KNOWLEDGE OF CONCEPTS: NATURAL WAY TO LEARN MATHEMATICS

Prof. SB DHAR

Albert Einstein observed, "Mathematics is the poetry of logical ideas, just as poetry can be as elegant as mathematical proofs." Logical idea that cannot refer back to any antecedent is called concept. The concept is a judgment. Concept does not state how a thing is there but it shows that the thing exists. For example: Tree or Life does not state that "there is a tree" or that "life is hard".

Mathematics comprises of skills and concepts. These skills and concepts are related and often build on one another. It may be possible that one student masters some and still struggles with others. For example, a child who has difficulty with basic multiplication facts may be successful in another area, such as geometry.

Memory, language, attention, and sequential ordering play significant role when children think with numbers. Mathematics being cumulative in nature, a number of brain functions are needed to work together to succeed because in solving problems, students need facts that are linked with one step to the next. For this, a student needs to use memory to recall rules and formulae. This recall is meaningful only when it brings the accurate facts with little mental effort.

Sometimes, while solving the problems:

(a) The students are unable to distinguish between the words like "add", "plus", and "combine" because they pretense the same meaning.

(b) The students do not understand the mathematical significance of the words like "hypotenuse" and "factor" etc. because these words do not occur in everyday conversations.

(c) The students understand the underlying concept undoubtedly, but they feel inability in recalling the proper mathematical term for the instructions involved in the problem.

To overcome these difficulties, the students are supposed to use the procedural memory. This is the method of doing mathematical operations step by step, such as performing a long division or multiplication. This memory helps students in transferring results from one step to the further steps consistently.

Memory skills help children to store concepts and skills and retrieve them for use in relevant applications.

For example, suppose a student knows the mathematical fact: $5 \times 3 = 15$.

If he is asked to solve the word problem, "There are five children. Each of them has three books. How many books are there in all?" To solve this problem, the student will have to rely on memory of the above multiplication fact and apply it to this particular case.

Skills develop sequentially. Sequential development means procedural approach to the learning. Students cannot begin to add numbers until they know that those *numbers represent quantities*. Students must be attentive while grabbing the concepts.

For example, children must learn to distinguish between a plus (+) and a minus (-) sign. After learning the concepts, it is a sequence of steps by which a frequently encountered problem is solved.

Students understand connections between counting and addition e.g., adding two is the same as counting on two. They use properties of addition to add whole numbers and to create and use increasingly sophisticated strategies based on these properties.

For example: Fifteen can be made as follows:

$15 = 0 + 15 = 1 + 14 = 2 + 13 = 3 + 12 = 4 + 11 = 5 + 10 = 6 + 9 = 7 + 8$.

One more important concept that students must note is that, in adding $3 + 4 + 5 = 12$, ***only two addition*** operations are done and ***not three*** because every mathematical operation is done between two quantities at a time. First we add 3 to 4 and then the result is added to 5 to reach the final answer 12. Similarly, in $7^4 = 7 \times 7 \times 7 \times 7$, only three multiplication operations are done in base 7, and not four.

Mathematical Concept *is defined as the 'why'. It is the workings behind the answer. It is the reason behind the answer. It is the answer of the question "why things occur?"*

In other words, concept includes knowledge of procedures. For example, if a student argues that 0.015 is a larger number than 0.05 because "15 is more than 5." It simply shows that he does not know the ***concept as well as the procedure*** to reach the result.

Concept is to be understood well to avoid common misunderstandings as well as inflexible knowledge and skill.

Students must know that

(a) *which* ideas are key
(b) *why* they are important
(c) *which* ideas are useful in problem solving
(d) *how* an idea or procedure is mathematical, and
(e) *how* to adapt previous experience and transfer it to new problems

Let us discuss some examples that explain the misunderstanding of the concepts. This happens only when the concepts are not practiced attentively.

(1) Concept: A more digit number is always bigger than the lesser digit numbers.

Fact: A 3-digit number is bigger than a 2-digit number.

Example: 3.24 is bigger than 4.6 because 3.24 has three digits and 4.6 has two digits.

Is it true, if not, Why?

Because for the first few years of learning, students come across with ***whole numbers (i.e. the set of numbers 0, 1, 2, 3, …)***. The 'digits' rule works with whole numbers only and will not work in this example because here decimal numbers are involved.

(2) *Concept: Product of numbers is always bigger than each of the numbers.*

Fact: When two numbers are multiplied together, the answer is always bigger than both the original numbers.

