SEQUENCES & SERIES

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A sequence is defined as the arrangement of numbers in which each term is related to its predecessor by a Uniform Law. There are evidences that B*abylonians* some 400 years ago, were knowing Arithmetic and Geometric Progressions. Indian mathematician *Aryabhata* (470AD) was the first to give formula for the sum of squares and cubes of natural numbers. In 598 AD, *Brahmgupta*; in 850AD, *Mahavira*; and in 1114-1185AD, *Bhaskar* also worked on these squares and cubes.

There are mainly four types of sequences in the 10+2 syllabus.

(a) Arithmetic Sequence
(b) Geometric Sequence
(c) Harmonic Sequence, and
(d) Arithmetico - Geometric Sequence

Problems are asked generally to find out n^{th} term of the sequence, identification of sequence, sum of finite number of terms, or infinite number of terms. Sometime typical problems are set to test learning of the various concepts of sequences, used in Theory of Equations, Probability, Trigonometry, Coordinate Geometry, and Calculus too.

Sequence

A sequence is a function whose domain is the set of Natural Number, i.e., $f:N\to X$ given by $f(n)=x_n$ for all $n\in N$.

A sequence whose Range is a subset of Real number is called a real sequence.

Series

If $a_1,a_2,a_3,\dots,a_n$ is a sequence then the expression $a_1+a_2+a_3+\dots+a_n$ is called a series.

Series can be finite if the number of terms is countable and infinite if the terms are uncountable.

Progression

The sequence whose terms follow the certain pattern are called progressions i.e., if there exists a formula for writing the n^{th} term of the sequence then it becomes a progression. All sequences do not form a progression.

Terms of a progression are connected either by + or – sign as the case may be.

Example:

If some sequence $<a_n>=(-1)^n$ is expressed then it consists of only two terms {-1,1} .

Facts related to A.P.

1. $T_1+T_2+T_3+\dots T_n$ is called in an Arithmetic Progression if $T_2-T_1=T_3-T_2=\dots=T_n-T_{n-1}=$ constant. This constant is called the Common Difference.

2. The n^{th} term of an AP , from beginning, is given by:

   $T_n=a+(n-1)d$, where a= first term, and d= common difference.

3. The n^{th} term from the end is given by:

   $T'_n=l+(n-1)(-d)$, where l = last term, and d is the common difference.

   Or, r^{th} term from the end = $(n-r+1)^{th}$ term from the beginning = a + (r-1)d where a is the first term and d is the common difference.

4. The sum to n terms of the AP :

$$S_n = \frac{n}{2}\{2a+(n-1)d\} = \frac{n}{2}(a+l)$$ where a = first term, d= common difference, l = last term or n^{th} term, n= number of terms.

5. A sequence is said to be an AP if the n^{th} term is a linear expression of n, i.e.

   $$T_n = An + B$$

6. A sequence is an AP if the sum to n terms is a quadratic expression of n, i.e.

   $$S_n = An^2 + Bn + C$$

7. If a constant is added to or subtracted from all the terms of an AP, the resultant series remains in AP.

   Example:

   If $a_1, a_2, a_3, \ldots a_n$ are in AP, and k is a constant then $a_1 \pm k$, $a_2 \pm k$, $a_3 \pm k, \ldots a_n \pm k$ will also be in AP.

8. If all the terms of an AP are multiplied or divided by a constant, the resultant series remains again in AP.

   Example:

   If $a_1, a_2, a_3, \ldots a_n$ are in AP, and k is a constant then ka_1 , ka_2 , ka_3 ,....,ka_n as well as $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \ldots, \frac{a_n}{k}$ will also be in AP.

9. If there are two APs and there corresponding terms are added or subtracted then the series formed by the new terms is again an AP with the new common difference of Sum of the two common differences or the difference of them as the case may be. But if the terms are multiplied together or divided by, then they do not form an AP.

