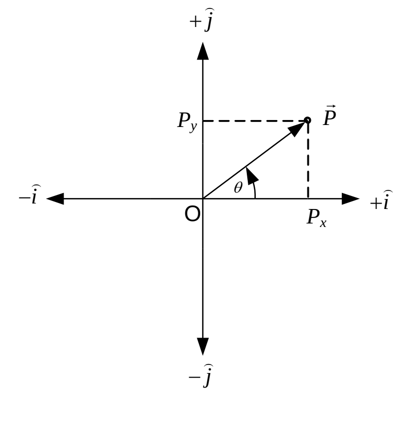
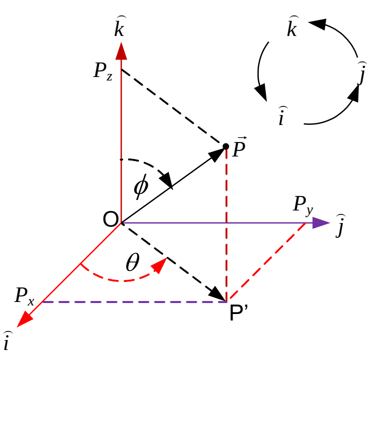
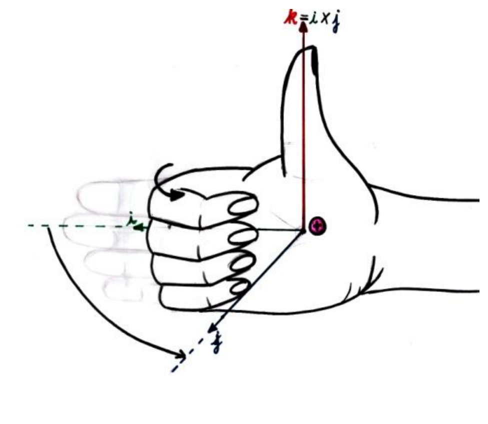
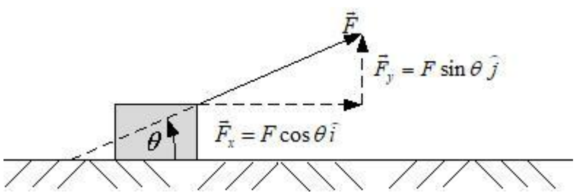
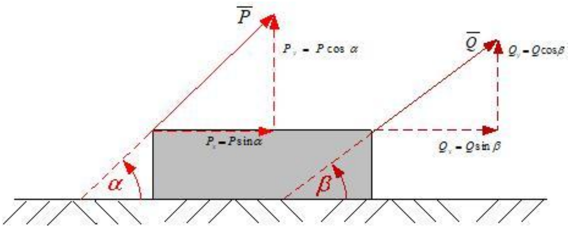
<tr><td>
</td><td>
</td><td rowspan="3"></td></tr>
<tr><td>2-D Coordinates</td><td>3-D Coordinates</td></tr>
<tr><td colspan="2">Cartesian Coordinate Systems</td></tr>
<tr><td colspan="2"></td><td>Right-Hand Thumb Rule for Vector Orientations in 3D</td></tr>
</table>

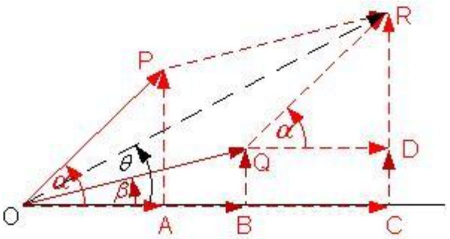
In 2D vector $P = \sqrt{P_x^2 + P_y^2}$, $P_x = P\cos\theta$ and $P_y = P\sin\theta$ are **resolution** of vectors along $\hat{i}$ and $\hat{j}$ directions and are called **orthogonal components and** have already been brought out earlier. It is to be noted that $\hat{i}$ and $\hat{j}$ directions are **orthogonal** (Perpendicular) to each other. These orthogonal components are independent to each other, since resolution of one orthogonal component along other is ZERO since $\cos\frac{\pi}{2} = 0$.***This proposition of orthogonal vectors is applicable to any system of vectors, coordinates and complex numbers***. *Vectors are always drawn to the scale to represent magnitude, and discrete direction*. In 3D space $OP' = \overrightarrow{OP}\sin\emptyset$, $OP_y = OP'\sin\theta$, $OP_x = OP'\cos\theta$, $OP_x = OP'\cos\theta$ and conversely, $OP' = \sqrt{{OP_x}^2 + {OP_y}^2}$ and $OP = \sqrt{{OP_x}^2 + {OP_y}^2 + {OP_z}^2} = \sqrt{OP'^2 + {OP_z}^2}$. Likewise, θ and ϕ can be expressed with the help of trigonometric ratios of P_x, P_y, P_z and OP'. A simple illustrations of vector in real life is as under, where vector $\vec{F}$ is resolved in to two components such that $\vec{F} = \vec{F}_x + \vec{F}_y$ where, $\vec{F}_x = F\cos\theta\,\hat{i}$ and $\vec{F}_y = F\cos\theta\,\hat{j}$.



It is pertinent to understand that during translational motion point of application of non-collinear vectors is the point of intersection of line representing the vectors or a directed line of length equivalent to its magnitude and parallel to it. In this case it is independent of the actual point of application. This would become clear when concept of resultant vector, parallelogram of vectors or triangle of vectors is considered. It is pertinent to indicate that point of application of vector, however, assumes importance in case of rotational effect of vectors, which would be analyzed in rotational mechanics in Physics.

In the figure a rectangular block, placed on a table, is subjected to Two forces P and Q at two different points at an angle with horizontal α and β, respectively. Combined effect of the Two forces requires resolution of the two forces in orthogonal directions, Horizontal and Vertical. Horizontal components of the Two Forces are co-linear, while, vertical components are displaced horizontally. In this example ***since there is no fixed point (a free body),*** *and hence combined effect of vertical components would be like vertical collinear forces; this would interact with gravitational force on the body*. Likewise, horizontal components of the two forces would cause horizontal motion of the block. Concept of free body would be dealt with in *Physics* separately. Accordingly, resultant of two forces on a free body is determined in as shown in the diagram. In this, using the above example of forces on a Free Body, the Two forces represented by $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are shown to act at one point O. Each of them are resolved into $\overrightarrow{OA} = OP\cos\alpha\,\hat{\imath}$ and $\overrightarrow{AP} = OP\sin\alpha\,\hat{\jmath}$, and $\overrightarrow{OB} = OQ\cos\beta\,\hat{\imath}$ and $\overrightarrow{BQ} = OP\sin\beta\,\hat{\jmath}$, respectively. With mathematical analogy vector $\overrightarrow{OP}$ together with its components are represented by $\overrightarrow{QR}$, $\overrightarrow{QD}$and $\overrightarrow{DR}$. Further, with similar analogy component $\overrightarrow{OA}$ is represented as $\overrightarrow{BC}$ and likewise $\overrightarrow{QB}$ is represented by $\overrightarrow{CD}$. Thus, the Two given vectors lead to a pair of additive collinear horizontal vectors $\overrightarrow{OB}$ and $\overrightarrow{BC}$ such that $\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC}$, an $\overrightarrow{CD} + \overrightarrow{DR} = \overrightarrow{CR}$. This leads to $\overrightarrow{OR} = \overrightarrow{OC} + \overrightarrow{CR}$. Here, $OC = OP\cos\alpha + OQ\cos\beta$ and $CR = OP\sin\alpha + OQ\sin\beta$.





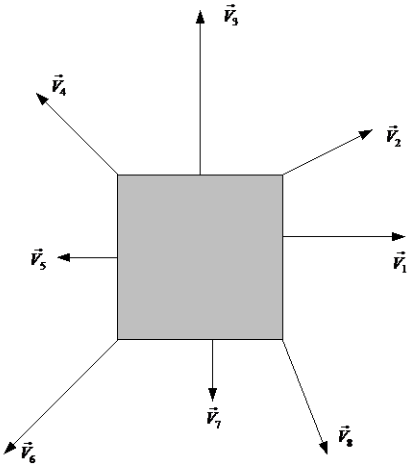
Thus, in parallelogram OQRP so constructed, sides OP and OQ represent two diverging vector at common vertex O. And diagonal OR from the common vertex represent $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PQ}$. These formulations, using trigonometric identities, lead to : $|R| = \left|\overrightarrow{OP} + \overrightarrow{OQ}\right| = \sqrt{(R\sin\theta)^2 + (R\cos\theta)^2}$=$\sqrt{(P\cos\alpha + Q\cos\beta)^2 + (Q\sin\beta + P\sin\alpha)^2}$. Here,

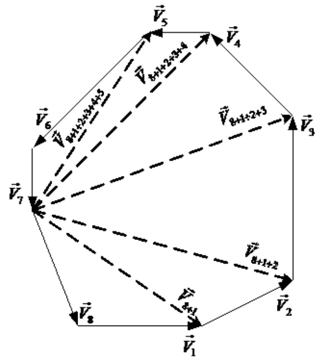
$$R = \sqrt{P^2(\cos^2\alpha + \sin^2\alpha) + Q^2(\cos^2\beta + \sin^2\beta) + 2PQ(\cos\alpha\cos\beta + \sin\alpha\sin\beta)} = \sqrt{P^2 + Q^2 + 2PQ\cos(\alpha - \beta)}$$

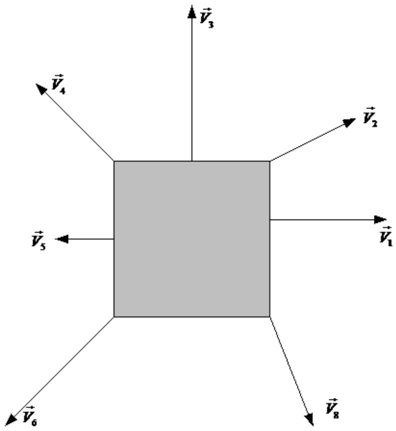
and $\theta = \tan^{-1}\left(\frac{Q\sin\beta + P\sin\alpha}{P\cos\alpha + Q\cos\beta}\right)$.

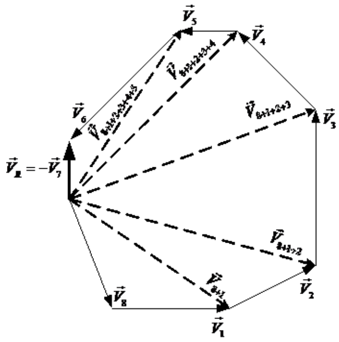
The parallelogram OQRP where, *one of vertex and corresponding two sides of parallelogram represent Two diverging Vectors, the resultant of the two vectors is te diagonal drawn from the Vertex, as shown in the figure is called* ***Parallelogram of Vectors*** where $\vec{R} = \vec{P} + \vec{Q}$. Further, if a vector $-\vec{R}$ together with $\bar{P} + \bar{Q}$, it would cancel the effect of $(\bar{P} + \bar{Q})$ and it is a state of equilibrium, where resultant force is ZERO. Thus, in the figure, ΔOQR, with its three sides representing the three constituent vectors, $\bar{P}, \bar{Q}$ and $-\bar{R}$. Accordingly, *in a system of three Vectors on a free body, if* - **a)** *a set of vectors with head of one coinciding with the tail of other are called* ***concurrent vectors***. **b)** *these vectors can be represented by three* ***concurrent vectors*** *forming a triangle, it is a state of equilibrium and this triangle is called* ***Triangle of Vector***, and **c)** *for a Triangle of vectors the constituent vectors must be Coplanar, it is a corollary*. It is similar to net effect of a number of magnets placed with matching of North pole to South pole of the another. This aspect would be brought out separately in Magnetism, a part of physics. A point inside the triangle so formed by the vectors, sees all the vectors are either Clockwise or Anti-Clockwise. *A normal convention is to place vectors for anti-clockwise concurrency*. On the similar lines subtraction, another mathematical operations, $\vec{R'} = \vec{P} - \vec{Q}$, requires to take $-\vec{Q}$ in a direction opposite to that of $\vec{Q}$.

Likewise, *in a system of many forces (**n** number) acting on a free body, in a state of equilibrium, can be represented by a **Polygon of Vectors** (having **n** sides)* as shown below. A vector, equal and opposite to any one side of the polygon is the resultant vector of the remaining (***n*-1**) vectors. This can be accomplished by taking resultant any two vectors, out of the given *n* vectors, and progressively determining resultant of the latest resultant vector and one of the remaining vectors. Vectors. It is seen that starting process of determining resultant vector $\vec{V}_{8+1} = \vec{V}_8 + \vec{V}_1$, *and taking it progressively a triangle of vectors is reached* $\vec{V}_{8+1+2+3+4+5} + \vec{V}_6 + \vec{V}_7 = 0$. *This is eventually a polygon of coplanar and concurrent vectors, under consideration representative of equilibrium condition of vectors.* Modifying this case by removing vector $\vec{V}_7$. The resultant of the vectors V_8, $V1$, V_2, V_3, V_4, V_5, and V_6 as shown in the figure. Resultant of vector starting $\vec{V}_8$ and $\vec{V}_1$ as done earlier, progressively a vector $\vec{V}_R$ is accomplished such that $\vec{V}_R = -\vec{V}_7 = \vec{V}_8 + \vec{V}_1 + \vec{V}_2 + \vec{V}_3 + \vec{V}_4 + \vec{V}_5 + \vec{V}_6$. This concept is an extrapolation of the concept of parallelogram of vectors when number of vectors is more than Two.

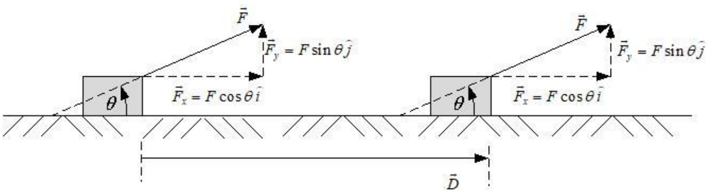




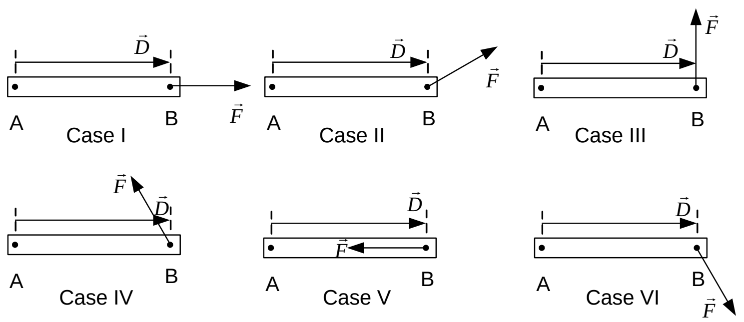




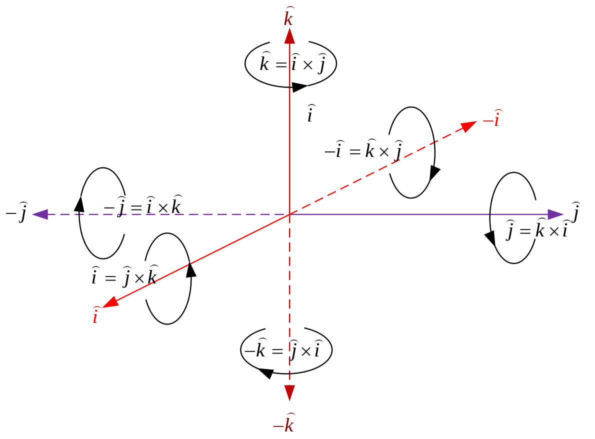
Multiplication of Vectors is unique and different from that elaborated in Number System. There are two types of multiplications called ***Dot Product*** and ***Cross Product***, and each is being discussed separately. The concept of Dot Product can be built by revisiting the concept of resolution of Vector, brought out above. Mathematically, ***Dot Product*** of two vectors is represented as: $\vec{F} \cdot \vec{D} = FD\cos\theta$ is a scalar quantity (equal to product of the magnitudes of the angles and cosine of angle between them), here, $0 \le \theta \le \pi$.

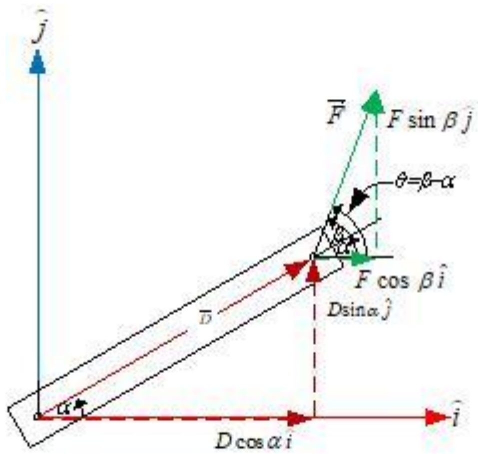


A physical realization of **Cross product of Vectors** is shown in the figure below, where a long rigid body is hinged at A, while Force F is applied at B which at a distance D from the point B. In case I, the effect of force is a Pull. While, in case Two, the force tends to cause turning of the body around Point A in anti-clockwise direction. This turning effect increases as the angle of force F with the line joining AB increases, and is maximum when the angle is 90^0, as shown in Case III. The turning effect tends to decrease as this angle is further decrease as shown in Case IV, till the Force is aligned to the line AB, where it becomes compressive as shown in Fig V. Thus Turning moment is seen to be following sinusoidal function of angle between D and F. Any further increase in angle beyond 180^0 to 360^0 as sown in case VI, undergoing a change with change in the angle as depicted in case I to V; the only difference is the direction of turning, which becomes Clockwise.



Mathematically, this physical observation of turning is represented by **Cross Product** of two vectors as $\bar{Z} = \bar{X} \times \bar{Y} = \overline{XY} \sin\theta$. Since, the turning is directional, either clockwise or anti clockwise, and has a magnitude corresponding to turning tendency, the result of Cross Product is a vector, unlike Dot Product. Generically represented as in the figure below (Left) with the help of unit vectors i, j and k. and a specific illustration is drwan from physical experience in figure (Right), where for force $\bar{F}$, displacement $\bar{D}$ from a fixed point called Fulcrum is resolved on *i-j* plane as both $\bar{F}$ and $\bar{D}$ are on this plane.





