THEORY OF EQUATIONS

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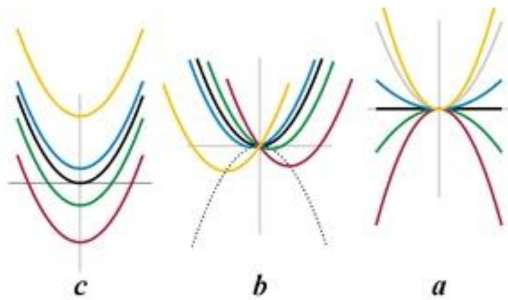
A polynomial when equated to zero makes an equation. The solution of the equation is called the root of the equation.

The graph of a quadratic function $f(x) = ax^2 + bx + c$ changes as a, b, c change. It looks like



Note: The values of the variable that makes a polynomial zero are called Zeros and not roots.

Example:

(a) A polynomial of one degree x+5=0 is called monomial equation. It has one solution.

(b) A polynomial of degree 2, $x^2+3x+2=0$ is called a quadratic equation. It has 2 solutions.

(c) A polynomial of degree 3, $x^3-6x^2+11x-6=0$ is called a cubic equation. It has 3 solutions.

Way to write the equation when roots are known

(a) If α and β are the roots of a quadratic equation, then the quadratic equation is written as
x^2 + (sum of the roots)x + (product of roots) = 0, or
$x^2 + (\alpha+\beta)x + (\alpha\beta) = 0$

(b) If α, β, and γ are the roots of a cubic equation, then the quadratic equation is written as
x^3 + (sum of the roots)x^2 + (sum of the products of two roots at a time)x + (product of all the three roots) = 0

$x^3 + (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x + (\alpha\beta\gamma) = 0$

Facts relating to the equation

1. A polynomial equation has at least one solution, i.e., it has at least one root.

2. A polynomial equation of degree *n* has *n* roots.

3. $ax^2 + bx + c = 0$ is called a quadratic equation if *a* is not zero.

4. If *a, b, c* are real then the equation is called with real coefficients.

5. The expression ($b^2 - 4ac$) is called the discriminant of the quadratic equation and is generally represented by D. It shows the nature of the roots of the equation.

6. If D > 0, the roots are real and distinct, if D=0, the roots are real and equal, if D<0, the roots are imaginary, if D = a perfect square, then the roots are rational provided *a, b, c* are rational otherwise, irrational.

7. If the sum of the roots of $ax^2+bx+c=0$ is zero (i.e., $a + b + c = 0$), then the roots are 1 and $\frac{c}{a}$ and in the case of $ax^2-bx+c=0$, the roots are -1 and $\frac{c}{a}$ if $a - b + c = 0$.

8. If a ***quadratic equation*** is satisfied by ***more than two roots*** (real or complex), then it is called the ***Identity*** and then $a = b = c = 0$.

9. If α, β be the roots of the equation $(x-a)(x-b)=c, c \neq 0$, then the roots of the equation $(x-\alpha)(x-\beta)+c=0$ are *a* and *b*.

10. If D_1 and D_2 are the Discriminant of two quadratic equations $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$ respectively and $D_1+D_2 \geq 0$ then at-least one of the D_1 and D_2 is ≥ 0 i.e. at least one of the equations has real roots.

11. If $D_1+D_2<0$, then at-least one of D_1 and D_2 <0, i.e. at least one of the given equations has imaginary roots.

12. If $D_1.D_2<0$, then the equation $(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)=0$ will have two real roots.

13. If $D_1.D_2>0$, then in the case (i) $D_1>0$ and $D_2>0$ i.e., the equation $(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)=0$ will have four real roots and in the case (ii) $D_1<0$ and $D_2<0$ the equation $(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)=0$ will have four complex roots.

14. If $D_1.D_2=0$, then in the case (i) $D_1>0$ and $D_2=0$ and (ii) $D_2>0$ and $D_1=0$ i.e., the equation $(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)=0$ will have two equal and two distinct real roots and in the case of (i) $D_1<0$ and $D_2=0$ and (ii) $D_2<0$ and $D_1=0$ i.e., the equation $(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)=0$ will have two equal real roots and two imaginary roots.

15. A polynomial equation $f(x)=0$ has exactly one real root equal to α if $f(\alpha)=0$ and $f'(\alpha)\neq 0$ and $f(x)=0$ has exactly two real roots α, β if $f(\alpha)=f'(\alpha)=0$ and $f''(\alpha)\neq 0$.

16. If $a=1$, b and c are integers, and roots are rational, then roots will be integers.

17. If a,b,c are rational and D is a perfect square then roots are rational.

18. To obtain the equation whose roots are reciprocal of the roots of a given equation: replace x by $\frac{1}{x}$.

19. To obtain the equation whose roots are negative of the roots of a given equation: replace x by –x.

20. To obtain the equation whose roots are square of the roots of a given equation: replace x by $\sqrt{x}$.

21. To obtain the equation whose roots are the n^{th} power of the roots of a given equation: replace x by $x^{\frac{1}{n}}$.