   Example:

   If $a_1, a_2, a_3, \ldots a_n$ and b_1 , $b_2, b_3, \ldots b_n$ are two APs then $a_1+ b_1$, a_2+b_2 , $a_3+b_3, \ldots, a_n+b_n$ and $a_1- b_1$, a_2-b_2 , $a_3-b_3, \ldots, a_n-b_n$ will also be in AP. But $a_1 b_1$, a_2b_2 , $a_3b_3, \ldots, a_nb_n$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_n}{b_n}$ will not be in AP.

10. If p^{th} term of an AP is q and the q^{th} term is p then $(p+q)^{th}$ term is 0 and the n^{th} term is (p+q-n).

11. In an AP if $pT_p = qT_q$, then $T_{p+q}=0$.

12. In an AP if S_p = q and S_q=p then S_{p+q}= -(p+q).

13. If $S_p=S_q$ then $S_{p+q}=0$

14. In a finite AP, the sum of equidistant terms from the beginning and the end is always constant and is equal to the sum of the first and the last term.

    Example:

    If $a_1, a_2, a_3, \ldots a_n$ are in AP then $a_1+ a_n= a_2+a_{n-1} = a_3+ a_{n-2}= \ldots\ldots =2a_1+(n-1)d$.

15. Three terms a, b, c are said to be in AP if $2b = a+c$ and b lies between a and c.

16. $T_r = \frac{T_{r-k} + T_{r+k}}{2}, 0 \le k \le n-r$.

17. 3 terms in AP are assumed to be $a-d, a, a+d$.

18. 4 terms in AP are assumed to be $a-3d, a-d, a+d, a+3d$.

19. If between a and b, n quantities $A_1, A_2, A_3, \ldots A_n$ are inserted and $a, A_1, A_2, A_3, \ldots A_n, b$ form an AP, then $A_1, A_2, A_3, \ldots A_n$ are called the Arithmetic Means.

20. $A_1 + A_2 + A_3 + ... + A_n = \frac{n}{2}(a+b)$

21. $A_1 = a + \frac{b-a}{n+1}$ and $A_2 = a + 2\left(\frac{b-a}{n+1}\right)$ and so on.

22. If three different quantities *a,b,c* will be in AP if $\frac{a-b}{b-c} = \frac{a}{a}$.

Facts relating to GP

1. Non-zero numbers $T_1, T_2, T_3, ...T_n$ are called to be in a geometric sequence if $\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = ... = \frac{T_n}{T_{n-1}} = r$; r is called the common ratio.

2. The n^th term of a GP is given by $T_n = ar^{n-1}$ where *a*= first term and r = common ratio, n = number of terms.

3. The sum to n terms of a GP is given by $S_n = \frac{a(r^n - 1)}{r-1}$.

4. The sum to infinite number of terms in a GP is possible if common ratio r < 1 and

   is given by $S_\infty = \frac{a}{1-r}$.

5. 3 terms of a GP are assumed to be $\frac{a}{r}, a, ar$.

6. 4 terms of a GP are assumed to be $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

7. If all the terms of a GP are multiplied or divided by the same non-zero constant, the series remains in GP.

8. The reciprocals of the GP form again a GP.

9. If each term of a GP is raised to the same power, the resulting sequence form GP.

10. In a finite GP, the product of equidistant terms from the beginning and the end is always constant and is equal to the product of the first and the last term.

11. Three non-zero terms *a, b, c* are in GP if $b^2 = ac$ and *b* lies between *a* and *c*.

12. If between *a* and *b*, n quantities $G_1, G_2, G_3, ...G_n$ are inserted and $a, G_1, G_2, G_3, ...G_n, b$ form a GP, then $G_1, G_2, G_3, ...G_n$ are called the Geometric Means.

13. If A and G are respectively the AM and GM, between *a* and *b*, then A>G.

14. Equation $x^2 - 2Ax + G^2 = 0$ has *a* and *b* as its roots.

15. No term of a G.P can be zero and so the common ratio cannot be zero.

16. If two GPs are there and their corresponding term are multiplied or divided by each other, then the new terms formed make again a GP.