The mathematical derivation of Cross Product, based on above identities and that $i \times i = 0$ and $j \times j = 0$, is as under-

$$\vec{D} = D(\cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}), \text{ and } \vec{F} = F(\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}) \therefore \vec{D} \times \vec{F} = \big(D(\cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath})\big) \times \big(F(\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath})\big)$$

$$= DF\big(\cos\alpha\cos\beta\,(\hat{\imath} \times \hat{\imath}) + \hat{\jmath} \times \hat{\imath}\sin\alpha\cos\beta + \cos\alpha\sin\beta\,(\hat{\imath} \times \hat{\jmath}) + \sin\alpha\sin\beta\,(\hat{\jmath} \times \hat{\jmath})\big)$$

$$= DF\big(-\sin\alpha\cos\beta\,\hat{k} + \cos\alpha\sin\beta\,\hat{k}\big) = (DF)\sin(\beta - \alpha)\,\hat{k} = (DF)\sin\theta\,\hat{k}$$

Here, $(\hat{\imath} \times \hat{\imath}) = (\hat{\jmath} \times \hat{\jmath}) = 0$ as per definition of *cross product of vectors*, since angle between collinear vector is zero, while magnitude of cross-product of unit orthogonal vector is One, while it signed value will be (+) ve or (-)ve depending upon their sequence of occurrence, elaborated in cross-product of vectors.

In case of multiplication of vectors, a single equation is formed, while either of multiplicand and multiplier are vectors (having two variables magnitude and angle), it does not comply with the requirement of solving simultaneous equations (*i.e. no equations must be equal to number of variables*), the equations are unsolvable. ***This is the reason that division of vectors is not possible.*** Further, *Dot Product* is a Scalar (non-spatial) quantity and therefore cannot be shown either in vector plane or space. *Cross Product* is just not vector but is in a direction orthogonal to the plane of multiplicand and multiplier vector, and therefore, its illustration involves three dimensional space. Nevertheless, *dot product being scalar, it follows Commutative Property of multiplication, while Cross Product being vector, does not comply with the Commutative Property of multiplication, while this property, together with associative and distributive properties are valid for addition and subtraction*. ***Point of application of vectors in cross product assumes importance due displacement between point of application of vector $\vec{F}$ and the fulcrum, while point of application of vectors is significant in Dot Product as elaborated earlier.***

Logarithm: It is an introduction to a new term ***Logarithm*** in the field of mathematics and is an extension of Theory of Indices, having an extensive utility in calculations involving large numbers. It starts with the mathematical definition as under -

Theorem-1*:* If $a^x = y$; then $x = log_a y$; it is read as $\boldsymbol{x}$ is equal to log (an acronym of logarithm) of y to the base a .
In a similar manner let $a^p = x$ and $a^q = y$, then $p = \log_a x$ and $q = \log_a y$.
Theorem-2: Further, $a^p \times a^q = x \times y = a^{p+q}; \therefore p + q = \log_a(x \times y) = log_a x + log_a y$.
Theorem-3: Extending this formulation, $(a^x)^n = a^{xn}; n \log p$

Theorem-3: Another case is of change of base and is: $a^x = p = b^y; x = \log_a p$ and $y = \log_b p$. Further, $a^{\frac{x}{y}} = b \therefore \log_a b = \frac{x}{y}$. It leads to $x = y \log_a b$. Substituting x and y, $\log_a p = (\log_b p)(\log_a b)$. This together with earlier derivation leads to : $(\log_b a)(\log_a b) = 1$, or $\log_a b = \frac{1}{\log_b a}$.

It is pertinent to note that $log_a a = 1$, and this can be verified from basic definition of logarithm.

In logarithmic calculations use of Common Log value of a number, under consideration is to be determined, and has two parts- **a) *Characteristic***, and **b) *Mantissa***. The common Log Table is used for determining Mantissa and it is with a base 10, as per definition of Logarithm. In respect of characteristic there are two guiding principles as under –

a. Characteristic is always an integer.

b. If the number ≥1, then its characteristic is ($\boldsymbol{k} = \boldsymbol{n}$-1), where $\boldsymbol{n}$ is number of digits on the left of decimal place.

c. In the number is <1, then -

i. Characteristic is $\boldsymbol{k}$ which is the place of first non-zero number after Decimal.

ii. Negative nature of characteristic is represented by $\bar{k}$

These two guiding principle can be combined into one, which requires conversion of the number in scientific notations. As an example 324.5691=3.245691x10^2 and 0.002839= 2.839x10^{-3} will characteristic of the +2 and –3 respectively. Here characteristic is the index of digit 10 in scientific representation of the two numbers. Since, log tables provide value of logarithm of number having Four digits only, hence digits are rounded to third place in the decimal. Rounding of Numbers and precision shall be elaborated in Chapter P-00, in Physics section, an integral part of this series.

In the log-table generally base is 10 and it complies with decimal system, which is most widely used. Accordingly, in logarithmic calculations, numbers are represented as –

$$324.5691 = 3.245691\text{x}10^2 \cong 3.246\text{x}10^2 \text{ and } 0.002839 = 2.839\text{x}10^{-3} = 2.839\text{x}10^{-3}.$$

This can be elaborated from Theorems of Logarithm as under-

$$\log_{10} 324.5691 = \log_{10}(3.245691 \times 10^2) \cong \log_{10} 3.245 + \log_{10}(10^2) = \log_{10} 3.245 + 2\log_{10} 10 = 2 + \log_{10} 3.245$$

Likewise, $\log 0.002839 = \log(10^{-3} \times 2.839) = -3 + \log 2.839 = \bar{3}.4532$

It is to be noted that mantissa is always positive but characteristic could be (+)ve or (–)ve based on index of 10 in representation of the number in scientific notation. In the second example above, (-)ve characteristic is represented as $\bar{3}$ and its logarithmic value is read as “bar 3 point 4532 “.

In case there is an index to number under calculations, multiply the log value with index. But, in case index is –ve steps are a little different and it would become clear from the examples below involving use of logarithmic table for calculations in following Steps –

Step-1: Convert all numbers in denominator to numerator by suitable changing their indices into scientific notations.

Step 2: Refer Log Table to note Logarithm of each of the multiplier,

a. Determine characteristic, from the associated numbers,

b. Refer Log-Table to determine mantissa

c. Apply index of each number to logarithms so noted-In case characteristic is negative convert the logarithmic value into a signed value.

Step 3: Add signed values of logarithm of all the multipliers.

Step 4: Convert log value, if it is (–) ve, into negative characteristic (with bar)and mantissa a (+)ve value.

Step 5: After calculating overall Logarithm, use Log-Table to determine Anti-Log.

Step 6: Place Decimal value as per characteristic, arrived at Stage (5) above.

These steps are implemented on typical calculation to demonstrate use of log-table, as shown below -

$\frac{234.589 \times .00197 \times 5.6381^{\frac{1}{5}}}{(3.895 \times 10^{-4}) \times 5789.173^2 \times 0.7936541^2}$. This expression is transformed product of Four digit s to numerator in scientific notations as $= 2.346 \times 10^2 \times 1.97 \times 10^{-3} \times 5.638^{\frac{1}{5}} \times 3.895^{-1} \times 10^4 \times (5.7892 \times 10^3)^{-2} \times (7.936 \times 10^{-1})^{-2}$.

Determination of logarithmic value of the calculation, element-wise is tabulated below -

S No.	Number in Scientific Notation	Logarithmic Value of base Number			Logarithmic Value of multiplicand alongwith its index
		Characteristic	Mantissa	Log Value	
1	2.346×10^2	2	3703	2.3703	2.3703
2	1.97×10^{-3}	-3	2945	$\bar{3}.2945$	$\bar{3}.2945$
3	$(5.638 \times 10^0)^{1/5}$	0	7511	0.7511	$0.7511 \times \frac{1}{5} = 0.1502$
4	$(3.895 \times 10^{-4})^{-1}$	-4	5905	$\bar{4}.5905$	$(\bar{4}.5905) \times (-1)$ $= (-4 + 0.5905) \times (-1)$ $= (-3.4095) \times (-1)$ $= 3.4095$
5	$(5.789 \times 10^3)^{-2}$	3	7626	3.7626	$3.7626 \times (-2)$ $= -7.5252 = \bar{8}.4748$
6	$(7.936 \times 10^{-1})^{-2}$	-1	8996	$\bar{1}.8996$	$(-1 + 0.8996)\text{x}(-2)$ $= -0.1003\text{x}(-2) = 0.2006$

Overall Characteristic = 2-3+0+3-8+0 = -6

Overall Mantissa = 0.3703 + 0.2945 + 0.1502 + 0.4095 + 0.4748 + 0.2006 = 1.8999

Overall logarithmic value : -6 + 1.8999$= -5 + 0.8999 = \bar{5}.8999$*;*

Logarithmically calculated= antilog ($\bar{5}.8999$)=(7925+16)x10^{-5}=7941+16x10^{-5}=7.925x10^3x10^{-5}=7.925x10^{-2}

This method in practical life with advent of scientific calculators and computation tools have become of classical importance, nevertheless, a drill into it is essentially advised to understand basic mathematics that goes into it. Slide Rule has been in use for scientific and engineering calculations upto Eighties of 20th Century. Basic principle used in design of Slide Rule is Log Table and thus it can be called a mechanised Log Table. It converts complete process of cascaded multiplications/divisions into an exercise of just movement of scale. User is just required to ascerta in decimal place in the result.

Logarithmic Series: This series is derived from exponential series and finds application in science, which shall be encountered as this journey proceeds. Derivation of logarithmic series is as under –

$$e^c = a\ ;\ c = \log a$$
$$e^{cy} = 1 + cy + \frac{c^2y^2}{!2} + \frac{c^3y^3}{!3} + \cdots$$
$$a^y = e^{cy} = 1 + y(\ln a) + \frac{y^2(\ln a)^2}{!2} + \frac{y^3(\ln a)^3}{!3} + \cdots$$

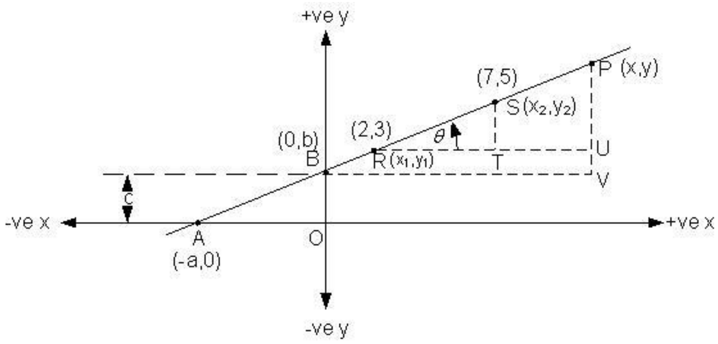
Let $1 + x = a|_{-1<x\leq 1}$, then using *Binomial Expansion* $a^y = (1+x)^y = 1 + yx + \frac{y(y-1)}{!2}x^2 + \frac{y(y-1)(y-2)}{!3}x^3 + \cdots$

Equivalence of two polynomials, for all values of a variable, has a necessary condition that coefficients of variable, in two polynomial, for its every index must be equal. Accordingly, equating coefficients of y in both the series would require splitting of RHS of the above polynomial and retain terms with only y we get a series $yx + y(-1)\frac{x^2}{2!} + y(-1)(-2)\frac{x^3}{3!}$...... Equating these coefficients of y with that in the earlier series it leads to –

$\ln a = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$, this is without prejudice to initial requirement that $-1 < x \leq 1$. Accordingly, the ***logarithmic series*** is: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$

Coordinate Geometry (Contd.): Introduction to Coordinate Geometry and its conventions have been made earlier in this chapter. However it was limited to representation of a point. It finds application in representation of any point, line, curve, shape or an expression that is drawn either on a plane or space. Such representations on a plane use two coordinates system and therefore are in the domain of 2-D coordinate geometry, while in space, requiring three Coordinates system, geometry is dealt with in 3-D coordinate geometry. Here, coordinate geometry is introduced for the purpose of treatment of relevance of various topics of physics in the chapters to follow. Details of coordinate geometry shall be dealt with separately in section Mathematics, am integral part of this series.

Lines: It is a trace of a point passing through two points and was introduced in geometry (G-02). In perspective of coordinate geometry the points are defined by their respective coordinates as shown in the figure below. Here, points are P (x_1,y_1) and Q (x_2,y_2) and corresponding coordinates have values are (2,3) and (7,5) respectively. The slope of the line is $\frac{\text{Length QR}}{\text{Length PR}} = \frac{y_2-y_1}{x_2-x_1} = \frac{5-3}{7-2} = \frac{2}{5} = 0.4$ .This slope is defined as $m = \tan\theta = \frac{y_2-y_1}{x_2-x_1}$, a trigonometric ratio. Hhere, $0 \leq \theta \leq \pi$ and $\theta \neq \frac{\pi}{2}$. The intercept of line on Y axis is shown as c, and in this instance it is +ve. The points of interception of the line x and y axis are A(-a,0) and B (b,0).



Taking a line passing through points **R** and **S**, whose coordinates are (2,3) and (7,5) in the figure shown above, slope of the line can be defined from ⊥Δ RST as : $m = \tan\theta = \frac{\text{ST}}{\text{RS}} = \frac{y_2-y_1}{x_2-x_1} = \frac{7-2}{5-3} = \frac{5}{2}$.

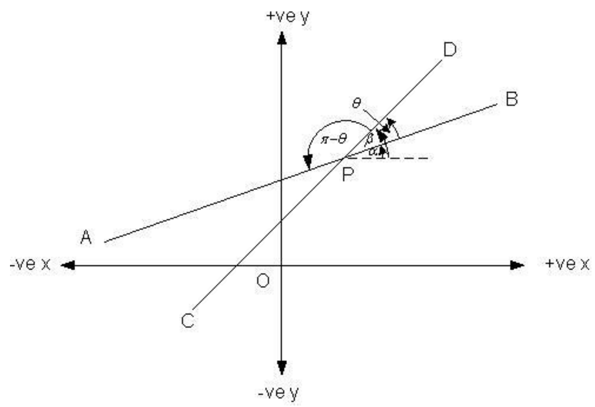
In coordinate geometry, a slope is notified as ***m***. This shall remain so for any other point on the line. Therefore, for a point **P**(x,y) on the line then as per geometry: $m = \frac{y_2-y_1}{x_2-x_1} = \frac{y-y_1}{x-x_1}$ .It leads to a general mathematical statement of straight line passing through Two points is: $\frac{y_2-y_1}{x_2-x_1} = \frac{y-y_1}{x-x_1}$; it is called ***First form of equation of a straight line.***

In case line is not parallel to either of the axes, i.e. x or y axis, then the line would intersect y axis say **B (**0,***b*****)**. Using this intercept ***b*** the equation of line can be written in another form as: $m = \frac{y-b}{x-0}$; $y - b = mx$; $y = mx + b$

Second form of the equation of a straight line is $y = mx + c$; where the intercept on y-axis is notified as ***c*** when $m \neq \infty$. Since, a line parallel to X-axis would have ***m*** = 0, accordingly equation of the line would be $y = c$. While, when $m = \infty$, equation of line is $x = c$.

There is another form of the equation which uses points **A (a,0)** and **B (0,b)** intercepts on y and x axis, and it is derived from First form of equation evolved above, and is $\frac{b-0}{0-a} = \frac{y-0}{x-a} \rightarrow bx - ab = -ay$. Dividing the equation by ab, it leads to $\frac{bx-ab}{ab} = -\frac{ay}{ab} \rightarrow \frac{x}{a} - 1 = -\frac{y}{b}$. In its final formulation, $\frac{x}{a} + \frac{y}{b} = 1$, and is called ***third form of equation of a straight line.***

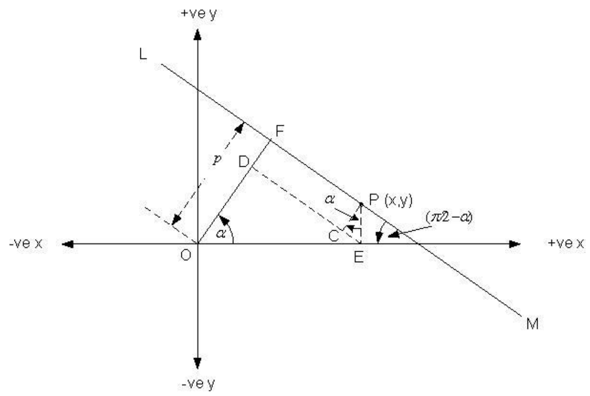
While, for ***a line parallel to y-axis***, it would have $m = \infty$, theoretically the line would tend to meet Y-axis at infinity and, therefore, $b \rightarrow \infty$. Accordingly, equation of line would be $x = a$, here $(a,0)$ is intercept of the line on X-axis.



Two lines having same slope (***m***) can be either collinear or parallel. If lines collinear then they will have equal values of ***c***, while parallel lines shall have unequal values of ***c***. If Two lines AB and CD are not parallel as shown in the figure, and they have slopes $m_1 = \tan\alpha$, and $m_2 = \tan\beta$. Then angle between the two lines $(\theta = \beta - \alpha)$, from the trigonometric identities brought out earlier in this chapter, is –

$$\tan\theta = \tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\alpha\tan\beta} = \frac{m_2-m_1}{1+m_1m_2}; \theta = \tan^{-1}\left(\frac{m_2-m_1}{1+m_1m_2}\right)$$

When line are orthogonal to each other, i.e. making angle $\theta = \frac{\pi}{2}$, then $\tan\theta = \infty$, necessary condition for this is $1 + m_1m_2 = 0$. Accordingly, for two lines having slopes m_1 and m_2 , the slopes are related as $m_1 = -\frac{1}{m_2}$.

