FIRST PRINCIPLE: THE SOUL OF MATHEMATICS

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It is not possible to prove every statement. The statements that are not proved are called first principles. The First Principle is defined as a basic, fundamental, self-evident proposition or assumption which cannot be deduced from any other proposition or assumption.

In mathematics, *First Principles* are referred to as *axioms* and *postulates*. First principle implies us to use definitions, or properties for solving the problems. Reasoning by first principle is supposed to be the best way to develop mental ability.

In arithmetic, numbers are represented by means of the *Arabic numerals* 0,1,2,3,4,5,6,7,8,9. In Algebra, letters as well as these numerals are used to represent the numbers. The signs of operations +, -, ×, ÷, = are used in arithmetic as well as in algebra.

Euclid was a Greek Mathematician. He is called the father of geometry. His work *Elements* is one of the most influential works in the history of mathematics. Euclid's *Elements* contains a lot of first principles for mathematics like *Definition, Conjecture, Axiom, Postulate, Theorem, Corollary, Lemma* etc. Let us discuss all these terms, one by one.

Definition: It characterizes the meaning of a word by giving all the properties and only those properties that must be true.

Definitions are a grab bag of

(a) Claim like : *a line is breadth less length*,

(b) Assertions like: *a diameter divides the circle in half*, *extremities of a surface are lines*, *the extremities of a line are points*, etc.

Some of the examples of definitions are:

(a) Parallel lines are straight lines that are in the same plane and do not meet, no matter how far extended in either direction.

(b) The point at which two lines meet is called the vertex of the angle.

(c) An angle is the inclination to one another of two straight lines that meet.

(d) If a straight line that stands on another straight line makes the adjacent angles equal, then each of those angles is called a right angle; and the straight line that stands on the other is called a perpendicular to it.


(e) An acute angle is less than a right angle. An obtuse angle is greater than a right angle.

(f) Figures are congruent when, if one of them were placed on the other, they would exactly coincide. Congruent figures are thus equal to one another in all respects.


(g) A circle is a plane figure bounded by one line, called the circumference, such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another. And that point is called the center of the circle. A diameter of a circle is a straight line through the center and terminating in both directions on the circumference. A straight line from the center to the circumference is called a radius; plural, *radii*.
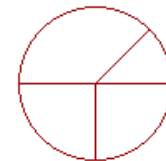
Note: A definition clarifies the idea of what is being defined, and gives it a name. A definition is required only to be understood. A definition is reversible, i.e., when the conditions of the definition are satisfied, then we may use that word.

Conjecture

Let us look at the following figures.
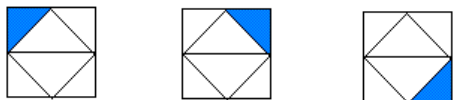
Using common sense, one can conclude that the colored triangle is rotating clock-wise in the pattern and therefore, the next figure in the pattern, may be as below:

This use of common sense is called inductive reasoning. This inductive reasoning leads us to make a *conjecture* about the next figure of the pattern.

Mathematically, *conjecture* is a statement that appears to be correct, but has not been proven formally. In other words, conjecture is a good guess or an idea about a pattern. It is an unproven claim. Once someone proves a conjecture, it becomes a **theorem**.

Goldbach's Conjecture is one of the oldest and best known problem in mathematics. It states:

Every even integer greater than 2 can be expressed as the sum of two primes, i.e.

4 = 2+2, 6 = 3+3, 8 = 3+5, 10 = 3+7,. …, 30 = 11+19, etc.

It has not been proved so far despite considerable effort.

There was another conjecture known as **Euler's sum of power**. It claimed in the 18th century that there were no whole numbers solutions for the equation: $x^4 + y^4 + z^4 = w^4$. It was correct until 1988 when Harvard Professor and American Mathematician **Noam Elkies** found a solution:
$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4.$$
He also proved that infinitely many solutions exist for this equation. Up to 200 years no one could neither prove nor disprove this conjecture.

Some important conjectures are:

(a) If two angles are vertical, then they are equal in measures.

(b) Linear pair of angles add up to 180 degrees.

(c) The sum of the interior angles in any triangle is 180 degrees.

(d) The sum of the measures of the interior angles in any convex quadrilateral is 360 degrees.

(e) If the triangle is an isosceles, then the base angles are congruent.

(f) A mid segment of a triangle is parallel to the third side and is half the length of the third side.

(g) If C is the circumference and D is the diameter of a circle, then there is a number π such that C = πD or C=2πr, where r is the radius of the circle.

(h) The length of the arc on a circle is given by
$$L = \left(\frac{m}{360}\right) \deg rees * circumference,$$ where m is the measure of the central angle.

(i) The shortest distance from a point to a line is measured along the perpendicular segment from the point to the line.


Postulate: It is a true statement. It does not require to be proved because it does not have any contradiction. It is used to derive the other logical statements to solve a problem. Postulates are also called Axioms.

Some important Postulates are:

(a) A line contains at least two points. We can stand side by side at the edge of something.

(b) A plane contains at least three non-collinear points. Three of us can stand on a parking lot so that I can see both of you-no one is hidden by the other two.