This 'rule' works for whole numbers, but fails when one or both of the numbers is less than one.

Example: Remember, the word 'times' substitutes the word 'of.'

So, $\frac{1}{2}$ times $\frac{1}{4}$ is the same as "a half of a quarter." It immediately demolishes the expectation that the product is going to be bigger than both original numbers. $8 \div \frac{1}{2} = 16$ explains that there are 16 halves in 8.

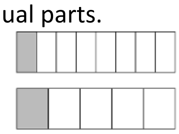
(3) *Concept: Fraction with smaller denominator and same numerator is bigger.*

Fact: $\frac{1}{5}$ is bigger fraction than $\frac{1}{8}$.

Generally, some of the pupils say $\frac{1}{8}$ is bigger than $\frac{1}{5}$ because they know that 8 is bigger than 5.

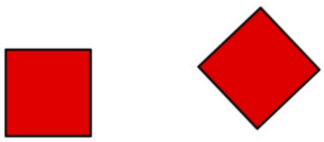
This reveals a gap in knowledge about what the bottom number i.e., the denominator, of a fraction does. It divides the top number i.e., the numerator, of course.

To clarify this, take two sheets of the same size of paper. Divide first sheet in 8 equal parts. Divide the other sheet in 5 equal parts.



Now, compare each single part of both sheets. Is doubt clear?

(4) *Concept: A figure does not change its shape or size if rotated through an angle.*



Fact: Square does not become a diamond after rotation.

We generally misrecognize the regular shapes when they are not in the upright position.

This is a misconception. Because we always draw a square, right-angled in the 'usual' position. If we draw facing a different direction, or just tilted over, the misconception arises.

Do not confuse with shape as a diamond. There is nothing like diamond in mathematics. It's either a square or a rhombus.

(5) *Concept: In multiplication of a number by 10, just a zero in the last of the number is added.*

Fact: $88 \times 10 = 880$

But, it is not true always.

Examples: $237 \times 10 = 2370$; $23.7 \times 10 = 23.70$; $0.237 \times 10 = 0.2370$, or $2/37 \times 10 = 2/370$?

Try to spot the correct one and remove the misconception 'just add zero' rule.

(6) *Concept: Proportion is used to compare a part to the whole.*

If it is asked that what proportion of the balls is blue in urn of 3 red and 2 blue balls?

Someone will say $\frac{2}{3}$ rather than $\frac{2}{5}$.

Why? Because they're comparing blue to red, not blue to all the balls.

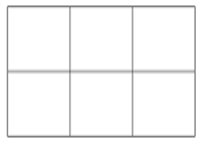
This misconception is due to non-understanding of difference between ***ratio and proportion***.

Ratio is used to compare quantitative relation between two amounts.

(7) *Concept: Perimeter is the continuous line forming the boundary of a closed figure, and area is the size of a surface*

Perimeter of a rectangle means to count the sides of the squares surrounding the shape.

Example: Perimeter of a two-by-three rectangle is 10 units and not the number of squares inside the rectangle i.e., 6 units.



Area of a rectangle means to count all the squares inside the rectangle, i.e. 6.

Mathematical Fact is something that needs to be memorized. The multiplication and addition tables are mathematical facts. They tell that 1 + 1 = 2 and that 2 × 2 = 4. There are no ifs, and/or buts about them.

In day to day calculation, procedural follow up is not practical in doing each and every problem. So, to save time and reach the result as early as possible, the need of memorization of facts is very essential. These facts are the ultimate results of the concepts. For example,

(a) In subtraction problems, for multi-digit numbers, students use the process of "borrow and regroup" without understanding why it is so.

(b) In multiplication problems, while multiplying two negative numbers and yielding a positive number does not bother them to care for the conceptual knowledge, *why* it is true.

The distinction between the concept and the fact is listed below:

(a) The concept is more general than the fact.

(b) The concept can be applied in so many more areas than the fact.

(c) The concept allows to solve so many more problems than just the fact.

(d) One fact lets solve one specific type of problem, but one concept covers a whole range of like problems.

(e) Concepts are understood, and facts are memorized.

While studying concepts, the concept of symmetry in nature cannot be ignored. Every one of us knows that only two things on this earth are perfect. One is the **Nature**, and the other is the **Mathematics**.

Perfect means, both cannot get better than what they are. Nature is a complex web of life and the lifeless. The very state of it being so perfect is the reason for life and also for death.