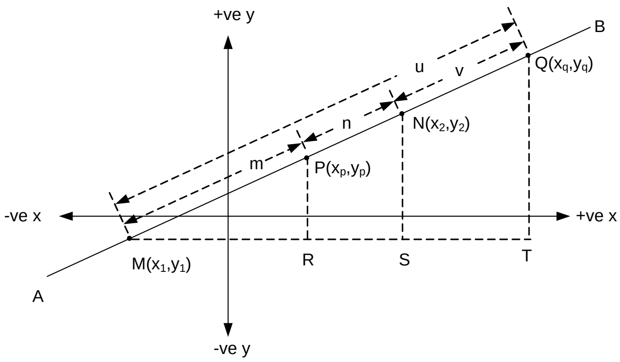


Fourth form of equation of a straight line uses perpendicular (also called normal) from origin O on to the given straight line, as shown in the figure below. Let, length of the normal OF is ***p*** and angle of the normal with x axis is **a**. Then coordinates of any point P (x,y) on the line LM, can be related to parameters of the normal OF, length ***p*** and angle $\boldsymbol{\alpha}$, polar coordinates of point of intersection of the normal F, is : $\mathrm{OF} = \mathrm{OD} + \mathrm{DF} = p = x\cos\alpha + y\sin\alpha$. This can be written as $x\cos\alpha + y\sin\alpha - p = 0$. Replacing the coefficients, it can be written as $Ax + bY + C = 0$. Since, these two equations are equivalent and hence their coefficients, from algebra shall be of same proportion, accordingly $\frac{\cos\alpha}{A} = \frac{\sin\alpha}{B} = \frac{-p}{C}$. It leads to $\cos\alpha = -\frac{pA}{C}$, $\sin\alpha = -\frac{pB}{C}$. Therefore from trigonometric identity $\frac{p^2A^2}{C^2} + \frac{p^2B^2}{C^2} = 1 \rightarrow p = \frac{C}{\sqrt{A^2+B^2}}$. Accordingly, $\cos\alpha = -\frac{AP}{C} = -\frac{A}{C}\bullet\frac{C}{\sqrt{A^2+B^2}} = -\frac{A}{\sqrt{A^2+B^2}}$, and likewise $\sin\alpha = -\frac{BP}{C} = -\frac{B}{\sqrt{A^2+B^2}}$. Thus, fourth form of equation can also be written as :

$$x\left(-\frac{A}{\sqrt{A^2+B^2}}\right) + y\left(-\frac{B}{\sqrt{A^2+B^2}}\right) - \frac{C}{\sqrt{A^2+B^2}} = 0 \rightarrow Ax + By + C = 0.$$

While evolving the above equation angular properties of triangles are used and are suitably depicted in the figure. This relation can also be used to arrive at equation of a line parallel to y-axis, having $\alpha = \frac{\pi}{2}$ radians. As a result, $\cos\alpha = 0$, and $\sin\alpha = 1$; accordingly, $x = p = C$, a constant, as proved earlier. On the similar lines, this equation can also be used to re-establish equation of a line parallel to x-axis, already proved using Second form of equation of a straight line.

Divisor of Two Points: There is always a line passing through two points say M (x_1,y_1)and N (x_2,y_2). An ***internal divisor point*** P is taken such that MP:NP::m:n, as shown in the figure below. It can be established that ΔMRP and ΔMSN are similar, and, therefore -

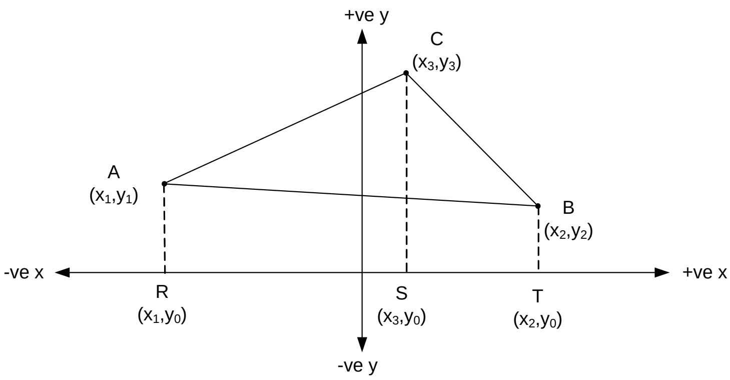


$\frac{m}{m+n} = \frac{\text{MP}}{\text{MN}} = \frac{\text{MR}}{\text{MS}} = \frac{x_p - x_1}{x_2 - x_1}$; $m(x_2 - x_1) = (m+n)(x_p - x_1)$. It leads to $m \cdot x_2 + n \cdot x_1 = (m+n)x_p$, Thus $x_p = \frac{m \cdot x_2 + n \cdot x_1}{m+n}$; likewise, $y_p = \frac{m \cdot y_2 + n \cdot y_1}{m+n}$.

Similarly, in the figure below, for an ***external divisor point*** Q such that MQ:NQ::u:v. On similar lines it can be shown that ΔMSN and ΔMTQ are similar, therefore –

$$\frac{u}{u-v} = \frac{\text{MQ}}{\text{MN}} = \frac{\text{MT}}{\text{MS}} = \frac{x_q - x_1}{x_2 - x_1} \text{ ; } u(x_2 - x_1) = (u-v)(x_q - x_1) \text{ ; } u \cdot x_2 - v \cdot x_1 = (u-v)x_q$$
$$\therefore x_q = \frac{u \cdot x_2 - v \cdot x_1}{u-v} \text{ ; likewise } ,y_q = \frac{u \cdot y_2 - v \cdot y_1}{u-v}$$

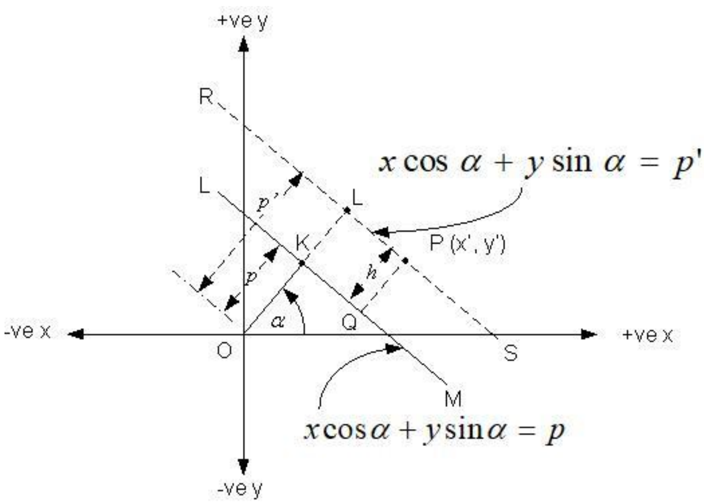
Area of a Triangle & Co-linearity: Triangle is an elementary shape which finds wide application in finite element analysis. Area of a triangle ABC, in the figure given below is, is always +ve value :



ar(ABC) = ar(ARSC) + ar(CSTB) - ar(ARTB)

$$= \frac{1}{2}(y_1 + y_3)(x_3 - x_1) + \frac{1}{2}(y_2 + y_3)(x_2 - x_3) - \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$$
$$= \frac{1}{2}\left(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right)$$

In addition to angular property of a triangle (i.e. sum of angles is π radians), there is another property of triangle called *non-linearity* (i.e. the three vertices of a triangle are non-collinear). Accordingly, *if area of a triangle is zero, it is possible if-and only-if the set of three vertices of the triangle are collinear.*

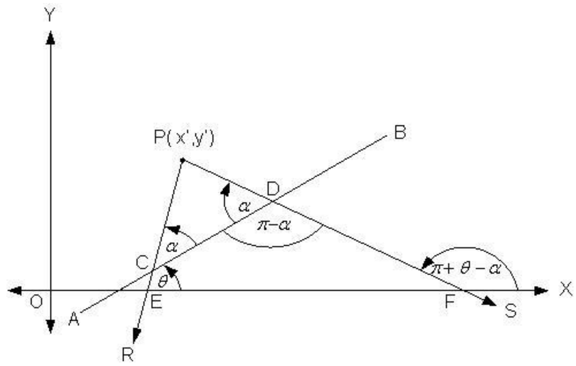


Different forms of equations of line and they are equivalent to each other. They only differ in respect of parameters defined for each of these. Use of equation of one form or the other depends upon the given problems and parameters, and converted suitably in the form required for specific purpose. There are some case studies in respect of straight lines, which find application not only in coordinate geometry but, other studies of maths and science.

One such case is ***Perpendicular distance of a point P from a given straight line*** is one of them. Let P (*x', y'*) be the point whose perpendicular distance of origin (0,0) from line

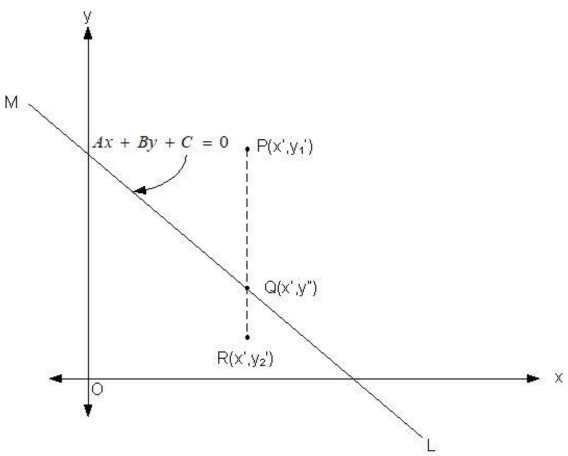
LM ($x\cos\alpha + y\sin\alpha = p$). A line RS is drawn parallel to LM is drawn passing through P whose equation would be ($x\cos\alpha + y\sin\alpha = p'$), where p' is the perpendicular distance of the line from origin 0. From these two equations for the point P on line RS, its perpendicular distance (h) from line LM would be $h = p' - p$. This, together with modified Fourth form of equation of a line can be written as $h = x'\cos\alpha + y'\sin\alpha - p = \frac{Ax+By+C}{\sqrt{A^2+B^2}}$. Here, $\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}$, $\sin\alpha = \frac{A}{\sqrt{A^2+B^2}}$, $-p = \frac{C}{\sqrt{A^2+B^2}}$.

Likewise, another case is of ***equation of lines from a point forming equal angle with a given line***. Let P be a point from which lines PR and PS forming an angle α with a given line AB ($y = mx + c$), and intercept X axis at points E and F, as shown in the figure below. Here, $m = \tan\theta$. The angles formed by lines PR and PS with X-axis are ∠PEX = $\theta + \alpha$ (sum of the Two angle) and ∠PFX = $\pi + \theta - \alpha$. (Property of exterior angle of a triangle). Accordingly, slope (m_1) of line PR = $\tan(\theta + \alpha) = \frac{m+\tan\alpha}{1-m\tan\alpha}$ and slope (m_2) of line PR = $\tan(\pi + (\theta - \alpha)) = \tan(\theta - \alpha) = \frac{m-\tan\alpha}{1+m\tan\alpha}$.



Using these calculated value of slopes of lines making an angle α with line AB, equation of line PR is $y - y' = \left(\frac{m+\tan\alpha}{1-m\tan\alpha}\right)(x - x')$ and line PS is $y - y' = \left(\frac{m-\tan\alpha}{1+m\tan\alpha}\right)(x - x')$.

Another case is of a point off a line as shown in the Figure below, where a point P (x',y_1') is so taken that and origin O is on the opposite side of the line ML. Projection of Point P, perpendicular to x- axis is point Q (x',y_2'), and it would satisfy equation of line ML i.e. $Ax' + By"$+C=0; $y" = -\frac{Ax'+C}{B}$. In the instant case $y" - y_1' < 0$, i.e. negative.

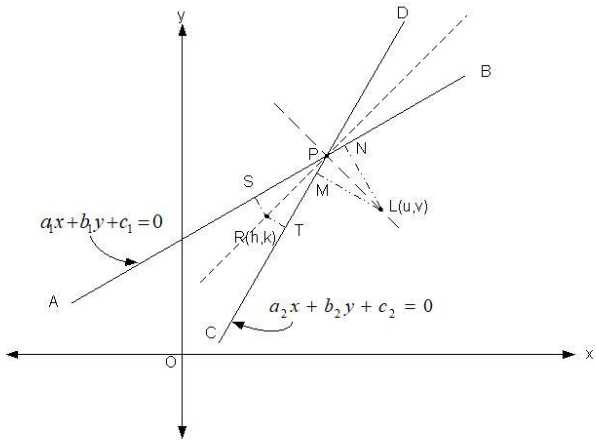


Another point R(x',y_2') below the line, and on the same side as that of the origin O, shall also have point Q as projection on the line ML, since x-coordinates of points P and R are same. Therefore, equation for projection of point P on line ML shall be valid for projection of point R on the line ML, which is same for both points P and R, in the instant case. Nevertheless, $y" - y_2' > 0$, positive. It is concluded that ***projection of a point on a line is +ve/-ve depending on whether the point under consideration is either on same or opposite side of the line with respect to origin***.

The logic derived above is used to determine equations of bisectors of two intersecting lines as shown in the figure below. Lines AB and CD are intersection at point at P such that ∠APC and ∠BPD are acute and obtuse angles, respectively. Points R and L are taken on acute and obtuse angle bisectors such that –

- Length of perpendiculars RS and RT from point R on line AB and CD are equal. Moreover this point R and origin are on same side of the lines AB and CD and therefore –

$$\frac{a_1h + b_1k + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2h + b_2k + c_2}{\sqrt{a_2^2 + b_2^2}}$$



- Likewise, length of perpendiculars LM and LN from point L on line AB and CD are equal. Moreover, this point L and origin are on same side of the lines AB but and on opposite side of the line CD and therefore –

$$\frac{a_1u+b_1v+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2u+b_2v+c_2}{\sqrt{a_2^2+b_2^2}}$$

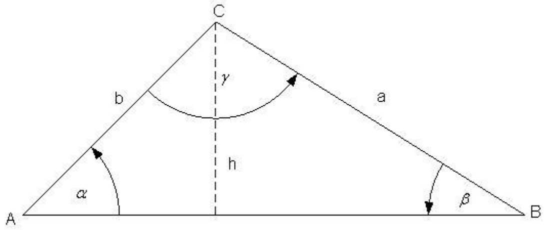
- Thus two sets of equation for acute and obtuse angle bisectors with in terms of generic coordinates (x,y) combine separate equation of acute and obtuse angle bisector, as derived above, and is -

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

Special cases viz. bisector to two straight lines, lines making an equal angle with a given straight lines shall be evolved as theorem or problems in a separate Chapter on Coordinate Geometry in Mathematics, an integral part of this manual.

Properties of Triangle (Trigonometry Contd...) : Triangle is an elemental shape, with minimum number of sides, and finds a wide application in the study of mathematics, physics and engineering. Area of a triangle on a plane, taking coordinates of vertices, has been derived above. There are other properties of a triangle, which needs to be appropriately introduced here to make the learning more conceptual and analytical.

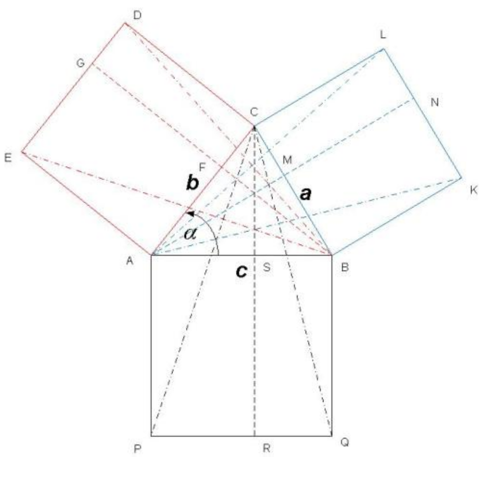
Ratio of Sine of any angle to the opposite side: It involves use of trigonometry, and it is applied on a triangle shown in the figure below. A perpendicular is drawn from vertex 'C' on the side 'c' opposite to the angle.



In ΔADC and ΔCDB; $h=b\sin\alpha=a\,\sin\beta$; accordingly, $\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}$. Likewise, constructing a perpendicular from vertex 'A' on the side 'a' opposite to it it would give $\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$. Further, using properties of ratio-proportion on the above two equations it gives a generalized expression : $\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$

Cosine of any Angle: It has been derived geometrically, with constructions on three sides of a scalene triangle, most general case, which will be reduced to Pythagoras Theorem, a specific case applicable to Right angle Triangle, an example of reverse engineering. In a ΔABC, deliberately its three sides shown in different colours, squares are constructed on its each side ΔABC with colours of corresponding sides. From each of the vertex ⊥ is drawn on opposite side and extended to opposite side of squares constructed on them. Each of the vertex is joined to ends of the opposite sides.

Now, $BC^2=\mathrm{ar(CBKL)}=\mathrm{ar(CMNL)}+\mathrm{ar(MBKN)}$. Further, $\mathrm{ar(CMNL)}=2\times\mathrm{ar(CAL)}$; being between || lines CL and AN and having common base CL.



Likewise, $\mathrm{ar(GFCD)}=2\times\mathrm{ar(DBC)}$; being between || lines DC and GB and having common base DC.

ΔDBC ≅ΔCAL; by SAS theorem since DC=CA, CB=CL and ∠DCA = ∠ACL=90^0+∠ACB.

Therefore, $\mathrm{ar(CMNL)}=\mathrm{ar(GFCD)}=AC^2-\mathrm{ar(EAFG)}$.

Likewise, it can be proved that :$\mathrm{ar(MBKN)}=\mathrm{ar(SRQB)}=AB^2-\mathrm{ar(APRS)}$.