17. If the terms are added or subtracted then they do not form a GP.

18. If $a_1, a_2, a_3, ..., a_n$ are in GP then $\log a_1, \log a_2, \log a_3, ..., \log a_n$ form an AP.

19. Three different non-zero quantities *a,b,c* are in GP if $\frac{a-b}{b-c} = \frac{a}{b}$.

Facts related to HP

1. $T_1, T_2, T_3, ...T_n$ of non-zero numbers is said to be in Harmonic Sequence if the reciprocals form an Arithmetic Sequence.

2. No term of an HP can be zero.

3. Harmonic Mean H between a and b is given by $H = \frac{2ab}{a+b}$.

4. $A \geq G \geq H$.

5. A, G, H are in GP.

6. If A,G,H are of three given numbers a,b,c then a,b,c are the roots of equation $x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$.

7. There is no formula to calculate the sum to n terms of an HP.

8. Three different non-zero quantities a,b,c are in HP if $\frac{a-b}{b-c} = \frac{a}{c}$.

Important Sums:

(a) $\sum \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\infty = \frac{\pi^2}{6}$

(b) $\sum \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\infty$

$= \frac{\pi^2}{8}$

(c) $\sum (-1)^{n-1} \frac{1}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...\infty = \frac{\pi^2}{12}$

(d) If nth term of a series $T_n = an^3 + bn^2 + cn + d$ then the Sum to n terms is given by $S_n = \sum T_n$

$= a\sum n^3 + b\sum n^2 + c\sum n + d\sum 1$

(e) $\sum n = 1 + 2 + ... + n = \frac{n(n+1)}{2}$

(f) $\sum n^2 = 1^2 + 2^2 + ... + n^2$

$= \frac{n(n+1)(2n+1)}{6}$

(g) $\sum n^3 = 1^3 + 2^3 + ... + n^3$

$= \left(\frac{n(n+1)}{2}\right)^2$

(h) $\sum n^4 = 1^4 + 2^4 + ... + n^4 =$

$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

(i) If Arithmetic Mean(A), $A = \frac{a+b}{2}$, Geometric Mean(G), $G^2 = ab$, and Harmonic Mean(H), $H = \frac{2ab}{a+b}$, then $G^2 = AH$

(j) The sequence of numbers 1, 1, 2, 3, 5, 8, ... is called a Fibonacci's sequence. This is determined by the conditions: $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ and $F_1 = 1 = F_2$

(k) a and b are given by $A \pm \sqrt{A^2 - G^2}$.

(l) $\left(\frac{a_1 + a_2 + a_3 + ... + a_n}{n}\right) \geq \sqrt[n]{a_1.a_2.a_3....a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_n}}$

Some Examples:

1. For any three positive real numbers *a,b,* and *c*, $9(25a^2+b^2)+25(c^2-3ac)=15b(3a+c)$, then find the relation between *a,b,* and *c*.

Solution:

$9(25a^2+b^2)+25(c^2-3ac)-15b(3a+c)=0$

$\Rightarrow 225a^2+9b^2+25c^2-45ab-15bc-75ac=0$

i.e., $\frac{1}{2}\left[(15a-3b)^2+(3b-5c)^2+(5c-15a)^2\right]=0$

$\Rightarrow (15a-3b)=0,\ 3b-5c=0,\ 5c-15a=0$

$\Rightarrow 15a=3b,\ 3b=5c,\ 5c=15a$

$\Rightarrow 15a=3b=5c$

$\Rightarrow \frac{a}{1}=\frac{b}{5}=\frac{c}{3}$ =k (say)

$\Rightarrow a=k, b=5k, c=3k$

$\Rightarrow$ *a,c, b* are in AP

2. Let *a,b, c* $\in$R. If $f(x)=ax^2+bx+c$ is such that *a+b+c=3* and $f(x+y)=f(x)+f(y)+xy$, $\forall$ x,y $\in$R. then find the value of $\sum_{n=1}^{10} f(n)$.