22. If equations $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$ are such that $a_1, a_2 \neq 0$, $a_1b_2 \neq b_1a_2$ and they are to have a common root, then $(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)=(c_1a_2-c_2a_1)^2$.

23. Quadratic expression $ax^2+2hxy+by^2+2gx+2fy+c$ can be resolved into two linear factors if $abc+2fgh-af^2-bg^2-ch^2=0$. It can be expressed in the determinant form as below:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

24. Between any two roots of a polynomial $f(x)=0$, there is always a root of its first derivative $f'(x)=0$.

25. If x^2+ax+b is an integer for every integer x, then $(a+b)$ is always an integer.

26. Every equation has a root real or imaginary.

27. The root is the value of the equation's variable which satisfies it.

28. The number of roots of an equation is equal to its degree and not more than in any case.

29. Equations with real coefficients may have imaginary roots but they occur in pair i.e. $a+ib$ and $a-ib$ are present together.

30. Equations with rational coefficients may have surds roots but they occur in pair i.e. $(a+\sqrt{b})$ and $(a-\sqrt{b})$ are always together.

31. Equations with all positive coefficients has no positive root i.e., $x^3+ax^2+x+b=0$ will have no positive root.

32. Equations having coefficients of all even powers of x as (-) and of all odd powers (+) will have no negative root i.e. $x^5-ax^4+bx^3-cx^2+dx-1=0$ will have no negative root.

33. Equations containing only even powers and all coefficients having same sign will have no real root i.e., $x^6+ax^4+bx^2+c=0$ will have no real root.

34. Equations containing only odd powers and all coefficients having same sign will have no real root except x =0, i.e. $x^5+ax^3+bx=0$ will have no real root except $x=0$.

35. An equation of odd degree has at least one real root whose sign is opposite to its last term.

36. An equation of even degree will have at least two real roots one negative and one positive whose last term is negative.

37. The maximum number of positive roots in an equation is equal to the change of sign of the coefficients of $f(x)$ and the maximum number of negative roots is equal to the change of sign in $f(-x)$. This is called the Descarte's Rule of sign.

38. Descarte's Rule of sign gives the maximum number of positive or negative real roots. It does not give the exact number of positive or negative real roots of $f(x)=0$.

39. In an equation $f(x)$ if for values a and b, $f(a)$ and $f(b)$ have different signs, then a root must exist between a and b.

40. $f(a)$ and $f(b)$ containing same signs has either no root or an even number of roots of $f(x)$ between a and b.

41. *f(a)* and *f(b)* containing different signs has an odd number of roots of *f(x)* between *a* and *b*.

42. If an equation has *r* equal roots, its first derivative *f'(x)* will have *(r-1)* equal roots.

43. To find an equation whose roots are enhanced by *m*, replace *x* by *x-m*.

44. To find an equation whose roots are diminished by *m*, replace *x* by *x + m*.

45. $ax^2+bx+c=0$ is positive or greater than 0 for all values of $x \in R$ iff a>0, D<0.

46. $ax^2+bx+c=0$ is negative or less than 0 for all values of $x \in R$ iff a<0, D<0.

47. Curve represented by the Quadratic expression cuts x-axis at two points iff D>0.

48. Curve represented by the Quadratic expression touches x-axis if D=0.

49. Curve represented by the Quadratic expression will not intersect x-axis if D<0.

50. Curve represented by the Quadratic expression will be completely above x-axis if a >0.

51. Curve represented by the Quadratic expression will be completely below the x-axis if a<0.

52. If the roots of the equation represented by $ax^2+bx+c=0$ are real α and β, and for a real k they are such that $\alpha < k < \beta$, then D>0, *a.f(k)<0*.

53. If k_1, k_2 are such that $k_1 < \alpha$, $\beta < k_2$, then $D \geq 0$, $a.f(k_1) > 0$, $a.f(k_2) > 0$, $k_1 < -(b/2a) < k_2$.

54. If $k < \alpha, \beta$ or $k > \alpha, \beta$; then $D \geq 0$, $a.f(k) > 0$.

55. If one of the roots lies in the interval (k_1, k_2) then $f(k_1)f(k_2)<0$.

56. If *a-b+c=0* then one root is -1 and the other root is (–c/a).

57. If $ax^2+bx+c=0$ and *a+b+c =0* then one root is always 1 and the other is (c / a).

58. If all the terms are with positive coefficients and no odd powers are there, it will have complex roots.

59. Every odd degree equation has at-least one real root.

60. α is a repeated root iff $f(\alpha)=0$ and $f'(\alpha)=0$.

61. If $(x-\alpha)^k$ divides f(x) then $(x-\alpha)^{k-1}$ divides f'(x).

62. If both roots are positive, then $\alpha+\beta>0$ and $\alpha\beta>0$.

63. If both roots are negative, then $\alpha+\beta<0$ and $\alpha\beta>0$.

64. If both roots are greater than k, then $D\geq0$, $(-b/2a) > k$, $a\,f(k) > 0$.

65. If both roots are less than k then $D\geq0$, $(-b/2a)< k$, $a\,f(k)>0$.

66. $a^{f(x)} > b$ where $a>0$

    (a) if $b>0$, then $f(x) > \log_a b$ if $a>1$ and $f(x) < \log_a b$ if $0<a<1$.