Combining the above two equations,

$$BC^2=\mathrm{ar(CMNL)}+\mathrm{ar(MBKN)}=AC^2-\mathrm{ar(EAFG)}+AB^2-\mathrm{ar(APRS)}$$

Rearranging, $BC^2=AC^2+AB^2-\mathrm{ar(EAFG)}-\mathrm{ar(APRS)}$

Following convention of naming sides opposite to the vertices of a triangle in lower case –

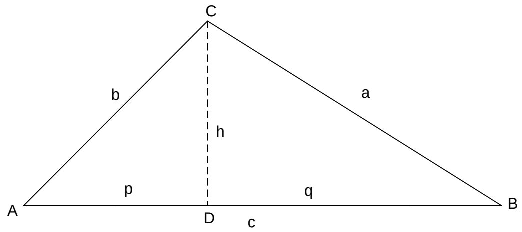
Length of side 'AB' $=c$, length of side 'BC' $= a$, and Length of side 'CA' $= b$. Thus, $a^2 = b^2 + c^2 - \text{ar(EAFG)} - \text{ar(APRS)}$

A close observation of two areas in the above equation reveals that, $\text{ar(EAFG)} = b(c\cos\alpha)$ and $\text{ar(APRS)} = \text{c}(b\cos\alpha)$. Accordingly, $\text{ar(EAFG)} + \text{ar(APRS)} = 2bc(\cos\alpha)$. Combining, this equation with te above it leads to general expressions : $\boldsymbol{a^2 = b^2 + c^2 - 2bc(\cos\alpha)}$**,** and $\boldsymbol{\cos\alpha = \frac{b^2+c^2-a^2}{2bc}}$

These general expressions in specific case when $\angle A=\alpha=90^0$ would lead to $\boldsymbol{a^2 = b^2 + c^2}$, *Pythagoras Theorem applicable to* $\perp\Delta$.

Area of a Triangle: It has been proved in Chapter II that area (A) of a ΔABC triangle is $=\frac{1}{2}(Height) \times (Base)$, accordingly, in the figure shown below -

$$A = \frac{1}{2}hc$$



Further, $\perp$ CD on line AB divides ΔABC in Two ∟Δs ADC and CDB and side AB in two parts AD and DB of lengths p and q, respectively. Accordingly as per Pythagoras theorem, $b^2 = p^2 + h^2$; $a^2 = q^2 + h^2$; Adding these Two equations we get –

$$a^2 + b^2 = p^2 + q^2 + 2h^2 = (p+q)^2 - 2pq + 2h^2 = c^2 - 2pq + 2h^2$$
$$= c^2 - 2p(c-p) + 2h^2 = c^2 - 2pc + 2p^2 + 2h^2 = c^2 - 2pc + 2(b^2 - h^2) + 2h^2$$
$$= c^2 + 2b^2 - 2pc$$

It leads to $p = \frac{b^2+c^2-a^2}{2c}$

Reverting back to basic equation of area of a triangle–

$$A^2 = \frac{1}{4}h^2c^2 = \frac{1}{4}(b^2 - p^2)c^2 = \frac{1}{4}(b-p)(b+p)c^2$$
$$=> \frac{1}{4}\left(b - \frac{b^2+c^2-a^2}{2c}\right)\left(b + \frac{b^2+c^2-a^2}{2c}\right)c^2$$
$$=> \frac{1}{16}(2bc - b^2 - c^2 + a^2)(2bc + b^2 + c^2 - a^2) = \frac{1}{16}(a^2 - (b-c)^2)((b+c)^2 - a^2)$$
$$=> \frac{1}{16}\big(a - (b-c)\big)\big(a + (b-c)\big)\big((b+c) - a\big)\big((b+c) + a\big)$$
$$=> \frac{1}{16}(a+c-b)(a+b-c)(b+c-a)(a+b+c)$$
$$=> \left(\frac{(a+c-b)}{2}\right)\left(\frac{(a+b-c)}{2}\right)\left(\frac{(b+c-a)}{2}\right)\left(\frac{(a+b+c)}{2}\right)$$

Let, $s = \frac{a+b+c}{2}$, and substituting this in above equation, and then rearranging them it leads to -

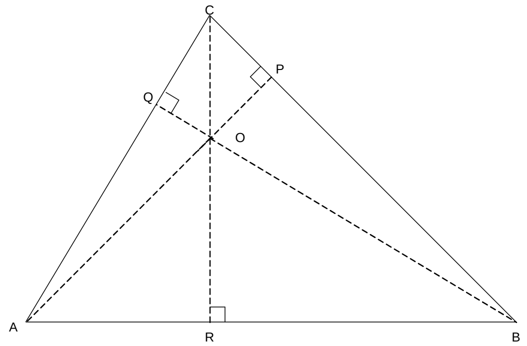
$$A^2 = (s-b)(s-c)(s-a)s = s(s-a)(s-b)(s-c)$$

Or, $\boldsymbol{A = \sqrt{s(s-a)(s-b)(s-c)}}$**.** This formula is known as ***Heron's formula***.

These identities would find extensive application in Physics, while, there are many such identities which find application in problem solving and would be dealt with in later Chapter in Trigonometry.

Further, there are sets of concurrent lines viz. altitudes, angle bisectors, medians, and perpendicular bisectors and points intersection of each set is called **orthocenter**, **incenter**, **centroid** and **circumcenter** respectively, which will find diverse applications in the forthcoming journey. Accordingly significance of each with analytical understanding shall be brought out later in this chapter, soon after introduction of Calculus, for building an integrated perspective.

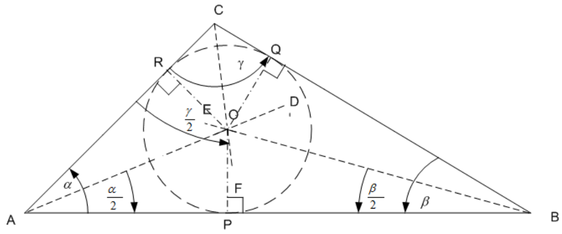
Orthocenter: It is the point at which altitudes run from each vertex and meet the opposite side; altitudes are at right angle to the corresponding sides. at a right angle.. In the triangle shown below point 'O' is orthocentre of ΔABC.As of yet no significant application of Orthocentre has been reported except an indirect application that triangle formed by joining base of perpendiculars drawn from vertices A,B and C on sides opposite to them viz. P,Q and R is shortest triangle inscribed inside the ΔABC. The ΔPQR is called Orthic Triangle. In acute angle triangle Orthocenter is inside the triangle, while in obtuse angle triangle it is outside. A property of Orthocentre in an acute angle triangle is as under –



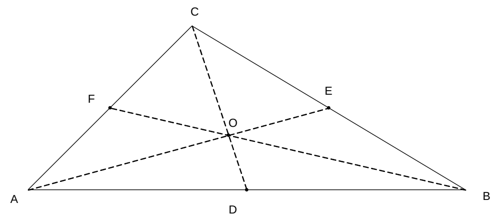
$$\frac{PO}{AO}+\frac{QO}{BO}+\frac{RO}{CO}=1, \text{ and } \frac{AO}{AP}+\frac{BO}{BQ}+\frac{CO}{CR}=2$$

This property can be proved using geometrical definition of area of a triangle, and is left as a case study for reader.

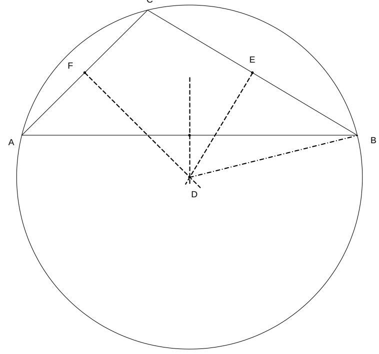
Incenter: It is a point of intersection of bisectors of each of the internal angles of a triangle. Each of the angle bisectors is equidistant from the two lines forming the angle. Accordingly , taking this incenter of circle as a centre, a circle can be drawn which has a radius of length equal to perpendicular from this point on any of the side of the triangle. This circle touches all the three sides of the triangle and is shown in the figure below.



Centroid: It is the point at which medians meet. Median is a line joining centre of each side to the vertex of the triangle opposite to it, and it divides the triangle in two parts, equal in area. This point is also called <u>*Centre of Gravity*</u> of the triangle, and it 2/3 from vertex and 1/3 from centre of the line opposite to it. This can be established geometrically and with ***integral calculus***, which is covered little later in this chapter. Centre of gravity would be covered in **Mechanics, a part of Physics section of Mentors' Manual.**.

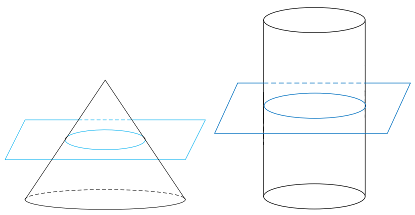


Circumcenter: It is a point on concurrent lines which are perpendicular bisectors of each side of triangle, which is equidistant from three vertices, and can be proved geometrically. Circumcenter can be within the circle if all interior angles are acute, while it may be outside if any of the interior angle is obtuse. In the instant case ∠ACB is obtuse, and one of the radial OB is shown in the figure.

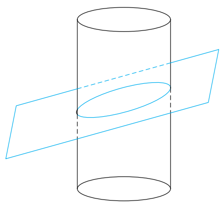


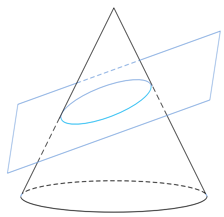
Locus : It is a trace of a point which satisfies specified conditions, which are defined in the form of an algebraic equation. Straight line is also a Locus, and equation of line in various forms is nothing but under specified conditions. Likewise, if a locus is not a straight line then it is called a ***Curve***. ***Conic sections***, *usually referred to as **Conic**, are special class of Curves mostly encountered in nature and engineering and generated by intersection of a plane with a cone or a set of inverted cones*. Conic sections can be realized in physical shapes and is called ***geometrical representation of Conics***. Trajectories or orbits of celestial bodies, planets and satellites are seen to be conforming to Conics and this representation is in domain of ***dynamics***. This shall be dealt with separately in **Mechanics, a part of Physics section of Mentors' Manual**.

Physical visualization of conic sections, as the name suggests, is made using cones, a three dimensional geometry as shown in the figure below. Cylinder is a special category of a cone having angle $\alpha = 0$ while radius of the base $r \neq 0$. of a frustum of a cone which has angle Zero and infinite height.



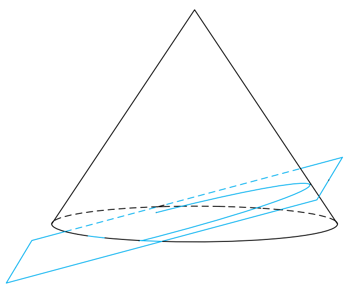
Circle: Perimeter of a section of a cone formed by a plain perpendicular to its axis



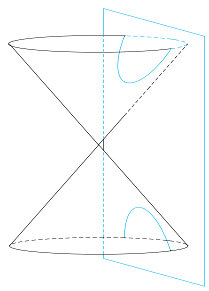


Ellipse: Perimeter of a section of a cylinder/one formed by a plain having an angle with the axis of cylinder/cone

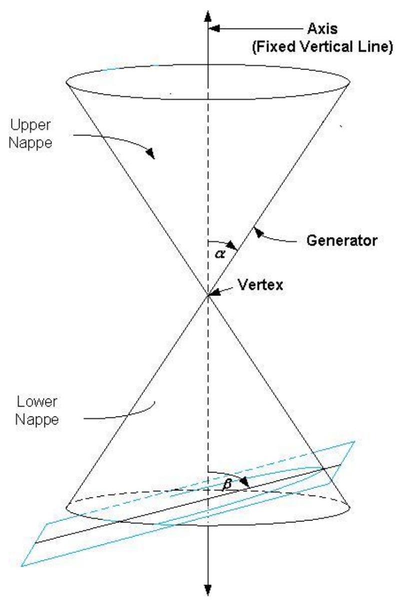
Ellipse *from a cylinder* | Ellipse *from a cone*



Parabola: Curved perimeter of a section of a cone formed by a plain inclined to the axis of cone and intersecting its base and surface on opposite sides of the axis



Hyperbola: Curved perimeter of a section of two inverted Cones formed by a plain having parallel to the axis of the two Cones but not coinciding with the axis



The above perspective of solids can be understood in more generic manner from the figure given here. Conic section has a fixed vertical line which is called ***Axis***. Point of intersection of a line inclined at an angle α with the Axis is called ***Vertex***. This angle is half of the angle of cone (2α). This inclined line is when rotated around the axis a pair of cones are generated; accordingly this inclined line is called ***Generator*** of cone. This cone has two parts ***Nappe***, the part above vertex is called ***Upper Nappe*** and the one below the vertex is called ***Lower Nappe***. Having visualized cones, conic sections are generated when a plane surface intersects a cone or a pair of cones. *Nature of conic section depends upon position of the plane and an angle β which it makes with axis*.

Point : When plane passed through Vertex and makes an angle $\beta = 90^0$

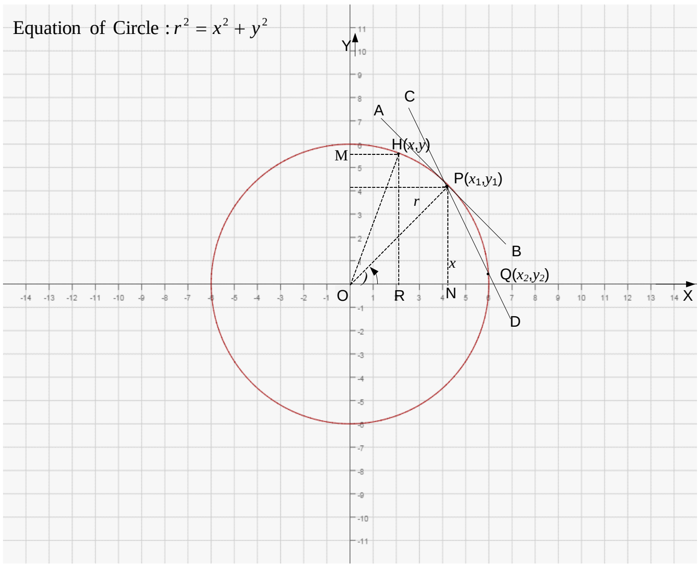
Circle : When plane does not pass through Vertex and makes an angle $\beta = 90^0$.

Ellipse : When plane does not pass through Vertex and makes an angle $\alpha < \beta < 90^0$
Parabola : When plane does not pass through Vertex and makes an angle $\beta > \alpha$
Hyperbola : When plane may or may not pass through Vertex and makes an angle $0 < \beta < \alpha$
Line : When plane passes through Vertex and makes an angle $\beta = \alpha$

It is time to see properties of conic sections from the perspective of Coordinate Geometry and as it progresses certain new terminologies would be added. It has a One or Two fixed point and corresponding One or Two fixed lines such that for any point on curve ratio of distance from fixed point and the line is fixed and is called ***eccentricity e***; value of the ***e*** decides nature of curve. For convenience of readers, naming convention of different points and lines has been adopted, to the possible extent, from the classical book of **Elements of Coordinate Geometry by S.L. Loney**. In this section simple concepts of conics and their characteristic equations are being elaborated. Equation of conics with having origin away from the pole and axes inclined to reference axes shall be dealt with in detail in an exclusive Chapter on **Coordinate Geometry** in **Mathematics Section** of **Mentors' Manual**.

Figures for elaboration of conic sections have been created using Graphing Calculator tool available online URL http://www.meta-calculator.com/online/) and using them figures, to the scale, for each conic section has been developed.

Circle: Circle is the *simplest conic section, a closed curve symmetrical about both X and Y axes, having a* ***fixed distance*** *of every point (x, y) on the curve from a fixed point called Centre; this centre is also called Focus of the circle.*



Taking a point H on the curve and forming $\perp\angle\Delta$OHM, by drawing $\perp$s on the two axes, by Pythagoras Theorem $\mathrm{OH}^2 = \mathrm{OR}^2 + \mathrm{OM}^2$; substituting in this equality values if these line segments it is $r^2 = x^2 + y^2$, where r is called ***Radius*** of the circle while $\boldsymbol{x}$ and $\boldsymbol{y}$ are coordinates of any point on the circle. This equation is called **Characteristic Equation of Circle**. Circle has its centre O as focus. In conic section it is difficult to define Directrix, which by mathematical interpretation is at stated to be at infinity. This interpretation requires defining eccentricity of ellipse having Major and Minor Axis, and accordingly for circle is deferred for a while until Ellipse is discussed.

Nevertheless, understanding of tangent and normal at a point on circle is relevant, which will have extensive applications; let the point be P(x_1,y_1). Since, every point on circle has a constant distance from the centre a line is drawn passing through the point and intersection the circle at Q(x_2,y_2).

Slope of the line is $m = \frac{y_2 - y_1}{x_2 - x_1}$. And equation of line AN passing through Points P and Q is –

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x + c \text{ ; and } y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_1 + c\text{; or } c = y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_1 = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$$

Since, these points are on the circle they would also satisfy equation of the circle, accordingly-

$$x_1{}^2 + y_1{}^2 = r^2 \text{ , and } x_2{}^2 + y_2{}^2 = r^2\text{; or } x_2{}^2 - x_1{}^2 + y_2{}^2 - y_1{}^2 = 0\text{; } x_2{}^2 - x_1{}^2 = -(y_2{}^2 - y_1{}^2)$$
$$\text{or,}(x_2 - x_1)(x_2 + x_1) = -(y_2 - y_1)(y_2 + y_1),$$

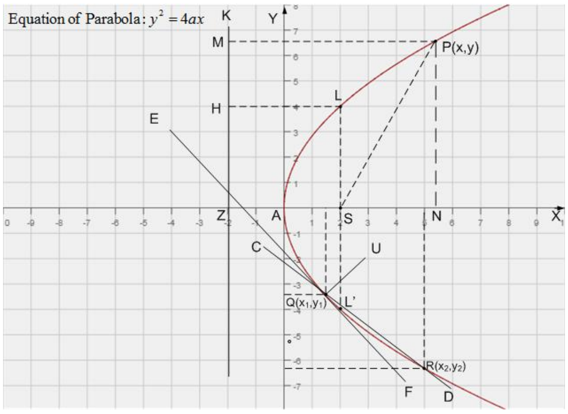
Hence, slope as derived above, turns out to be: $m = -\frac{(x_2 + x_1)}{(y_2 + y_1)}$. Here, definition of ***tangent*** of a curve at any point becomes pertinent, it is a line which touches the curve only at the point under consideration. Accordingly, $x_1 = x_2$ and $y_1 = y_2$, therefore $m = -\frac{2x_1}{2y_1} = -\frac{x_1}{y_1}$. Accordingly, equation of tangent would be $y - y_1 = \left(-\frac{x_1}{y_i}\right)(x - x_1)$. It leads to

$y \cdot y_1 - y_1^{\,2} = -x \cdot x_1 + x_1^{\,2} \rightarrow x \cdot x_1 + y \cdot y_1 = x_1^{\,2} + y_1^{\,2} = r^2$. Thus, **general equation of tangent to a circle** at any point (x_1, y_1) is $x \cdot x_1 + y \cdot y_1 = r^2$.