(c) Through any two points, there is exactly one line. There is only one point above your head where I can drop something so that it hits you.

(d) If two points lie in a plane, then the line joining them lies in that plane. I can move in a straight line to you if you are standing in the same parking lot as I.

(e) If two planes intersect, then their intersection is a line. The edge of our house where the south and west meet is a line.

(f) The measure of an angle is unique positive number.

(g) There is one and only one angle bisector for any given angle.

(h) The measure of any line segment is a unique positive number.

(i) If two parallel lines are cut by a transversal, then corresponding angles are congruent.

Equality of Cooresponding Angles
A = E, C = G, B = F, D = H
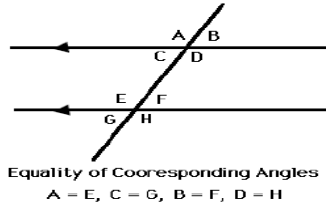
(j) If x is point on AB then AX+XB=AB

(k) Through a point not on a line, exactly one line is parallel to the given line.
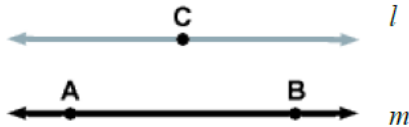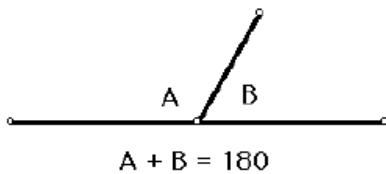

Line *l* is the only line parallel to *m*, but passing through C.

(l) If two angles are vertical angles, then they are congruent.

Angle A = Angle B
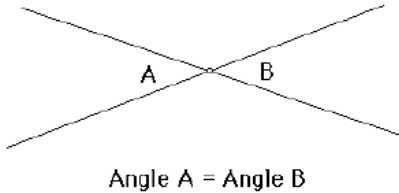
(m) If two angles form a linear pair, then the measure of the angles add up to 180 degrees.

A + B = 180

(n) All right angles are equal to one another.

(o) If a straight line that meets two straight lines makes the interior angles on the same side less than two right angles, then those two straight lines, if extended, will meet on that same side.
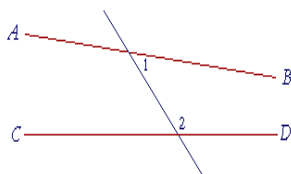
That is, if angles 1 and 2 together are less than two right angles, then the straight lines *AB, CD*, if extended far enough, will meet on that same side; which is to say, AB, CD are not parallel.

Axiom: Logical axiom is a proposition or a statement, which is considered as universally true. Non logical axiom is sometimes called postulate.

Some axioms are:

(a) Things equal to the same thing are equal to one another.

(b) If equals are joined to equals, the wholes will be equal.

(c) If equals are taken from equals, what remains will be equal.

(d) Things that coincide with one another are equal to one another.

(e) The whole is greater than the part. Equal magnitudes have equal parts; equal halves, equal thirds, and so on.

Theorem: A theorem is a mathematical statement that is proved using rigorous mathematical reasoning. A theorem is a major result.

Some of the Theorems are as under:

(a) If two lines are perpendicular, then they meet to form right angles.

(b) If two lines meet to form a right angle, then these lines are perpendicular.

(c) If two angles are supplementary to the same angle, then the angles are congruent, i.e. if 1 is supplementary to 2, 3 is supplementary to 2, then 1 is equal to 3.

(d) The sum of the measure of the angles in every triangle is 180 degrees.

A + B + C = 180
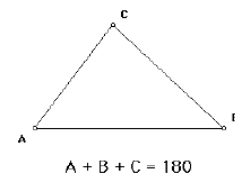
Corollary: A *corollary* is a result in which proof relies on a given theorem, i.e., Corollary is dependent on another theorem.

Lemma: A *lemma* is a small result. It is a helping theorem. It is of lesser importance than a theorem.

Let us see one example that elaborates the inter-relation between Theorem, Corollary and Lemma.
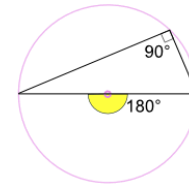
Theorem: *An inscribed angle a° is half of the central angle.*
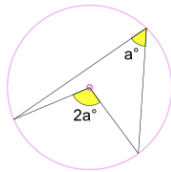
Corollary of the above theorem: Keeping the endpoints fixed, the angle a° is always the same, no matter where it is on the circumference. It is abbreviated as ***Angles Subtended by the Same Arc are equal.***
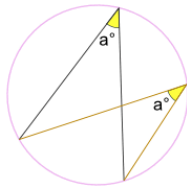
Lemma of *Angle at the Center Theorem:* **Angle inscribed in a semicircle is always a right angle.**

Note: Every mathematical theorem starts as a conjecture or a postulate before it was tested and accepted as proven mathematical facts

Rene Descartes, a French mathematician said, "I think; therefore I am". I shall advise my students to remember that it is not enough to have a good mind but the main thing is to use it well. Rising questions during the learning brands the understanding faultless, as the origin of wisdom rests in the queries. So, think wisely and learn systematically using first principles.

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