Solution:

Given

$f(x)=ax^2+bx+c$ ....(i)

Putting x=1, f(1)=a+b+c=3 (given)

Put, x=1, y=1 in (i)

f(2)=f(1)+f(1)+1=2(3)+1=7

put, x=1,y=2 in (i)

f(3)=f(1)+f(2)+2=3+7+2=12

Similarly, f(4)=18..

Hence,

$\sum_{n=1}^{10} f(n)$ =f(1)+f(2)+f(3)+...to 10 terms

S_n=3+7+12+18+....

Make a series whose sum can be deduced using method of difference.

S_n=3 + 7 + 12 + 18 + ... + T_n ....(ii)

S_n= 3 + 7 + 12 + 18 + ...+ T_n ....(iii)

Subtract (iii) from (ii)

0=3+4+5+....-T_n

$\therefore T_n$= 3+4+5+....to n terms

$=\frac{n}{2}\{2\times3+(n-1)\times1\}=\frac{n(n+5)}{2}$

$S_n=\sum T_n=\frac{1}{2}\sum(n^2+5n)$

$=\frac{1}{2}\left(\sum n^2+5\sum n\right)=\frac{n(n+1)(n+8)}{6}$

Putting, n=10, S_{10}=330

3. The Fibonacci sequence is defined by $a_1=a_2=1$ and $a_n=a_{n-1}+a_{n-2}$, n>2. Find $\frac{a_{n+1}}{a_n}$ for n=5.

Solution:

Given $a_1=1$, $a_2=1$

Putting n=3, $a_3=a_2+a_1=2$

Putting n=4, $a_4=a_3+a_2=3$

Putting n=5, $a_5=a_4+a_3=5$

Putting n=6, $a_6=a_5+a_4=8$

$\therefore \left(\frac{a_{n+1}}{a_n}\right)_{n=5}=\frac{a_6}{a_5}=\frac{8}{5}$

4. In an AP if m^{th} term is *n* and the n^{th} term is *m*, where $m\neq n$, find the p^{th} term.

Solution:

Let the AP be *a, a+d, a+2d,....*

Then

T_m: $a+(m-1)d=n$ ....(i)

T_n: $a+(n-1)d=m$ ....(ii)

From (i)-(ii)

$(m-n)d=-(m-n)\Rightarrow d=-1$

$\Rightarrow a=n-(m-1)(-1)=n+m-1$

$\therefore T_p=a+(p-1)d=n+m-1+(p-1)(-1)=n+m-1-p+1=n+m-p$

5. In an AP, if p^{th} term is $\frac{1}{q}$ and q^{th} term is $\frac{1}{p}$, prove that the sum of first pq terms is $\frac{1}{2}(pq+1)$ where $p \neq q$.

Solution:
Let the AP be $a, a+d, a+2d, \ldots$

T_p: $a+(p-1)d = \frac{1}{q}$ ....(i)

T_q: $a+(q-1)d = \frac{1}{p}$ ....(ii)

From (i)-(ii)

$(p-q)d = \frac{1}{q} - \frac{1}{p} = \frac{(p-q)}{pq} \Rightarrow d = \frac{1}{pq}$

$\Rightarrow a = \frac{1}{q} - \frac{(p-1)}{pq} = \frac{p-p+1}{pq} = \frac{1}{pq}$ n

$\therefore S_{pq} = \frac{pq}{2}\{2a+(pq-1)d\} =$

$\frac{pq}{2}\left\{\frac{2}{pq}+(pq-1)\frac{1}{pq}\right\} = \frac{pq+1}{2}$

6. The sums of n terms of two arithmetic progressions are in the ratio 5n+4 : 9n+6. Find the ratio of their 18th terms.

Solution:
Let the two APs be
$a_1, a_1+d_1, a_1+2d_1, \ldots$ And $a_2, a_2+d_2, a_2+2d_2, \ldots$
Given that

$$\frac{S'_n}{S''_n} = \frac{\frac{n}{2}\{2a_1+(n-1)d_1\}}{\frac{n}{2}\{2a_2+(n-1)d_2\}} = \frac{\{2a_1+(n-1)d_1\}}{\{2a_2+(n-1)d_2\}} =$$

$$\frac{\left\{a_1+\frac{n-1}{2}d_1\right\}}{\left\{a_2+\frac{n-1}{2}d_2\right\}} = \frac{5n+4}{9n+6}$$