Further, slope m' of a normal to a line $m' = -\frac{1}{m}$, and therefore slope of normal at any point on the circle is $m' = -\frac{1}{m} = \frac{y_1}{x_1}$. Accordingly, equation of normal to a circle at any point (x_1, y_1) is $y - y_1 = m'(x - x_1) \rightarrow y - y_1 = \frac{1}{\frac{x_1}{y_1}}(x - x_1)$. It leads $y - y_1 = \frac{y_1}{x_1}(x - x_1) \rightarrow y \cdot x_1 - y_1 \cdot x_1 = y_1 \cdot x - y_1 \cdot x_1 \rightarrow y_1 \cdot x - y \cdot x_1 = 0$ Thus **generic equation of normal to a circle at any point** (x_1, y_1) is $y_1 \cdot x - y \cdot x_1 = 0$.

Accordingly, *a radial in a circle is always perpendicular, also called orthogonal, to the tangent at that point*, and is in consistence with geometrical inference, discussed earlier in the chapter-II

Parabola: This conic section, is a curve open on one side, but symmetrical about its axis. Its has an eccentricity $e = 1$, i.e. ratios of distances of every point on the curve from fixed point Focus S (PS) and fixed line Directrix L (LM) is equal to one, or PS=PM. Equation of Parabola is stated to be $y^2 = 4ax$. Here, a is the distance of the focus (AS) and directrix (AZ) from vertex A (a,0), or AS=AZ=a.



Deriving the equation of parabola from these parameters, by definition PM , the distance from directrix $(x + a)$, is equal to PS, the distance from focus $\sqrt{(= y^2 + (x - a)^2)}$.

Solving this equality, with the help of $\perp\angle\Delta$SPN, leads to –

$$(x + a)^2 = y^2 + (x - a)^2$$

$$\text{or, } x^2 + a^2 + 2ax = y^2 + x^2 + a^2 - 2ax$$

Accordingly, $y^2 = 4ax$ is the characteristic equation of the curve Parabola.

Further, drawing a perpendicular to X-axis at Focus S, it intersects Parabola at points L and L'. Since, it is symmetrical about X-axis, LL'=2SL and the line segment LL' is called ***Latus Rectum***. As per the definition of parabola,$\text{LS} = \text{LM} = y = \text{SZ}$. While, distance of vertex A (0 ,0) from focus and directrix $\text{AS} = \text{ZA} = a = x$ and from characteristic equation, it leads to: $y^2 = 4a \times a = 4a^2$, or, $y = 2a = \text{SZ} = \text{ZA} + \text{AS} = \text{ZA} + a$; or $\text{ZA} = a$

Thus, distance vertex (or pole) of the parabola is from Focus and Directrix is equal to a and it is its characteristic parameter.

It is the time to consider equation of tangent and normal at any point on parabola, on the lines similar to that done for circle. Accordingly, equation of line CD passing through points $\text{Q}(x_1, y_1)$ and $\text{R}(x_2, y_2)$ is, $y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x + c$. Points Q and R satisfy characteristic equation of parabola and hence, $y_1^2 = 4ax_1$and $y_2^2 = 4ax_2$. Therefore, $y_2^2 - y_1^2 = 4a(x_2 - x_1)$, and hence,$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1}$.

Since, tangent touches curve at only one point and, therefore, $y_1 = y_2$ and hence, $= \frac{4a}{2y_1} = \frac{2a}{y_1}$. Thus, equation of tangent, line EF becomes: $y = \left(\frac{2a}{y_1}\right)x + c$. Value of c for tangent at point $\text{Q}(x_1, y_1)$ comes to $c = y_1 - \left(\frac{2a}{y_1}\right)x_1 = \frac{y_1^2 - 2ax_1}{y_1} = \frac{4ax_1 - 2ax_1}{y_1} = \frac{2ax_1}{y_1}$. Accordingly, general equation of the tangent parabola is $y = \left(\frac{2a}{y_1}\right)x + \frac{2ax_1}{y_1} \rightarrow y \cdot y_1 = 2a(x + x_1)$.

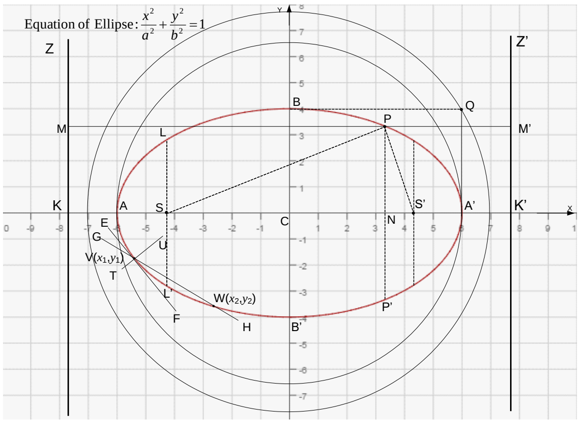
Since slope of normal at the point is $m' = -\frac{y_1}{2a}$, its equation would be –

$y = \left(-\frac{y_1}{2a}\right)x + c$; and value of c for the normal is $c = y_1 + \left(\frac{y_1}{2a}\right)x_1 = \left(\frac{y_1}{2a}\right)(2a + x_1)$. Accordingly, equation of the normal, line QU, is $y = \left(-\frac{y_1}{2a}\right)x + \left(\frac{y_1}{2a}\right)(2a + x_1) \rightarrow y - y_1 = \left(\frac{y_1}{2a}\right)(x_1 - x)$.Thus, **general equation of normal to a parabola at any point** (x_1, y_1) on it is $y - y_1 = -\left(\frac{y_1}{2a}\right)(x - x_1)$

ELLIPSE: It is a conic section in the form of a closed curve which is symmetrical about two perpendicular axes; longer axis is called Major Axis and smaller one is called Minor Axis, as shown in the Figure below. The major axis has two Focus symmetrical about centre of the Ellipse, while Two Directrix are parallel to Minor Axis; both are placed symmetrical about Centre C of the ellipse. Ellipse has eccentricity ($\boldsymbol{e}$) i.e. ratio of distance of any point on the ellipse from the Focus and directrix on the same side of Centre is less than One ($\boldsymbol{e}$<1).

Accordingly, taking a point P on the conic $\frac{\mathrm{PS}}{\mathrm{PM}} = \frac{\mathrm{PS}'}{\mathrm{PM}'} = e$. Likewise, in respect of distances of the vertices, $\frac{\mathrm{SA}}{\mathrm{AK}} = \frac{\mathrm{SA}'}{\mathrm{KA}'} = e$; $\mathrm{SA} = e \cdot \mathrm{AK}$ and $\mathrm{SA}' = e \cdot \mathrm{AK}'$. Adding these two identities, and using property of symmetry of ellipse about its axes it leads to $\mathrm{SA} + \mathrm{SA}' = 2a = e(\mathrm{AK} + \mathrm{AK}') = e(\mathrm{KK}') = 2e \cdot \mathrm{CK}$; hence, $a = e.\mathrm{CK}$.

Likewise, subtracting the two identities, it leads to $\mathrm{SA} - \mathrm{SA}' = \mathrm{SS}' = 2\mathrm{CS} = e(\mathrm{AK} - \mathrm{AK}') = e(\mathrm{AA}') = 2e \cdot \mathrm{CA} = 2e \cdot a$; hence $\mathrm{CS} = ae$. Linear eccentricity is the distance between Centre of the conic and focus i.e. CS.



Further, CA=CA'=a and CB=CB'=b. Using the identities in respect of point P on ellipse –

$\mathrm{SP} = e \cdot \mathrm{PM} = e \cdot (\mathrm{NC} + \mathrm{CK}) = e \cdot \left(\frac{\mathrm{a}}{\mathrm{e}} + \mathrm{CN}\right) = e \cdot \left(\frac{\mathrm{a}}{\mathrm{e}} + \mathrm{x}\right) = a + ex$; and,

$\mathrm{S}'\mathrm{P} = e \cdot \mathrm{PM}' = e \cdot (\mathrm{CK}' - \mathrm{CN}) = e \cdot \left(\frac{\mathrm{a}}{\mathrm{e}} - \mathrm{CN}\right) = e \cdot \left(\frac{\mathrm{a}}{\mathrm{e}} - \mathrm{x}\right) = a - ex$

Adding these two equations, $SP + S'P = e \cdot 2 \cdot \frac{\mathrm{a}}{\mathrm{e}} = 2\mathrm{a}$, or *sum of the focal radii of a point on ellipse is constant and equal to length of its major axis*$(= 2a)$ .It is another property if an ellipse.

Now, applying Pythagoras theorem to $\perp\angle\Delta\mathrm{SPN}$, $\mathrm{SP}^2 = \mathrm{SN}^2 + \mathrm{NP}^2 = (\mathrm{SC} + \mathrm{CN})^2 + \mathrm{NP}^2$. Replacing the length with their corresponding coordinate values, it leads to $(a + ex)^2 = (ae + x)^2 + y^2$

or, $y^2 = (a + ex)^2 - (ae + x)^2 = a^2(1 - e^2) + x^2(e^2 - 1)$

or, $\frac{y^2}{a^2(1-e^2)} = (a + ex)^2 - (ae + x)^2 = a^2(1 - e^2) + x^2(e^2 - 1)$,

or $\frac{y^2}{a^2(1-e^2)} = 1 - \frac{x^2}{a^2}$; or, $\frac{y^2}{a^2(1-e^2)} + \frac{x^2}{a^2} = 1$;or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The last form is characteristic equation of an ellipse where, $b^2 = a^2(1 - e^2)$.Thus with the eccentricity and semi-major axis are defined, semi-minor axis can be determined, alternatively with semi- major and semi-minor axis defined, eccentricity $e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$.

Circle is a special case of an Ellipse where both major and minor axes are equal i.e. $a = b$, *which would lead to* $e = 0$, *and therefore directrix of circle would lie at infinity.*

Further, extending the analogy of tangent and normal at point V (x_1,y_1) as done for earlier conics, first a line is considered also passing through another point W(x_2,y_2). Equation of the line is $y - y_1 = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x - x_1)$.

Further, Points V and W shall also satisfy equation of ellipse, and accordingly –

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \text{ ; and } \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1.$$

Subtracting two equations, $\left(\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2}\right) - \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) = 0 \rightarrow \left(\frac{y_2^2}{b^2} - \frac{y_1^2}{b^2}\right) = -\left(\frac{x_2^2}{a^2} - \frac{x_1^2}{a^2}\right) \rightarrow \frac{(y_2-y_1)(y_2+y_1)}{b^2} = -\frac{(x_2-x_1)(x_2+x_1)}{a^2}$.

It resolves into $\frac{(y_2-y_1)}{(x_2-x_1)} = -\left(\frac{b}{a}\right)^2 \frac{(x_2+x_1)}{(y_2+y_1)}$.

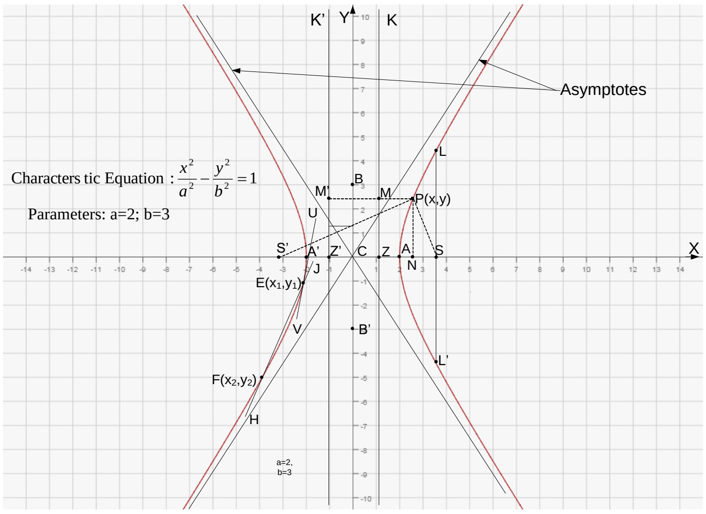
According to definition of tangent EF at a point (x_1, y_1) there is an equivalence $x_1 = x_2$ and $y_1 = y_2$ and hence its equation would be $y - y_1 = -\left(\frac{b}{a}\right)^2 \frac{(2x_1)}{(2y_1)}(x - x_1) = \left(\frac{b}{a}\right)^2 \left(\frac{x_1}{y_1}\right)(x_1 - x)$. Thus general equation of the tangent to an ellipse at any point (x_1, y_1) is $y = -\left(\frac{b}{a}\right)^2 \left(\frac{x_1}{y_1}\right) x + \left(y_1 + \left(\frac{b}{a}\right)^2 \left(\frac{x_1^2}{y_1}\right)\right)$, having slope $m = -\left(\frac{b}{a}\right)^2 \left(\frac{x_1}{y_1}\right)$.

Further, slope of normal TU at the point (x_1, y_1) would be $m' = -\frac{1}{m} = \left(\frac{a}{b}\right)^2 \left(\frac{y_1}{x_1}\right)$; and equation would be –

$$y - y_1 = \left(\frac{a}{b}\right)^2 \left(\frac{y_1}{x_1}\right)(x - x_1); y = \left(\frac{a}{b}\right)^2 \left(\frac{y_1}{x_1}\right)x + \left(y_1 - \left(\frac{a}{b}\right)^2 \left(\frac{y_1}{x_1}\right)x_1\right)$$

In the figure Circle drawn with AA' as diameter is called ***Auxiliary Circle***, while circle drawn with centre same as that of Ellipse and auxiliary circle, but with a radius $r = \sqrt{a^2 + b^2}$ is called ***Director Circle***. More about ellipse shall be elaborated in an exclusive chapter, in Matheatics section of Mentors' manual.

HYPERBOLA: *It is a conic section in the form of a pair of open curves which are symmetrical about two axes but on one axis two foci lie symmetrical about the centre of the conic, while on other axis the one of curves is a mirror image of the other as shown in the figure below. Further, eccentricity (**e**), the ratio of distance of any point on the conic from one of the focus and that from fixed line i.e. Directrix on the same side w.r.t. to centre C is fixed and greater than One (**e**>**1**).*



In the figure C is the Centre of the hyperbola which is symmetrical about the two axes, while S and S' are twin foci, K and K' are twin fixed lines Directrix, and A and A' are twin vertices of the hyperbola. By definition of conic $\mathrm{AS} = e \cdot \mathrm{AZ}$ and $\mathrm{AS}' = e \cdot \mathrm{AZ}'$. Adding these two identities, and symmetrical property of conic $\mathrm{SS}' = \mathrm{AS} + \mathrm{AS}' = e \cdot (\mathrm{AZ} + \mathrm{AZ}') = e \cdot \mathrm{AA}'$; or $\mathrm{SS}' = 2\mathrm{CS} = 2ae$; alternatively $\mathrm{CS} = ae$. And subtracting two equations we get $\mathrm{AS} - \mathrm{AS}' = \mathrm{AA}' = 2e(\mathrm{AZ}' - \mathrm{AZ}) = 2\mathrm{e}(\mathrm{ZZ}')$, or $2a = 2e \cdot \mathrm{CZ}$; alternatively $\mathrm{CZ} = \frac{a}{e}$.

It is the time to determine characteristic equation of hyperbola for which taking a point P(x,y) is taken and forming a $\perp\angle\Delta$SPN by taking foot of a perpendicular from it on X axis and Focus S. Applying Pythagoras Theorem: $\mathrm{PS}^2 = \mathrm{PN}^2 + \mathrm{NS}^2$.

Geometrically, $\mathrm{PS} = e \cdot \mathrm{PM}$, $\mathrm{PM} = \mathrm{ZN} = \mathrm{CN} - \mathrm{CZ} = x - \mathrm{CZ}$; $\mathrm{NS} = \mathrm{CS} - \mathrm{CN}$; $\mathrm{PN} = y$. While, LL' is ***Latus Rectum***. Applying above identities, $\mathrm{PS} = e \cdot \left(x - \frac{a}{e}\right) = ex - a$, and $\mathrm{NS} = x - ae$ Accordingly, $(ex - a)^2 = (x - ae)^2 + y^2$; or $(ex)^2 + a^2 - 2aex = (ae)^2 + x^2 - 2aex + y^2 \rightarrow a^2(1 - e^2) = x^2(1 - e^2) + y^2 \rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} \rightarrow 1 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$; here $b = a\sqrt{e^2 - 1}$.

In final form $1 = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ is called **characteristic equation of hyperbola** having characteristic parameters a, b and e if any of the two parameters are known the third parameter can be determined with an identity of hyperbola $e = \sqrt{\left(\frac{b}{a}\right)^2 + 1}$. While AA' corresponding to parameter a called **transverse axis**, the BB' on Y-axis correspond to parameter b is called **conjugate axis**.

At this point, it is relevant to determine tangent at any point and for that a line is taken to be passing through Two points E(x1,y1) and F(x2,y2) are taken on the conic. Hence, $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$; and $\frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = 1$. Subtracting identity for E(x1,y1) from F(x2,y2) it leads to $\left(\frac{x_2^2}{a^2} - \frac{y_2^2}{b^2}\right) - \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}\right) = 0 \rightarrow \frac{(x_2-x_1)(x_2+x_1)}{a^2} - \frac{(y_2-y_1)(y_2+y_1)}{a^2} = 0 \rightarrow \frac{(x_2-x_1)(x_2+x_1)}{a^2} - \frac{(y_2-y_1)(y_2+y_1)}{a^2} = 0$; or $\frac{(x_2-x_1)(x_2+x_1)}{a^2} - \frac{(y_2-y_1)(y_2+y_1)}{a^2} = 0$; or $m = \left(\frac{y_2-y_1}{x_2-x_1}\right) = \left(\frac{a}{b}\right)^2 \left(\frac{x_2+x_1}{y_2+y_1}\right)$.