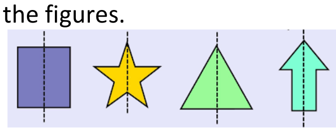
Mathematics is man-made, to understand nature and beyond. Mathematics is no different from nature. Any real-world problem can be solved in mathematics and any mathematical solution is effective in the real world.

According to Euclid, ***"The laws of nature are but the mathematical thoughts of God."***

Mathematically symmetry means that one shape becomes exactly like another when we move it in some way: flip or slide. In other words, two objects are said to be symmetrical if they are of the same size and shape, with one object having a different orientation from the first. i.e., when a figure has two sides that are mirror images of one another.

Symmetry is there when it is possible to draw a line through a picture of the object and along either side the image would look exactly the same.

For example in the figures below, the vertical lines are the line of symmetry of the figures.

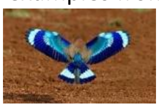


There are two types of symmetry: *bilateral and radial*.

(a) **Bilateral symmetry :** In this type of symmetry, the object has two sides that are the mirror images of each other. For example:

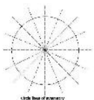

Human body is the best example of a living being which has bilateral symmetry.

Other examples from the Nature are as under:



(b) **Radial symmetry :**Symmetry about a central axis is called the radial symmetry. There are numerous lines of symmetry in this type of objects.

Circle is a simple diagram where there are infinite number of symmetries. Each diameter is the line of symmetry in the circle.



Some natural examples of this type of symmetry are:



Fibonacci, an Italian mathematician is supposed to be the most talented western mathematician of the middle age. He was born around 1170. He developed a very beautiful sequence 1, 1, 2, 3, 5, 8, 13, 21, 34… It is a series of numbers in which each number is the sum of the previous two numbers.

The beautiful thing about this sequence is that if a series of squares with lengths equal to the numbers of this sequence is traced, a line through the diagonals of each square will form a *Fibonacci spiral*. In nature many many Fibonacci spirals can be seen like,


Learning is not necessarily an outcome of teaching. It depends upon practice. Children learn to do well only what they practice doing. When a student starts learning, his brain remains focused on the basic computations. When he becomes automatic with the facts, his brain becomes able to focus on other aspects of the task like the challenges of place value, decimals, or fractions etc. This automation with the basic facts frees the brain to focus on other mathematical processes. *Committing basic mathematical facts to memory* speeds up mathematical tasks.

Students who have committed basic facts to memory are able in performing critical mental mathematical tasks comfortably. They estimate answers prior to solving the problems. They are able to compare their estimates to the actual answers, and determine the reasonableness of their solutions.

Automaticity is the quick and effortless recall of mathematical facts. There remains no need to count every object, no need to think about related facts, or no need to extend patterns. The answer becomes automatically known. Automaticity with basic facts should be a goal for our students, to save their precious time, but alone it is not enough.

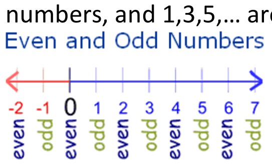
Let us know about some concepts:

(a) ***Representation of Numbers :***Numbers are fundamental to mathematics. There are different symbols to represent them in the world. The International system use ten distinct symbols to represent them: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

(b) ***Representation of Fractions :***It is a partial representation of a number. For example: One third of two is given by $\frac{2}{3}$, or, two-third of one is $\frac{2}{3}$

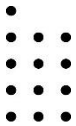
(c) ***Recognition of even and odd numbers :***All numbers are either even or odd. If they are equally split in groups of **twos,** they are **even;** otherwise odd.

2,4,6,…are even numbers, and 1,3,5,… are odd numbers.



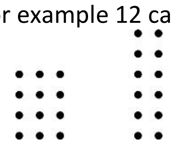
(d) ***Recognition of prime numbers :***Prime numbers are numbers that are divisible by 1 and the number itself. Prime numbers are still a mystery in mathematics as its n^{th} term formula has not been found. Prime numbers cannot be arranged in a rectangular form.

For example 13 is a prime number. We cannot arrange all the dots to make any way a rectangle. One dot is always left.



(e) ***Recognition of composite numbers :***The numbers that have more than 2 factors are called composite numbers. 6 is a composite number as it has 1, 2, 3, and 6 as its factors. 12 is also a composite number as its factors are 1, 2, 3, 4, 6, and 12.