    (b) $x \in D_f$ if $b \leq 0$

67. $a^{f(x)}=1, a<0, a \neq 1 \Rightarrow f(x)=0$.

    Example:

    $5^{x^2+5x+6}=1$.

    $\Rightarrow x^2+5x+6=0$ and may be evaluated.

68. $f(a^x)=0 \Rightarrow f(t)=0$ where $t=a^x$.

    $2^{2x}-6.2^x+8=0$ assume $2^x= t$ and proceed.

69. $a^{f(x)}+ b^{f(x)}=c$ where $a,b,c \in R$ and $a^2+b^2=c$. obviously f(x)=2 gives the solution.

    Example:

    $3^{x-4}+5^{x-4}=34 \Rightarrow 3^2+5^2=34 \Rightarrow x-4=2$ is a solution.

70. If the equation is of the form $(x-a)^4+(x-b)^4=A$ then the substitution $t=\frac{(x-a)+(x-b)}{2}$ makes it solvable.

Some Important Questions

1. Solve $x^2+2=0$

   Solution:

   $x^2=-2 \Rightarrow x^2=2i^2 \Rightarrow x=i\sqrt{2}, -i\sqrt{2}$

2. If, for a positive integer n, the quadratic equation $x(x+1)+(x+1)(x+2)+\ldots+(x+\overline{n-1})(x+n)=10n$ has two consecutive integral solutions, then find the value of n.

   Solution:

$$x(x+1)+(x+1)(x+2)+\ldots+(x+\overline{n-1})(x+n)$$
$$=10n$$

$$\Rightarrow \sum_{r=1}^{n}(x+r-1)(x+r)=10n$$

$$\Rightarrow \sum_{r=1}^{n}\{x^2+(2r-1)x+(r^2-r)\}=10n$$

On solving, $x^2+nx+\left(\frac{n^2-31}{3}\right)=0$

Let the roots be α and α+1, then

$$2\alpha+1=-n \Rightarrow \alpha=-\frac{n+1}{2} \quad \text{...(i)}$$

$$\alpha(\alpha+1)=\frac{n^2-31}{3} \quad \text{...(ii)}$$

From equations (i) and (ii), eliminate α,

$n^2=121 \Rightarrow n=11$

3. Let p, q be integers and let α, β be the roots of the equation $x^2-x-1=0$, where $\alpha\neq\beta$. For $n=0,1,2,3,....$, let $a_n=p\alpha^n+q\beta^n$. If a and b are rational numbers and $a+b\sqrt{5}$, then $a=0=b$.
(i) If $a_4=28$ then show that the value of $p+2q$ is 12.

Solution:

$$a_4=p\alpha^4+q\beta^4 \quad \text{(i)}$$

If α is a root of the given equation then

$$\alpha+\beta=1 \quad \text{(ii)}$$

$$\alpha^2=\alpha+1 \Rightarrow \alpha^4=(\alpha+1)^2$$

$$=\alpha^2+2\alpha+1=3\alpha+2 \quad \text{(iii)}$$

Similarly,

$$\beta^4=3\beta+2 \quad \text{(iv)}$$

From equations (i),(ii), (iii), and (iv)

$$a_4=p\alpha^4+q\beta^4=p(3\alpha+2)+q(3\beta+2)$$

$$=p(3\alpha+2)+q(3-3\alpha+2)$$

$$=p(3\alpha+2)+q(5-3\alpha)$$

$$=\alpha(3p-3q)+2p+5q$$

Given that $a_4=28$

But $\alpha=\frac{1+\sqrt{5}}{2}$

Hence,

$$\frac{1+\sqrt{5}}{2}(3p-3q)+2p+5q=28$$

$$\Rightarrow \frac{1+\sqrt{5}}{2}(3p-3q)+2p+5q-28=0$$

$\Rightarrow p=q$ and $2p+5q-28=0$

$\Rightarrow p=q=4$

Therefore, $p+2q=12$

(ii) Prove that the value of $a_{12}=a_{11}+a_{10}$.

Solution:

Given that $a_n=p\alpha^n+q\beta^n$

Also, $\alpha^2=\alpha+1$

$$\Rightarrow \alpha^n=\alpha^{n-1}+\alpha^{n-2}$$

Similarly,

$$\beta^n=\beta^{n-1}+\beta^{n-2}$$

Hence,

$$a_n=p\alpha^n+q\beta^n$$
$$=p(\alpha^{n-1}+\alpha^{n-2})+q(\beta^{n-1}+\beta^{n-2})$$
$$=(p\alpha^{n-1}+q\beta^{n-1})+(p\alpha^{n-2}+q\beta^{n-2})$$
$$=a_{n-1}+a_{n-2}$$

On putting, n=12, we get $a_{12}=a_{11}+a_{10}$

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