Further equation of line passing through the Two points, under consideration, is $y - y_1 = \left(\frac{y_2-y_1}{x_2-x_1}\right)(x - x_1)$. It leads to $y - y_1 = \left(\frac{a}{b}\right)^2 \left(\frac{x_2+x_1}{y_2+y_1}\right)(x - x_1)$.

By definition of tangent points E(x_1,y_1) and F(x_2,y_2) are coincident and hence, $x_1 = x_2$ and $y_1 = y_2$ and hence slope of tangent is $m = \left(\frac{a}{b}\right)^2 \left(\frac{x_1}{y_1}\right)$. Accordingly ***characteristic equation of tangent of a hyperbola at any point*** (x_1,y_1) is $y - y_1 = \left(\frac{a}{b}\right)^2 \left(\frac{x_1}{y_1}\right) x - x_1$

The slope of normal, as proved earlier $m^{'} = -\frac{1}{m} =$ -$\left(\frac{b}{a}\right)^2 \left(\frac{y_1}{x_1}\right)$. And accordingly, ***characteristic equation of normal at Point*** (x_1,y_1) is $y - y_1 = -\left(\frac{b}{a}\right)^2 \left(\frac{y_1}{x_1}\right) x - x_1$.

It is the time to know characteristic equation of Asymptotes which is defined as tangent to the hyperbola at infinity. Take $y = mx + c$ as the equation of asymptote. Substituting value of y from this equation in the equation of the conic we get $\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$; or, $b^2x^2 - a^2(m^2x^2 + c^2 + 2mxc) = a^2b^2$; or, $(b^2 - a^2m^2)x^2 - (2mca^2)x - a^2(b^2 + c^2) = 0$. Since asymptotes meet the conic at infinity i.e. roots of equation should be $x = \infty$. Easier way is to transform equation into $z = \frac{1}{x}$ such that roots of the equation $z = 0$. Accordingly, $a^2(b^2 + c^2)z^2 + (2mca^2)z - (b^2 - a^2m^2) = 0$; and roots of the equation are $z = \frac{-(2mc\,a^2) \pm \sqrt{4m^2c^2a^4 + 4a^2(b^2+c^2)(b^2-a^2m^2)}}{a^2(b^2+c^2)}$.

Essential condition for $z = 0$ is $(2mca^2) = \pm\sqrt{4m^2c^2a^4 + 4a^2(b^2 + c^2)(b^2 - a^2m^2)}$; squaring both the sides it leads to $4m^2c^2a^4 = 4m^2c^2a^4 + 4a^2(b^2 + c^2)(b^2 - a^2m^2) \rightarrow 4a^2(b^2 + c^2)(b^2 - a^2m^2)$. Or; either $b^2 - a^2m^2 = 0$, or $b^2 + c^2 = 0$ or $a = 0$. For hyperbola to exist $a \neq 0$. The second condition is also not true since $b \neq 0$. This concludes that slope of asymptotes is $m = \pm\sqrt{\frac{b}{a}}$. Further, for this condition to exist where $a \neq 0$ or $m \neq 0$, essential requirement is $c = 0$.

Accordingly important characteristics of Asymptotes, distinctly shown in the figure, are –

a. They are pair of tangent to the conic at infinity have –
b. correspondence in diagonally opposite quadrants,
c. The tangents intersect each other at Centre C.
d. These tangents are called Asymptotes, marked explicitly in the figure.

Definitions used in Conic Sections, as they have appeared progressively during elaboration of each are summarily consolidated here –

Focus: It is fixed point, a reference point, from which distance of any point on curve goes in defining a Conic.

Directrix: It is a fixed line, reference line from which distance of any point on curve goes in defining Conic.

Eccentricity: It is the ratio of distance of a point on Conic from Focus and Directrix. It forms the basis of classification of a Conic.

Axis: It is a line passing through Focus and perpendicular to Directrix, and is eventually a fixed line in a Conic, since the reference point and line i.e. Focus and Directrix are also fixed.

Vertex: It is the point of intersection of Conic with the Axis.

Linear eccentricity : It is the distance between the Centre and the Focus (or one of the two foci).

Latus Rectum: It is double of section of chord of a Conic which perpendicular to axis and passes through S. Half Latus Rectum is distance between Focus and point of intersection of the Conic and line perpendicular to the axis drawn on Focus.

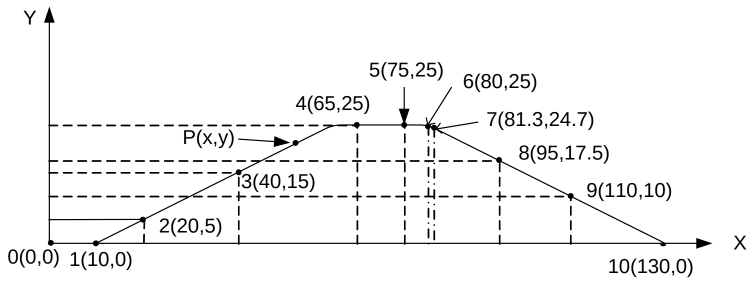
Focal Parameter: It is the distance from the focus (or one of the two foci) to the Directrix.

There are three Basic classifications of Conics. Comparison of parameters that discriminate Conics is shown in the table below.

Particulars	**Ellipse**		**Parabola**	**Hyperbola**
	Ellipse	**Circle**		
Curves	It is a single closed curve	It is a single closed curve, a special case of Ellipse	`It is single open curve	It is a set of two Open curves
Focus	It has Two Foci	Centre of Circle	`It has One Focus	It has Two Foci
Axis	It has Two axes, Major axes passes through its Two Focii, while Minor Axis is perpendicular to Major Axis, at its centre.	It has no axis, however, a line perpendicular to plane of circle at its Centre can be called axis for the purpose of definition.	It has one axis	Iy has two axes
Nature of Curve	Closed	Closed	Open	Open
Characteristic Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$x^2+y^2=r^2$	$y^2=4ax$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Eccentricity (e)	$\sqrt{1-\frac{b^2}{a^2}}$	0	1	$\sqrt{1+\frac{b^2}{a^2}}$
Linear eccentricity (c)	$\sqrt{a^2-b^2}$	0	-	$\sqrt{a^2+b^2}$
Latus Rectum ($2l$)	$\frac{b^2}{a}$	a	4a	$\frac{b^2}{a}$
Focal Parameter	$\frac{b^2}{\sqrt{a^2-b^2}}$	∞	$2a$	$\frac{b^2}{\sqrt{a^2+b^2}}$

In respect of conics their characteristic equations, parameters, tangents and normal at any point on it have been brought out above with the consideration of Coordinate Geometry. Defining tangent and normal, not only for conic rather for any curve, becomes much simpler with the help of differential calculus and is brought out little later in this chapter. Moreover, It finds wide application in physics. Further details of conics, together with its general equation, shall be separately covered in Coordinate Geometry a chapters on Mathematics Section of Mentors' Manual.

<u>**CALCULUS:**</u> Prior to Newton, algebraic analysis of observations was based on finite change in one of the parameter. Newton, studied the changes in observations, in a different manner, based on ***infinitesimal change*** in any parameter (x) such that it is ***very small change and tending to be ZERO, represented as $\Delta x \to 0$ but not equal to Zero***, and thus created a new branch of mathematics ***Calculus***. A beginning of Calculus is being illustrated through a very simple example of slope of a flyover (which represents a curve), encountered in day-to-day life. ***Slope (m)*** in common parlance is defined as ratio of rise in height (Δy) for



a certain distance traced in a horizontal plane (Δx) and is mathematically expressed as $m = \frac{\Delta y}{\Delta x}$. This slope ***(m)***, over a distance range Δx, assumes different value depending the value of Δx. This shall become explicit with the figure, and the corresponding table, shown below -

Observations in the figure above have been brought out in the Table below.

S No.	Range (in terms of points on the curve)	Δx	Δy	Slope at a point $m = \frac{\Delta y}{\Delta x}$	Remarks
1	1-2	10 (=20-10)	5 (=5-0)	0.50	Normal & actual
2	1-3	30 (=40-10)	15 (=15-0)	0.50	
3	1-4	55 (=65-10)	25 (=25-0)	0.45	Error in calculated value, less than actual
4	1-5	65 (=65-10)	25 (=25-0)	0.38	Increase in error in calculated slope
5	1-6	70 (=80-10)	25 (=25-0)	0.35	Further increase in error calculated slope
6	1-7	71.3(81.3-10)	24.7 (=24.7-0)	0.35	Error in calculated slope. Despite actual slope becoming –ve, calculated value is +ve
7	1-8	85 (=95-10)	17.5 (=17.5-0)	0.21	
8	1-9	100 (=110-10)	10 (=10-0)	0.10	
9	1-10	120 (=130-10)	0 (=0-0)	0	
10	2-3	20 (=40-20)	10 (=15-5)	0.50	Same as at S.N. 1 & 2
11	3-4	25 (-65-40)	10 (=25-15)	0.40	Calculated slope reduction is due to change of slope, in-between the points
12	4-5	10 (=75-65)	0 (=25-25)	0	Points are on horizontal lin
13	5-6	5 (=80-75)	0 (=25-25)	0	
14	6-7	1.3 (=81.3-80)	-0.3(=24.7-25)	-0.23	Reflects change of slope to –ve, as actual
15	7-8	13.7 (=95-81.3)	-7.2(=17.5-24.7)	-0.53	Nearly -0.5; observation error is reflected
16	8-9	15 (=110-95)	-7.5 (=10-17.5)	-0.50	Actual-ve slope
17	9-10	20 (=130-110)	-10 (=0-10)	-0.50	
18	3-5	35 (=75 -40)	10 (=25-15)	0.29	Effect of change of slope is reduced due to increase in Δx

The table above shows that calculated value of slope, as defined above $\left(m = \frac{\Delta y}{\Delta x}\right)$ depends upon the value of Δx,whenever there is change in slope in between points under consideration. Therefore, smaller the value of Δx, lesser is the error in calculated value of the slope in the region. Thus, calculated value is tending to be more accurate and following inferences are drawn -

1. Slope is prospective change at any point and mathematically $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$, i.e. $x_2 > x_1$, and thus it is forward trend.
2. Larger the value of Δx more is the error in calculated value of slope and thus actual change in slope, in the interval under consideration, is suppressed.
3. Accurate determination of slope at a point is possible when $\boldsymbol{\Delta x \to 0}$ ***but*** $\boldsymbol{\Delta x \neq 0}$; because the moment $\boldsymbol{\Delta x = 0}$, mathematically slope becomes ∞, an indeterminate quantity, and is not actual.
4. This gives rise to ***concept of limit***, an important consideration in mathematical analysis. Details of limit and continuity shall be dealt with in a separate chapter on mathematics.

Here, it is pertinent to describe function, dependent variable and independent variable. In the instant case trace of a curve on a plane is defined in terms of variable x. Accordingly, ordinate of any point on the curve (y) is determined with the given characteristic equation of the curve using corresponding value of x. Little before characteristic equations for various conics have been defined, taking a case of circle its equation $x^2 + y^2 = r^2$ can be written as $y = \pm\sqrt{r^2 - x^2}$. This in mathematical language is called as *y is a function of x such that* $y = \pm\sqrt{r^2 - x^2}$ *; where y is a dependent variable and x is an independent variable.*

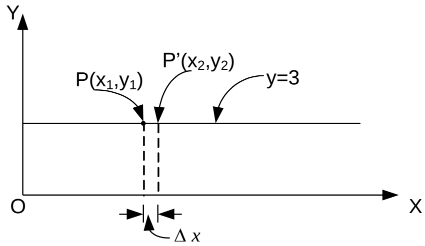
Accordingly, *differential coefficient of a function, in language of mathematics is* $\frac{dy}{dx} = \frac{\Delta y}{\Delta x}_{|\Delta x \to 0} = \frac{y_2 - y_1}{x_2 - x_1}_{|\Delta x \to 0}$.

This definition of differential coefficient can be applied to any function with one independent variable. Elaboration and application of the concept is done with the following cases-

Differentiation of a constant (y=3) as per definition, called First Principle, is -

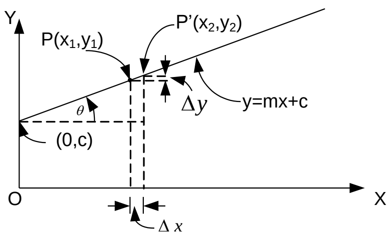
$$\frac{dy}{dx} = \frac{d}{dx}c = \frac{\Delta y}{\Delta x}_{|\Delta x \to 0} = \frac{(y_2 - y_1)}{(x_2 - x_1)}_{|\Delta x \to 0} = \frac{((y_1 + \Delta y) - y_1)}{((x_1 + \Delta x) - x_1)}_{|\Delta x \to 0} = \frac{(\Delta y)}{(\Delta x)}_{|\Delta x \to 0}$$

$$= \frac{0}{(\Delta x)}_{|\Delta x \to 0} = 0$$



With the given nature of function y, Δy=0 for any value of Δx and, therefore, $\frac{d}{dx}c = 0$..

Taking another case of inclined straight line $y = mx + c$, where m and c are constants and, therefore, to determine $\boldsymbol{\frac{dy}{dx}}$;



$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x}_{|\Delta x \to 0} = \frac{(y_2 - y_1)}{(x_2 - x_1)}_{|\Delta x \to 0} \frac{((mx_2 + c) - (mx_1 + c))}{((x_1 + \Delta x) - x_1)}_{|\Delta x \to 0}$$

$$= \frac{(mx_2 - mx_1)}{(\Delta x)}_{|\Delta x \to 0} = \frac{(m(x_1 + \Delta x) - mx_1)}{\Delta x}_{|\Delta x \to 0}$$

It leads to $\frac{dy}{dx} = \frac{m\,\Delta x}{\Delta x}_{|\Delta x \to 0} = m$

This leads to Four logical conclusions **(a)**$\boldsymbol{\frac{d}{dx}x = 1}$, **(b)**$\boldsymbol{\frac{d(mx)}{dx} = m\frac{d(x)}{dx} = m}$, **(c)**$\boldsymbol{\frac{d}{dx}(mx + c) = m}$, *and* (d)$\frac{d}{dx}y = m = tan\theta$, ***slope of the function at the infinitesimal triangle. It is also called slope of tangent at any point of a curve.***

Determine$\boldsymbol{\frac{dy}{dx} = \frac{d}{dx}x^n}$: requiresuse of Binomial Theorem and is as under –

$$\frac{d}{dx}x^n = \frac{(x+\Delta x)^n - x^n}{\Delta x}_{|\Delta x \to 0} = \frac{x^n\left(1+\frac{\Delta x}{x}\right)^n - x^n}{\Delta x}_{|\Delta x \to 0} = \frac{x^n\left(\left(1+\frac{\Delta x}{x}\right)^n - 1\right)}{\Delta x}_{|\Delta x \to 0} = x^n\frac{\left(\left(1+n\frac{\Delta x}{x}+\frac{n(n-1)}{2}\left(\frac{\Delta x}{x}\right)^2 \ldots\right)-1\right)}{\Delta x}_{|\Delta x \to 0}$$

$$= x^n\frac{\left(\left(1+n\frac{\Delta x}{x}+\frac{n(n-1)}{2}\left(\frac{\Delta x}{x}\right)^2 \ldots\right)-1\right)}{\Delta x}_{|\Delta x \to 0} = x^n\frac{\left(n\frac{\Delta x}{x}+\frac{n(n-1)}{2}\left(\frac{\Delta x}{x}\right)^2 \ldots\right)}{\Delta x}_{|\Delta x \to 0} = x^n\left(\frac{n}{x}+\frac{n(n-1)}{2}\frac{\Delta x}{x^2}\ldots..\right)_{|\Delta x \to 0} = \frac{n}{x}x^n = \boldsymbol{nx^{n-1}}$$

Differential coefficient of trigonometric function $\boldsymbol{sin\,\theta}$ ***and*** $\boldsymbol{cos\,\theta}$ requires use of trigonometric identities, already brought out earlier in this chapter, and is as under -

$$\frac{d\sin\theta}{d\theta} = \left.\frac{\sin(\theta+\Delta\theta)-\sin\theta}{\Delta\theta}\right|_{\Delta\theta\to0}$$
$$= \left.\frac{\sin\theta\cos\Delta\theta+\cos\theta\sin\Delta\theta-\sin\theta}{\Delta\theta}\right|_{\Delta\theta\to0}$$

Using the limit,

$$\frac{d\sin\theta}{d\theta} = \left.\frac{\sin\theta+\cos\theta\sin\Delta\theta-\sin\theta}{\Delta\theta}\right|_{\Delta\theta\to0}$$
$$= \left.\frac{\cos\theta\sin\Delta\theta}{\Delta\theta}\right|_{\Delta\theta\to0} = \cos\theta\left.\frac{\sin\Delta\theta}{\Delta\theta}\right|_{\Delta\theta\to0} = \boldsymbol{\cos\theta}$$

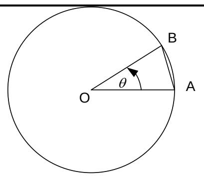
$$\frac{d\cos\theta}{d\theta} = \left.\frac{\cos(\theta+\Delta\theta)-\cos\theta}{\Delta\theta}\right|_{\Delta\theta\to0}$$
$$= \left.\frac{\cos\theta\cos\Delta\theta-\sin\theta\sin\Delta\theta-\cos\theta}{\Delta\theta}\right|_{\Delta\theta\to0}$$

Using the limit,

$$\frac{d\cos\theta}{d\theta} = \left.\frac{\cos\theta-\sin\theta\sin\Delta\theta-\cos\theta}{\Delta\theta}\right|_{\Delta\theta\to0}$$
$$= -\left.\frac{\sin\theta\sin\Delta\theta}{\Delta\theta}\right|_{\Delta\theta\to0} = -\sin\theta\left.\frac{\sin\Delta\theta}{\Delta\theta}\right|_{\Delta\theta\to0} = -\boldsymbol{sin\theta}$$