Composite numbers can be arranged in a rectangular form. For example 12 can be arranged as below:



Note: Every number is either composite or prime. Only exception is 1 which is neither composite nor prime.

(f) ***Writing of big numbers :***In writing big numbers in words, the word "and" is not used. The comma (,) is also not used. The comma (,) is used in separating the periods in writing numbers in *digits* only. The comma is called separator.

There are two place value charts:

(i) ***International place value chart:***In this system all the digits are grouped in ***THREES*** from extreme Right to the Left.

The periods are: Ones, Thousands, Millions, Billions, Trillions, Quadrillions, Quintillions,…

Place Value Chart								
Millons			Thousands			Ones		
Hundred Million	Ten Million	Million	Hundred Thousands	Ten Thousands	Thousands	Hundred	Tens	Ones
100,000,000	10,000,000	1,000,000	100,000	10,000	1,000	100	10	1

For example: The 9-digit number 123456789 is written as 123,456,789 and 11-digit number 12345678901 is written as 12, 345, 678, 901.

Note: In writing the numbers in words, ***plurals of periods****, and* ***separators*** *both are not used.*

If 123, 456, 789 is written as: ***One hundred twenty three millions, four hundred fifty six thousands, seven hundred eighty nine***, it is not correct.

The correct way of writing this number is: ***One hundred twenty three million four hundred fifty six thousand seven hundred eighty nine***.

(ii) ***Hindu-Arabic place value chart:*** In this system, the periods are: Ones, Thousands, Lakhs, Crores, and so on…

Crores		Lakhs		Thousands		Ones		
Ten Crores (TC) (10,00,00,000)	Crores (C) (1,00,00,000)	Ten Lakhs (TL) (10,00,000)	Lakhs (L) (1,00,000)	Ten Thousands (TTh) (10,000)	Thousands (Th) (1000)	Hundreds (H) (100)	Tens (T) (10)	Ones (O) (1)

→ Periods   Places ←

The digits are grouped in ***THREES***, then ***TWOS***, then ***TWOS*** and so on from Right to Left.

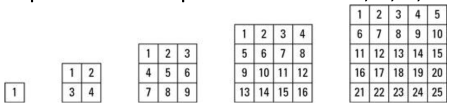
For example: 123456789 is written as 12, 34, 56, 789.

It is written in words as:

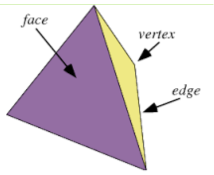
Twelve crore thirty four lakh fifty six thousand seven hundred eighty nine.

(g) ***Recognition of Square Numbers :***Square number is a number when it is multiplied by itself. 4 is a square number as it is found by 2×2.

Example of other square numbers: 1, 4, 9, 16, 25 …



(h) ***Euler's formula for Polyhedron :***Polyhedron is a three-dimensional solid with flat surfaces, such as cube, pyramid or soccer ball.



Euler was a Swiss mathematician. He gave a formula for polyhedrons.

(Number of Faces) + (Number of Vertices) – (Number of Edges) = 2

For example: A cube has six (6) faces, eight (8) vertices and twelve (12) edges.

So, 6 + 8 – 12 = 2

(i) Coordinates

Rene Descartes, a French mathematician started the **Cartesian coordinate system in 1637, showing** a *two-dimensional* picture of **plane.**

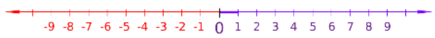
It consists of two intersecting number lines. Forming a cross, this rectangular graph can determine any point in a plane by using two numbers: the *x-coordinate* and the *y-coordinate*.

(j) ***Dimensions :*** Dimension is defined as the minimum number of coordinates needed to specify any point within it.

For example: A number line is one-dimensional.

John Wallis was an English mathematician who created this graph in the 17th century.

Often integers are shown as specially marked points evenly spaced on the line. For example integers from -9 to 9 are marked as below:

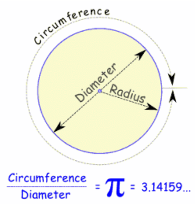


Real Number line is denoted by R.

(k) ***Algebra :***Algebra studies *structure*, *relation*, and *quantity*. It is generalization of arithmetic by substituting concrete numbers with symbols. The ancient Babylonians experimented with a form of algebra some 3,000 years ago, but the Arabs perfected this specialized branch of mathematics after the Persian Muhammad Ibn Mūsā al-khwārizmī's publishing his great treatise *Al-Jabr* (820 AD). *Al-jabr* means "reunion."