To find the ratio of 18th terms, $a_1+\frac{n-1}{2}d_1$ should look like a_1+17d_1. It means $\frac{n-1}{2}=17$, or n=35.

Therefore $\frac{\left\{a_1+\frac{n-1}{2}d_1\right\}}{\left\{a_2+\frac{n-1}{2}d_2\right\}} =$

$$\left(\frac{5n+4}{9n+6}\right)_{n=35} = \frac{5\times 35+4}{9\times 35+6} = \frac{179}{321}$$

7. The sum of first three terms of a GP is $\frac{13}{12}$ and their product is -1. Find the common ratio and the terms.

Solution:

Let the three terms of a GP be $\frac{a}{r}, a, ar$.

Given $a^3=-1 \Rightarrow a=-1$

Also, $\frac{a}{r}+a+ar = \frac{13}{12}$

$\Rightarrow -\frac{1}{r}-1-r = \frac{13}{12}$

$\Rightarrow 12r^2+25r+12=0$

$\Rightarrow (4r+3)(3r+4)=0$

$\Rightarrow r = -\frac{3}{4}$ or $-\frac{4}{3}$

Hence the terms are $\frac{4}{3}, -1, \frac{3}{4}$

8. If AM and GM of two positive numbers a and b are 10 and 8 respectively, then find the numbers.

Solution:

Given that $\frac{a+b}{2}=10$ and $\sqrt{ab}=8$

$\Rightarrow a+b=20$, and $ab=64$

Using $(a-b)^2=(a+b)^2-4ab$

$\Rightarrow (a-b)^2=400-256=144$

$\Rightarrow a-b=12$

Solving $a-b=12$ and $a+b=20$,

$a=16$ and $b=4$

9. The sum of two numbers is 6 times their geometric means, show that the numbers are in the ratio $(3+2\sqrt{2}):(3-2\sqrt{2})$

Solution:

Let the numbers be a and b.

Given that

$$a+b=6\sqrt{ab}$$

$$\Rightarrow \frac{a+b}{\sqrt{ab}}=\frac{6}{1}$$

On using componendo and dividendo

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}}=\frac{6+2}{6-2}=\frac{2}{1}$$

$$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2}=\frac{2}{1}$$

On taking square roots

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{2}}{1}$$

Using componendo and dividend

$$\frac{2\sqrt{a}}{2\sqrt{b}}=\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\Rightarrow \frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

On squaring both sides

$$\frac{a}{b}=\frac{3+2\sqrt{2}}{3-2\sqrt{2}}$$

10. Find the sum of the following series upto n terms: $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$

Solution:

The n^{th} term of the series is

$$T_n=\frac{1^3+2^3+3^3+\ldots+n^3}{1+3+5+\ldots(2n-1)}=\frac{\sum n^3}{n^2}$$

$$=\frac{\left(\frac{n(n+1)}{2}\right)^2}{n^2}=\frac{(n+1)^2}{4}$$

$$\Rightarrow S_n=\sum T_n=\frac{1}{4}\left(\sum n^2+2\sum n+\sum 1\right)$$

$$=\frac{1}{4}\left(\frac{n(n+1)(2n+1)}{6}+\frac{2n(n+1)}{2}+\frac{n}{1}\right)$$

$$=\frac{n}{24}\left((n+1)(2n+1)+6(n+1)+6\right)$$

$$=\frac{n}{24}\left(2n^2+9n+13\right)$$

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