Here, it uses an important derivation of limit $\left.\frac{\sin\theta}{\theta}\right|_{\theta\to0} = 1$ which is possible both geometrically and using trigonometric series, where $\boldsymbol{\theta}$ ***is in radians***. It is as under -

Geometric Derivation



In this figure length of arcAB = OA·θ, a geometrical inference. When, θ is infinitesimal -

In this figure length of arcAB = OA·θ, a geometrical inference. When, θ is infinitesimal –

1. Chord AB tends to coincide the arc AB,
2. Sum of other two ∠s of the ΔABC tends to be 180^0, hence $\angle OAB \to 90^0$.
3. $\sin\theta|_{\theta\to0} = \frac{\text{AB}}{\text{OA}} = \frac{\text{OA}\cdot\theta}{\text{OA}} = \theta$
4. $\left.\frac{\boldsymbol{\sin\theta}}{\boldsymbol{\theta}}\right|_{\boldsymbol{\theta\to0}} = \frac{\theta}{\theta} = \mathbf{1}$

Trigonometric Series Based Derivation

$$\left.\frac{\sin\theta}{\theta}\right|_{\theta\to0} = \left.\frac{\theta-\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}-\frac{\theta^7}{\lfloor 7}\ldots}{\theta}\right|_{\theta\to0}$$
$$=> 1-\frac{\theta^2}{\lfloor 3}+\frac{\theta^4}{\lfloor 5}-\frac{\theta^6}{\lfloor 7}\ldots.\Big|_{\theta\to0}$$

Since θ is infinitesimal all terms containing θandwith higher degree would tend to be Zero, hence-

$$\left.\frac{\boldsymbol{\sin\theta}}{\boldsymbol{\theta}}\right|_{\boldsymbol{\theta\to0}} = \mathbf{1}$$

Determination of $\boldsymbol{\frac{d}{dx}e^x}$ requires expansion of exponential series and it is - $e^x = 1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\cdots\left.\frac{x^r}{\lfloor r}\right|_{r\to\infty}$. From first principle,

$$\frac{d}{dx}e^x = \left.\frac{e^{x+\Delta x}-e^x}{\Delta x}\right|_{\Delta x\to0} = e^x\left.\frac{\left(1+\Delta x+\frac{(\Delta x)^2}{\lfloor 2}+\frac{(\Delta x)^3}{\lfloor 3}+\frac{(\Delta x)^4}{\lfloor 4}+\cdots\right)-1}{\Delta x}\right|_{\Delta x\to0}$$

$$= e^x\left.\frac{\Delta x+\frac{(\Delta x)^2}{\lfloor 2}+\frac{(\Delta x)^3}{\lfloor 3}+\frac{(\Delta x)^4}{\lfloor 4}+\cdots}{\Delta x}\right|_{\Delta x\to0} = e^x\left(1+\frac{\Delta x}{\lfloor 2}+\frac{(\Delta x)^2}{\lfloor 3}+\frac{(\Delta x)^3}{\lfloor 4}+\cdots\right)\Big|_{\Delta x\to0}$$

Since, $\Delta x\to0$ and hence all terms in the above expression contain Δx and its higher order would be Zero and hence - $\boldsymbol{\frac{d}{dx}e^x = e^x}$

Determination of $\frac{\mathbf{d}}{\mathbf{dx}}\boldsymbol{\log_e x}$ involves use of logarithmic series $\left(\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\ldots\right)$

$$\frac{\mathrm{d}}{\mathrm{dx}}\log_e x = \left.\frac{\log_e(x+\Delta x)-\log_e x}{\Delta x}\right|_{\Delta x\to0} = \left.\frac{\log_e\left(\frac{x+\Delta x}{x}\right)}{\Delta x}\right|_{\Delta x\to0} = \left.\frac{\log_e\left(1+\frac{\Delta x}{x}\right)}{\Delta x}\right|_{\Delta x\to0}$$

$$= \frac{\left(\frac{\Delta x}{x}\right) - \frac{\left(\frac{\Delta x}{x}\right)^2}{2} + \frac{\left(\frac{\Delta x}{x}\right)^3}{3} - \frac{\left(\frac{\Delta x}{x}\right)^4}{4} \dots}{\Delta x}\Bigg|_{\Delta x \to 0} \quad = \left(\frac{1}{x}\right) - \frac{\frac{\Delta x}{x^2}}{2} + \frac{\frac{(\Delta x)^2}{x^3}}{3} - \frac{\frac{(\Delta x)^3}{x^4}}{4} \dots \Bigg|_{\Delta x \to 0}$$

Imposing the limit $\Delta x \to 0$ all terms with higher degree of Δx would become Zero and hence the expansion reduces to -

$$\frac{\mathbf{d}}{\mathbf{dx}} \boldsymbol{\log_e x} = \frac{\mathbf{1}}{\boldsymbol{x}}$$

The above two cases are interesting applications of limits.

Differential Coefficient of $\log_a x$ requires use of logarithmic derivations, brought out earlier in section on Logarithm , and is as under-

$\log_a x = \log_e x \cdot \log_a e$, and $\log_e a \cdot \log_a e = 1$, hence, $\log_a e = \frac{1}{\log_e a \cdot}$ and $\log_a x = \frac{1}{\log_e a \cdot} \log_e x \cdot$

It has been seen earlier that $\frac{d}{dx}\left(Cf(x)\right) = C\frac{d}{dx}f(x)$. In the instant case $\boldsymbol{a}$is a constant and hence $\boldsymbol{\log_e a}$ is also a constant. Accordingly, $\frac{d}{dx}\log_a x = \frac{d}{dx}\left(\frac{1}{\log_e a \cdot}\log_e x\right) = \left(\frac{1}{\log_e a \cdot}\right)\frac{d}{dx}(\log_e x) = \left(\frac{1}{\log_e a \cdot}\right) \cdot \frac{1}{x} = \frac{1}{x} \cdot \log_a e$

Now is the time to determine differential coefficient of sum, product and quotient of two functions, because in real life problems a function in isolation rarely occurs. It is, therefore, analysed below –

Differential Coefficient of Sum of Functions:

Let, $f(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$

$$\frac{d}{dx} f(x) = \frac{d}{dx}(f_1(x) + f_2(x) + f_3(x) + \cdots)$$

$$= \frac{(f_1(x + \Delta x) + f_2(x + \Delta x) + f_3(x + \Delta x) + \cdots) - (f_1(x) + f_2(x) + f_3(x) + \cdots)}{\Delta x}\Bigg|_{\Delta x \to 0}$$

$$=> \frac{(f_1(x + \Delta x) + f_2(x + \Delta x) + f_3(x + \Delta x) + \cdots) - (f_1(x) + f_2(x) + f_3(x) + \cdots)}{\Delta x}\Bigg|_{\Delta x \to 0}$$

$$=> \frac{f_1(x + \Delta x) - f_1(x)}{\Delta x}\Bigg|_{\Delta x \to 0} + \frac{f_1(x + \Delta x) - f_1(x)}{\Delta x}\Bigg|_{\Delta x \to 0} + \frac{f_1(x + \Delta x) - f_1(x)}{\Delta x}\Bigg|_{\Delta x \to 0} \dots$$

$$\frac{d}{dx}(f_1(x) + f_2(x) + f_3(x) \dots) = \frac{d}{dx}f_1(x) + \frac{d}{dx}f_3(x) + \frac{d}{dx}f_3(x) + \cdots$$

Differential Coefficient of Product of Two Functions:

Let, $f(x) = f_1(x) \cdot f_2(x)$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(f_1(x).f_2(x)\right) = \frac{f_1(x + \Delta x).f_2(x + \Delta x) - f_1(x).f_2(x)}{\Delta x}\Bigg|_{\Delta x \to 0}$$

$$=> \frac{f_1(x + \Delta x).f_2(x + \Delta x) - \left(f_1(x).f_2(x + \Delta x) - f_1(x).f_2(x + \Delta x)\right) - f_1(x).f_2(x)}{\Delta x}\Bigg|_{\Delta x \to 0}$$

$$=> \frac{f_1(x + \Delta x).f_2(x + \Delta x) - f_1(x).f_2(x + \Delta x)}{\Delta x} + \frac{f_1(x).f_2(x + \Delta x) - f_1(x).f_2(x)}{\Delta x}\Bigg|_{\Delta x \to 0}$$

$$=> \left(\frac{f_1(x + \Delta x) - f_1(x)}{\Delta x}\right).f_2(x + \Delta x)\Bigg|_{\Delta x \to 0} + f_1(x) \cdot \left(\frac{f_2(x + \Delta x) - f_2(x)}{\Delta x}\right)\Bigg|_{\Delta x \to 0}$$

Imposing limit $\Delta x \to 0$, in the above expression it reduces to

$$\boldsymbol{\frac{d}{dx}\left(f_1(x)\cdot f_2(x)\right) = \left(\frac{d}{dx}f_1(x)\right).f_2(x) + f_1(x)\cdot\left(\frac{d}{dx}f_2(x)\right)}$$

Differential Coefficient of Quotient of Two Functions:

Let, $f(x) = \frac{f_1(x)}{f_2(x)}$

$$\frac{d}{dx}f(x) = \left.\frac{\frac{f_1(x+\Delta x)}{f_2(x\Delta x)} - \frac{f_1(x)}{f_2(x)}}{\Delta x}\right|_{\Delta x\to 0} = \left.\frac{\frac{f_1(x+\Delta x)\cdot f_2(x) - f_1(x)\cdot f_2(x\Delta x)}{f_2(x+\Delta x)\cdot f_2(x)}}{\Delta x}\right|_{\Delta x\to 0}$$

$$=> \left.\frac{\frac{f_1(x+\Delta x)\cdot f_2(x) - \left(f_1(x)\cdot f_2(x) - f_1(x)\cdot f_2(x)\right) - f_1(x)\cdot f_2(x\Delta x)}{f_2(x+\Delta x)\cdot f_2(x)}}{\Delta x}\right|_{\Delta x\to 0}$$

$$=> \frac{f_2(x)\cdot\left.\left(\frac{f_1(x+\Delta x)-f_1(x)}{\Delta x}\right)\right|_{\Delta x\to 0} - f_1(x)\cdot\left.\frac{f_2(x+\Delta x)-f_2(x)}{\Delta x}\right|_{\Delta x\to 0}}{f_2(x+\Delta x)\cdot f_2(x)|_{\Delta x\to 0}}$$

Imposing the limits the above expression reduces to -

$$\boldsymbol{\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{f_2(x)\cdot\frac{d}{dx}\left(f_1(x)\right) - f_1(x)\cdot\frac{d}{dx}\left(f_2(x)\right)}{\left(f_2(x)\right)^2}}$$

Likewise, differentiations of special cases which are generally encountered in mathematical analysis are –

Differentiation of function of a function:

Let, $f(x) = f_1\left(f_2(x)\right) = f_1(t)$; here $t = f_2(x)$, and $t + \Delta t = f_2(x + \Delta x)$

$$\frac{d}{dx}f(x) = \left.\frac{f(t+\Delta t) - f(t)}{\Delta x}\right|_{\Delta x\to 0} = \left.\frac{f(t+\Delta t) - f(t)}{\Delta t}\cdot\frac{\Delta t}{\Delta x}\right|_{\Delta x\to 0}$$

$$=> \left(\left.\frac{f(t+\Delta t) - f(t)}{\Delta t}\right|_{\Delta t\to 0}\right)\left(\left.\frac{f_2(x+\Delta x) - f_2(x)}{\Delta x}\right|_{\Delta x\to 0}\right)$$

$$\boldsymbol{\frac{d}{dx}f(x) = \frac{d}{dx}f_1(t) = \frac{d}{dt}f_1(t)\cdot\frac{d}{dx}t = \frac{d}{dt}f_1(t)\cdot\frac{d}{dx}f_2(x)}$$

The expression together with final result shows intermediate forms also, for the convenience of correlation should there be any intermediate form encountered by the reader in handling different problems of physics or mathematics.

Differentiation of Parametric Function: There are situations when both $\boldsymbol{y}$ and x are expressed in terms of a third parameter say $\boldsymbol{t}$, say $\boldsymbol{y = f_1(t)}$ and $x = \boldsymbol{f_2(t)}$

$$\frac{dy}{dx} = \frac{dy}{dx}\cdot\frac{dt}{dt} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

Differentiation of an Implicit Function: It is a case of an equation involving of two variables and can be best illustrated with an example, say $x^4 + y^3 + xy^2 + c = 0$, then to determine $\frac{dy}{dx}$, the equation id differentiated as sum of functions and the result is –

$$4x^3 + 2\,y^2\frac{dy}{dx} + \left(x \cdot 2y\frac{dy}{dx} + y^2\right) + 0 = 0$$

$$2y(y + x)\frac{dy}{dx} = -(y^2 + 4x^3)$$

$$\therefore\ \boldsymbol{\frac{dy}{dx} = -\frac{(y^2 + 4x^3)}{2y(y + x)}}$$

Infinite Series: Let there be an infinite series of the kind $y = x^{x^{x\,\ldots infinity}}$.
In this case say, $y = x^y$; then $y = \log_x y = \log_e y \cdot \log_x e = \frac{\log_e y}{\log_e x}$, or $y \log x = \log y$
Differentiating both sides of the equation w.r.t. x , it leads to -
$\frac{dy}{dx} \cdot \log x + y \cdot \frac{1}{x} = \frac{1}{y} \cdot \frac{dy}{dx}$; or $\left(\frac{1}{y} - \log x\right) \cdot \frac{dy}{dx} = \frac{y}{x}$; or $\frac{dy}{dx} = \frac{y^2}{x(1-y\log x)}$

These are basic derivations of differentiations, which are used to solve different problems involving differential calculus. Since, *the concept involves ratio of difference in dependent variable for an infinitesimal change in the independent variable, to analyze a process or phenomenon, it is called **differential calculus***. Some standard results of differentiation are brought out in the table below.

Standard Results of Differentiation

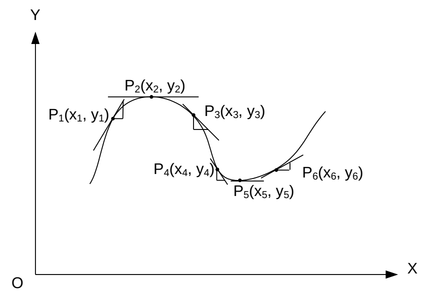
$\frac{d}{dx}x^n = nx^{n-1}$	$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$
$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$
$\frac{d}{dx}\log_e x = \frac{1}{x}$	$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}$
$\frac{d}{dx}\log_a x = \left(\frac{1}{\log_e a\,\cdot}\right) \cdot \frac{1}{x} = \frac{1}{x} \cdot \log_a e$	$\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2}$
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1 + x^2}$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\operatorname{cosec}^{-1} x = -\frac{1}{x\sqrt{x^2 - 1}}$
$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$	$\frac{d}{dx}\sinh x = \cosh x$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\cosh x = \sinh x$

Applications of Differential Calculus: *Determination of rate of change or acceleration of a process, which in kinematics is equivalent to velocity and acceleration, and maximum* (*Maxima*) *and minimum* (*Minima*) *value of a function* are some of the important application of differential calculus. ***Rate of change*** of a process is the $\frac{dy}{dx} = \dot{y}$ or slope of the process function y, or single derivative, w.r.t. its independent variable x. While, ***acceleration of the process*** is the $\frac{d^2y}{dx^2} = \ddot{y}$, rate of change (slope) of slope of y i.e. $\frac{dy}{dx}$, or double derivative w.r.t. its independent variable x. *Like this depending on the actual change which is being considered there can be derivative of any order say* ***nth Order Derivative*** *and is expressed as* $\frac{d^n y}{dx^n}$.

As equation of tangent and normal at any point (x_1, y_1) of the curve, first step is to determine slope of tangent $m = \frac{d}{dx}$y, and that of normal is $m' = -\frac{1}{m}$ where ***y*** is characteristic equation of any curve, including conic sections covered in section on Coordinate Geometry, Next step is direct equation of tangent [$y - y' = m(x - x')$] and the equation of normal is $[y - y' = m'(x - x')]$. It is that simple.

Maxima and Minima: In the figure below a simple curve with one crest and one valley is shown taking six points P_1, P_2...P_6, slope at each point, on the figure above, are summarized as under-

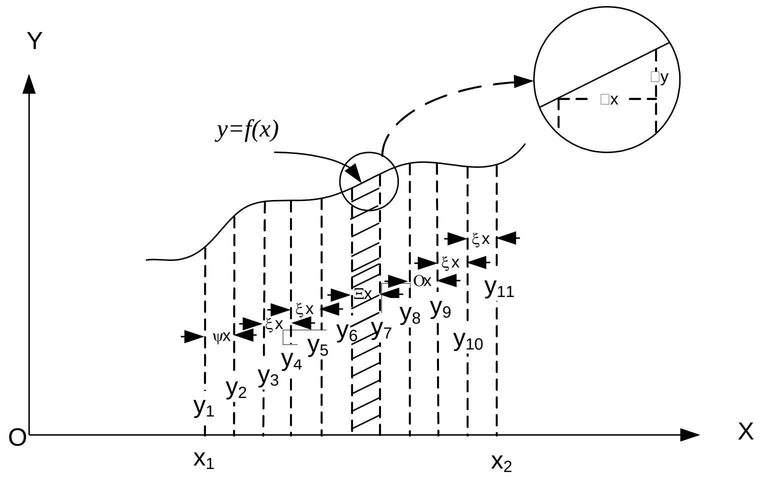
Point	Δx	Δy	Slope (m) $=\frac{\Delta y}{\Delta x} = \dot{y}$	Change of Slope (Δm)	Rate of Change of Slope ($\dot{m}$) $=\frac{\Delta m}{\Delta x} = \ddot{y}$	Conclusion
1	+ve	+ve	+ve	-ve	N/A**	N/A
2	+ve	Zero	**Zero**	-ve	**-ve**	**Maxima**
3	+ve	-ve	-ve	-ve	N/A**	**N/A**
4	+ve	-ve	-ve	+ve	N/A**	N/A
5	+ve	Zero	**Zero**	+ve	**+ve**	**Minima**
6	+ve	+ve	_+ve	N/A*	NA**	N/A



Note: * Since differentiation is prospective and at this point no prospective value of slope is tabulated and hence is marked N/A.