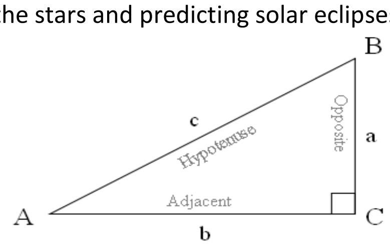
(l) ***Set:*** Georg Cantor and Richard Dedekind in 1870s initiated the study of modern set theory.
They formulated that ***a set can include another set but it cannot include itself***.
A={1, 2, 3, 4, ….}

(m) ***Pi (π):*** It is a mathematical constant. *It stands for ratio of circumference of a circle to its diameter.* Its approximate value is 3.1415926535….
Pi is a *transcendental* number. No equation can ever define it.



The Ancient Greek mathematician Archimedes of Syracuse (287-212 BC) is largely considered to be the first to calculate an accurate estimation of the value of pi.

(n) ***Trigonometric Ratios :*** **Trigonometry** (from Greek *Τριγονομετρία "tri* = three" + "*gon* = angle" + *metr[y] = to measure")* deals with *ratios* of the *sides* of *right triangles.*
It was firstly developed as a navigation method. Trigonometry emerged over 4,000 years ago in ancient Egypt, Mesopotamia and the Indus Valley. But the Greek Hipparchus (circa 150 BC) compiled the first trigonometric table using the sine for solving triangles. With this table, Hipparchus became the greatest astronomer of antiquity, mapping the stars and predicting solar eclipses.



SOH-CAH-TOA.
Sine = **O**pposite ÷ **H**ypotenuse
Cosine = **A**djacent ÷ **H**ypotenuse
Tangent= **O**pposite ÷ **A**djacent

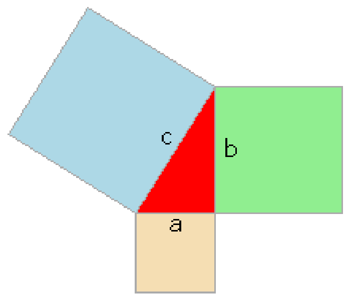
(o) ***Logarithm :*** A **logarithm** states the *power* by which a *base* (usually 10) must be raised to produce a given number. Logarithms express the *ratio* of related numbers. By converting arithmetical progression into geometric progression, they make multiplication and division as simple as addition and subtraction.

It is defined mathematically as following:
If $z = x^y$, then $y = \log_x z$. It is read as "y is equal to logarithm (or log) of z at base x.
Example: $1000 = 10^3 \Rightarrow 3 = \log_{10} 1000$. It may be read as the logarithm of 1000 to the base 10 is 3.

John Napier of Scotland, first propounded logarithms in his landmark book *Mirifici Logarithmorum Canonis Descriptio* (1614).
Logarithms are the *inverse* or opposite of exponentials, just as subtraction is the opposite of addition and division is the opposite of multiplication.

(p) ***Equality(=) :*** The equal sign or equality sign is a mathematical symbol used to indicate equality. It was invented by ancient Indian mathematician *Brahmgupt*.
Equality means one thing is mathematically the same as another. Mathematical statement with equality sign is an equation.
For example: 2x + 3y = 12, Or, Three times of Ten = Thirty

(q) ***Pythagoras Theorem :*** This theorem is named after the Greek mathematician Pythagoras. According to this theorem:
The square of the *hypotenuse* of a right triangle equals the sum of the squares on the other two sides. The pink and green squares have the same combined area as the blue square.



I believe that if one can understand the mathematical facts well, then the learning of mathematics will certainly become simpler, and this simplicity will play a lead role in developing the necessary skills of keeping ideas well arranged.

Dr S.B. Dhar, one of the member of Think Tank, Editor (Education Section). He is an eminent mentor, analyst and connoisseur of Mathematics from IIT for preparing aspirants of Competitive Examinations for Services & Admissions to different streams of study at Undergraduate and Graduate levels using formal methods of teaching shared with technological aids to keep learning at par with escalating standards of scholars and learners. He has authored numerous books – Handbook of Mathematics for IIT JEE, A Textbook on Engineering Mathematics, Reasoning Ability, Lateral Wisdom, Progress in Mathematics (series for Beginner to Class VIII), Target PSA (series for class VI to class XII) and many more. e-Mail ID: maths.iitk@gmail.com

—00—