** Decision on Maxima or minima depends upon +ve or –ve value of $\ddot{y}$, while $\dot{y} = 0$ at the point under consideration, a precondition. Therefore, wherever $\dot{y} \neq 0$, the entry in the column is marked N/A.

INTEGERAL CALCULUS: Study of this topic starts with a curve whose area is to be determined. Elaboration of the concept of integration is with a function is plotted and is shown in the figure below –



Area under the curve is $= y_1 \cdot \Delta x + y_2 \cdot \Delta x \ldots y_{10} \cdot \Delta x = \sum_{k=1}^{k=10} y_k \cdot \Delta x = \int_{x=x_1}^{x=x_{10}} y dx$. The last part of the equation is considering Δx to be infinitesimal and is shown as integration as per language and convention of mathematics. It implies that all infinitesimal elements are summated (integrated) and thus gives rise to ***Integral Calculus***.

Moreover, from ***Differential Calculus*** we have $f(x) = \frac{d}{dx}F(x) = \frac{\Delta F(x)}{\Delta x}\Big|_{\Delta x \to 0}$; accordingly, $f(x) \cdot \Delta x = \Delta y$; in Integral Calculus it is generally expressed as $\int \boldsymbol{f(x) \cdot dx = F(x)}$ and precise representation is $\int \boldsymbol{f(x)dx = F(x) + C}$; *here C is constant of integration which depends upon initial condition which could be different for different situation and is thus indefinite.* *Accordingly, expression* $\int \boldsymbol{f(x)dx = F(x) + C}$*is called* ***indefinite Integral.*** In case zone of integration is known with boundary values of independent variable then integration is $\int_{x_1}^{x_2} \boldsymbol{f(x)dx} = [\boldsymbol{F(x)}]_{x_1}^{x_2} = [(\boldsymbol{F(x_2)} + \boldsymbol{C}) - (\boldsymbol{F(x_1)} + \boldsymbol{C})] = [\boldsymbol{F(x_2)} - \boldsymbol{F(x_1)}]$ *is called* ***Definite Integral,*** and is elaborated in the beginning of this section. The constant of integration since reflect effect of cumulative change from initial condition, hence ***integration is stated to be historical in nature. However, in definite integration this constant of integration is automatically eliminated.***

Thus, Integration is inverse of Differentiation and accordingly Standard Forms of Integration, without constant of integration are –

$\int x^n\,dx = \frac{x^{n+1}}{n+1}$	$\int \sec^2 x\,dx = \tan x$	$\int \frac{1}{\sqrt{1+x^2}}dx = \tan^{-1} x$
$\int \frac{1}{x}dx = \log x$	$\int \text{cosec}^2 x\,dx = -\cot x$	$\int \left(-\frac{1}{\sqrt{1+x^2}}\right)dx = \cot^{-1} x$
$\int e^x\,dx = e^x$	$\int \sec x \tan x\,dx = \sec x$	$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1} x$
$\int a^x\,dx = \frac{a^x}{\log a}$	$\int \text{cosec}\, x \cot x\,dx = -\,\text{cosec}\, x$	$\int \left(-\frac{1}{x\sqrt{x^2-1}}\right)dx = \text{cosec}^{-1} x$
$\int \sin x\,dx = -\cos x$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x$	$\int \cosh x\,dx = \sinh x$
$\int \cos x\,dx = \sin x$	$\int \left(-\frac{1}{\sqrt{1-x^2}}\right)dx = \cos^{-1} x$	$\int \sinh x\,dx = \cosh x$

Integration of sum of functions is equal to sum of integrals of constituent functions. It is illustrated as below: Let $f(x) = f_1(x) + f_2(x) \ldots f_n(x)$ and $\int f(x)\,dx = F(x)$, $\int f_1(x)\,dx = F_1(x)$, $\int f_2(x)dx = F_2(x)$.. and $\int f_n(x)dx = F_n(x)$. Accordingly, $\frac{d}{dx}F(x) = f(x)$, $\frac{d}{dx}F_1(x) = f_1(x)$, $\frac{d}{dx}F_2(x) = f_2(x)$, ... and $\frac{d}{dx}F_n(x) = f_n(x)$. Therefore, $d\big(F(x)\big) = f(x) \cdot dx$, $d\big(F_1(x)\big) = f_1(x) \cdot dx$, $d\big(F_2(x)\big) = f_2(x) \cdot dx$ ... and $d\big(F_n(x)\big) = f_n(x) \cdot dx$.

$$\int f(x)dx = \int \big(f_1(x) + f_2(x) \ldots f_n(x)\big)dx = \int f_1(x)dx + \int f_2(x)dx + \ldots \int f_n(x)dx$$
$$= F_1(x)+F_2(x)..F_n(x)$$

Integration of Product of a constant and a Function: Let $y = c \cdot f(x)$ Then $\frac{d}{dx}y = y' = c \cdot \frac{d}{dx}f(x) \to y' \cdot dx = c \cdot d\big(f(x)\big)$. Integrating both sides $\int y' \cdot dx = c \cdot \int d\big(f(x)\big) \to \int y' \cdot dx = c \cdot f(x)$

Integration of Product of Two Functions: Let $f(x) = f_1(x) \cdot f_2(x)$, then-

$$\frac{d}{dx}(f_1(x) \cdot f_2(x)) = f_1(x) \cdot \frac{d}{dx}f_2(x) + \frac{d}{dx}f_1(x) \cdot f_2(x)$$

$$=> \frac{d}{dx}(f_1(x) \cdot f_2(x)) = f_1(x) \cdot \dot{f}_2(x) + \dot{f}_1(x) \cdot f_2(x)$$
$$=> d(f_1(x) \cdot f_2(x)) = \big(f_1(x) \cdot \dot{f}_2(x) + \dot{f}_1(x) \cdot f_2(x)\big)\,dx$$
$$=> \int d\big(f_1(x) \cdot f_2(x)\big) = \int \big(f_1(x) \cdot \dot{f}_2(x) + \dot{f}_1(x) \cdot f_2(x)\big)\,dx$$
$$=> f_1(x) \cdot f_2(x) = \int \big(f_1(x) \cdot \dot{f}_2(x) + \dot{f}_1\big)\,dx + \int \big(\dot{f}_1(x) \cdot f_2(x)\big)\,dx$$
$$=> \int \big(\dot{f}_1(x) \cdot f_2(x)\big)\,dx = f_1(x) \cdot f_2(x) - \int \big(f_1(x) \cdot \dot{f}_2(x)\big)\,dx$$

Let $\dot{f}_1(x) = \emptyset(x)$; then, $\int \emptyset(x)dx = f_1(x)$; making these substitution in the above equation –

$$=> \int \left(\emptyset(x)\cdot f_2(x)\right)dx = \left(\int \emptyset(x)dx\right)\cdot f_2(x) - \int \left(\left(\int \emptyset(x)dx\right)\cdot f_2'(x)\right)dx$$

This derivation becomes handy in handling cases arising in solving various problems.

Methods of Integration: Problems that are generally encountered do not involve straight application of standard results and therefore need arises of handling with a bit of mathematical skill and are broadly as under –

Substitution Method: Let the problem is $\int \frac{\boldsymbol{a}}{(\boldsymbol{bx}+\boldsymbol{c})^{\boldsymbol{n}}}dx$; then a substitution $bx + c = u$ reduces the problem to $\int \frac{\boldsymbol{a}}{\boldsymbol{u}^{\boldsymbol{n}}}dx$. This creates another problem that the function has an independent variable u while the integration is w.r.t. $\boldsymbol{x}$. Mathematically, differentiating both sides of the substation leads to $u = b\cdot dx$, or $dx = \frac{du}{b}$. Making another substitution for dx, the problem gets simplified to a simple integration of standard form as under –

$$\int \frac{\boldsymbol{a}}{(\boldsymbol{bx}+\boldsymbol{c})^{\boldsymbol{n}}}dx = \int \frac{\boldsymbol{a}}{\boldsymbol{u}^{\boldsymbol{n}}}dx = \int \frac{\boldsymbol{a}}{\boldsymbol{u}^{\boldsymbol{n}}}\frac{du}{b} = \frac{a}{b}\int \frac{\boldsymbol{1}}{\boldsymbol{u}^{\boldsymbol{n}}}du = \frac{a}{b}\cdot\frac{\boldsymbol{u}^{\boldsymbol{n}+\boldsymbol{1}}}{\boldsymbol{n}+\boldsymbol{1}}$$

Substituting the valve of $\boldsymbol{u}$ in the integration result in $\int \frac{\boldsymbol{a}}{(\boldsymbol{bx}+\boldsymbol{c})^{\boldsymbol{n}}}dx = \frac{a}{b}\cdot\frac{(\boldsymbol{bx}+\boldsymbol{c})^{\boldsymbol{n}+\boldsymbol{1}}}{\boldsymbol{n}+\boldsymbol{1}}$.

Substitution method requires a sharp observation to foresee the results of substitution towards problem solving and not a closed loop or further complicating the problem. Beginners, need not get scared rather practice problems of integration to develop a proficiency available at references provided at the end of chapter.

Handling substitution in case of definite integral requires little care in handling limits and is elaborated with the following example-$\int_1^3 \frac{x^2}{2x^3+5}dx$. In this case let $u = 2x^3 + 5$, then $du = 6x^2dx$, or $x^2dx = \frac{du}{6}$

Now if, $= 1 \to u = 2\cdot 1^3 + 5 = 7$, and $x = 3 \to u = 2\cdot 3^3 + 5 = 59$.

This problem can be solved in two alternatives to it –

Alternative 1: Use of limit after integration

$$\int_1^3 \frac{x^2}{2x^3+5}dx = \left[\int \frac{1}{u}\cdot\frac{du}{6}\right]_{x=1}^{x=3} = \frac{1}{6}\left[\int \frac{1}{u}du\right]_{x=1}^{x=3} = \frac{1}{6}[\log u + C]_{x=1}^{x=3} = \frac{1}{6}[\log(2x^3+5)]_{x=1}^{x=3}$$

$$=> \frac{1}{6}\log\frac{2\cdot 3^3+5}{2\cdot 1^3+5} = \frac{1}{6}\log\frac{59}{7}$$

Alternative II: Change of limit with substitution

$$\int_1^3 \frac{x^2}{2x^3+5}dx = \int_7^{59} \frac{1}{u}\cdot\frac{du}{6} = \frac{1}{6}\int_7^{59} \frac{1}{u}du = \frac{1}{6}[\log u]_7^{59} = \frac{1}{6}\log\frac{59}{7}$$

Choice of alternative is based on comfort of student/mentor or convenience based on problem to be solved, the result remains unchanged.

Partial Fractions: In case of integration of a fraction it is possible to convert a fractions into sum of simple factions, partial fractions, which has two possibilities **-a)** non-repetitive partial fractions and **b)** repetitive partial factions. Both the cases are explained with one example each –

a) Non-Repetitive Factors:

$$\int \frac{1}{x^2-a^2}dx = \int \frac{1}{(x-a)(x+a)}dx = \int \left(\frac{A}{(x-a)} + \frac{B}{(x+a)}\right)dx$$

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{(x-a)} + \frac{B}{(x+a)} = \frac{A(x+a)+B(x-a)}{(x-a)(x+a)} = \frac{x(A+B)+a(A-B)}{(x-a)(x+a)}$$

On equating coefficients of numerator in above equality, *A+B=0 and A-B=1* and thus $A = \frac{1}{2}$, and $B = \frac{-1}{2}$.Accordingly,$\int \frac{1}{x^2-a^2}dx = \int \frac{1}{(x-a)(x+a)}dx = \frac{1}{2}\int \frac{1}{(x-a)}dx$-$\frac{1}{2}\int \frac{1}{(x+a)}dx = \frac{1}{2}\log\frac{x-a}{x+a} = \log\sqrt{\frac{x-a}{x+a}}$

b) Repetitive Factors:

$$\int \frac{1}{(x-a)^2(x+a)}dx = \int \left(\frac{Ax+B}{(x-a)^2} + \frac{C}{(x+a)}\right)dx = \int \frac{Ax}{(x-a)^2}dx + \int \frac{B}{(x-a)}dx + \int \frac{C}{(x+a)}dx$$

Using substitution$(u = x - a)$, and other elaboratopns above, the solution comes to –

$$=> \int \frac{A(u+a)}{u^2}du + \int \frac{B}{(x-a)}dx + \int \frac{C}{(x+a)}dx = \int \left(\frac{A}{u} + \frac{aA}{u^2}\right)du + B\log(x-a) + C\log(x+a)$$

$$=> \log(x-a) - \frac{aA}{u} + \log\frac{(x-a)^B}{(x+a)^C} = -\frac{aA}{(x-a)} + \log\frac{(x-a)^{B+1}}{(x+a)^C}$$

Successive Reduction:

$$\int \sin^2 x\,dx = \int \sin x \cdot \sin x\,dx = \sin x \int \sin x\,dx - \int \left(\frac{d}{dx}\sin x \int \sin x\,dx\right)dx$$

$$=> \sin x \cdot (-\cos x) - \int (\cos x \cdot (-\cos x))dx = -\sin x \cdot \cos x + \int \cos^2 x\,dx$$

$$=> \frac{1}{2}\sin 2x + \int (1-\sin^2 x)dx = \frac{1}{2}\sin 2x + x - \int \sin^2 x\,dx$$

Or, $2\int \sin^2 x\,dx => \frac{1}{2}\sin 2x + x$; or $\int \sin^2 x\,dx => \frac{1}{4}\sin 2x + \frac{x}{2}$

This way successive integration, wherever applicable, can be applied to function of any degree.

More about integration shall be discussed in separate chapter on Calculus in Mathematics Section of Mentors' Manual. Basic concept of Integration would find extensive application in study of Physics right through, starting from first chapter on Kinematics. **Standard Results of Integration** are tabulated below.

$\frac{d}{dx}F(x) = f(x);\ \int f(x)dx = F(x) + C$	$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C$
$\int f_1(x)\cdot f_2(x)dx = f_1(x)F_2(x) + \int \left(\frac{d}{dx}f_1(x)F_2(x)\right)dx$	$\int \sqrt{x^2-a^2}dx = \frac{1}{2}\left\{x\sqrt{x^2-a^2} - a^2\cosh^{-1}\frac{x}{a}\right\} + C$
$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\frac{x-a}{x+a} + C$	$\int \sqrt{x^2+a^2}dx = \frac{1}{2}\left\{x\sqrt{x^2+a^2} - a^2\sinh^{-1}\frac{x}{a}\right\} + C$
$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$	$\int \sqrt{a^2-x^2}dx = \frac{1}{2}\left\{x\sqrt{a^2-x^2} - a^2\sin^{-1}\frac{x}{a}\right\} + C$
$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\frac{x+a}{x-a} + C$	$\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$
$\int \frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\frac{x}{a} + C = \log\left\{x + \sqrt{x^2-a^2}\right\} + C$	$\int \frac{1}{\sqrt{(2ax-x^2)}}dx = \mathrm{vers}^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{1}{\sqrt{x^2+a^2}}dx = \sinh^{-1}\frac{x}{a} + C = \log\left\{x + \sqrt{x^2+a^2}\right\} + C$	

Properties of Definite Integration	
$\int_a^b f(x)dx = \int_a^b f(t)dt = F(b) - F(a)$	a. $\int_{-a}^{a} f(x)dx = 0\ ; f(x) = -f(-x);$ if f(x) is an odd function b. $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx\ ;$ if $f(x)$is an even function
$\int_a^b f(x)dx = -\int_b^a f(x)dx$	
$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$,where $a < c < b$	a. $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$; if $f(2a-x) = f(x)$ b. $\int_0^{2a} f(x)dx = 0$; if $f(2a-x) = -f(x)$
$\int_0^a f(x)dx = \int_0^a f(a-x)dx$	

More about integration shall be discussed in separate chapter on Calculus. Basic concept of Integration would find extensive application while calculating Moment of Inertia and Gravitational Force and Potential in next Chapter-IV on Mechanics

Solution of Differential Equations is a special application of Integral Calculus. Everything in nature is dynamically changing and this dynamic relationship of dependent variable in terms of derivatives of different order w.r.t. independent variables is expressed in the form of Differential Equation. It is pertinent to know that ***an nth order differential equation will have n constants*** and are determined based on boundary conditions. It may be a bit out of context at this point to elaborate and appreciate the concept at this stage. Accordingly, in the first encounter to Differential Equation will occur while dealing with Simple Harmonic Motion (SHM), in Mechanics, Physics Section of this Mentors' Manual, and there the concept of solution of Differential Equation shall be elaborated.

Summary: Topics covered in this Chapter on Foundation Mathematics is aimed at to facilitate understanding of concepts of Physics. Necessity of this chapter is based on actual experience of mentoring. In addition to algebra, trigonometry etc. Vectors, Trigonometry, Differentiation and Integration find extensive application in understanding How and Why of Physics. Despite, in Physics Section, any of these concepts are considered essential in mathematics shall be elaborated in-line to add to conceptual clarity and lucidity of understanding.

It is essential to caution students that mere understanding the mathematical concepts brought out above is not sufficient, unless each concept is practiced by solving problems in reference books on mathematics. Mentors and students would find that, after practice recalling a right formula, and that too correctly, is not a burden on memory, rather it becomes intuitive.

If a reader feels need of additional elaboration, or finds typographical error(s) or any ambiguity in illustrations, it is requested to please revert back to subhashjoshi2107@gmail.com, with specific observations, and if possible it is requested to cite relevant reference. Changes/corrections, if any, incorporated in this non-commercial document based on your inputs shall be individually acknowledged.

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