LINEAR INEQUALITY

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We have seen the problems like,
Solve:
(a) $x+3>1$
(b) $5x<25$
(c) $5\geq x+1$
(d) $3x-1\leq 2$
We know that
(a) **>** symbol is called ***greater than***. It means that **5>4**. It is read as ***five is greater than four***.
(b) **<** symbol is called ***less than***. It means that **3<4**. It is read as ***three is less than four***.
(c) ≥ symbol is called **greater or equal to**. It means that ***either the left hand quantity is greater than the right hand quantity or it is equal***.
Remember: if a quantity is greater than a quantity, it cannot be equal to it. So, it is a combination of two statements: Either greater than or equal to. If 8>3, it cannot be equal to 3.
(d) ≤ symbol is called less than or equal to. It means that ***either the Left hand quantity is less than the Right hand quantity or it is equal***.
Remember: if a quantity is less than a quantity, it cannot be equal to it. So, it is a combination of two statements: Either less than or equal to. If 7< 10, it cannot be equal to 10.

Definition

In-equal means quantities that are not equal. The property of being two or more quantities unequal is called inequality.

(a) A quantity **x** is said to be greater than another quantity **y** if **(x-y)** is **positive** or **greater than 0**. This is written as **(x-y) >0** or **x > y**.
(b) A quantity **x** is said to be smaller than another quantity **y** if **(x-y)** is **negative** or **less than zero**. This is written as **(x-y)<0** or **x < y**.

Some Properties

1. If x > y then y < x
2. If x > y then –x < - y.
3. If a > b then $\left(\frac{a}{b}\right) > \left(\frac{a+x}{b+x}\right)$.
4. If a < b then $\left(\frac{a}{b}\right) < \left(\frac{a+x}{b+x}\right)$.
5. The arithmetic mean is greater than Geometric mean i.e. $\frac{x+y}{2} > \sqrt{xy}$.
6. If a,b,c are positive quantities then $\left(x^2+y^2+z^2\right) > (xy+yz+zx)$.
7. If x > y, y> z then x > z.
8. If x > y then x + k > y + k.
9. If x < y then x – k < y – k.
10. If x > 0, and a > b > c then then $a^x > b^x$.
11. If a > 1, x>y>0 then $a^x > b^y$.
12. If a > 1, and x > y then $\log_a x > \log_a y$.

Special Inequalities

Weierstrass' inequality

If $x_1, x_2, ..., x_n$ are n positive real numbers then for all n≥2,
$(1+x_1)(1+x_2)....(1+x_n) > 1+x_1+x_2+....+x_n$.
If $x_1, x_2, ..., x_n$
are ***n*** positive real numbers less than 1, then,
$(1-x_1)(1-x_2)....(1-x_n) > 1-x_1-x_2-....-x_n$

Cauchy-Schwarz Inequality

If $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ are 2n real numbers then,

$$(x_1y_1+x_2y_2+....+x_ny_n)^2 \leq (x_1^2+x_2^2+....+x_n^2)(y_1^2+y_2^2+....+y_n^2)$$

if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = ... = \frac{x_n}{y_n}$.

Tchebychef's Inequality

If $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ are any real numbers such that,

(a) $x_1 \geq x_2 \geq x_3 \geq .... \geq x_n$, and

$y_1 \geq y_2 \geq y_3 \geq .... \geq y_n$

Then

$$\frac{1}{n}\sum x_i y_i \geq \left(\frac{1}{n}\sum a_i\right)\left(\frac{1}{n}\sum b_i\right)$$

(b) $\frac{1}{n}\sum x_i y_i \leq \left(\frac{1}{n}\sum a_i\right)\left(\frac{1}{n}\sum b_i\right)$

If $x_1 \geq x_2 \geq x_3 \geq .... \geq x_n$,and

$y_1 \leq y_2 \leq y_3 \leq .... \leq y_n$

Bernoulli's Inequality

$(1+x)^n \geq 1+nx$, $x \in R$, $x \geq -1$, $n \in N$

Some Hinted Solutions

1. ***Given $n^4 < 10^n$ for a fixed positive integer $n \geq 2$, prove that $(n+1)^4 < 10^{n+1}$.***

Hint:

Use the inequality: $\left(\frac{n+1}{n}\right)^4 = \left(1+\frac{1}{n}\right)^4 \leq \left(1+\frac{1}{2}\right)^4 < 10$

Therefore,
$(n+1)^4 \leq n.10 < 10^{n+1}$.

2. ***Find the interval for x when $\log_{0.3}(x-1) < \log_{0.09}(x-1)$.***

Hint:

The expression is true for x > 1.
Assume the given expression True and reach the result $(x-1)^2 > (x-1)$
i.e. $(x-1)(x-2) > 0$
Hence the limit is $(2,\infty)$

3. ***If $a^2 + b^2 + c^2 = 1$ then find the limit (ab + bc +ca) lies in.***

Hint:

We know that
$(a+b+c)^2 \geq 0$
$\Rightarrow a^2 + b^2 + c^2 + 2(ab+bc+ca) \geq 0$

$\Rightarrow 1+ 2(ab+bc+ca) \geq 0$

$\Rightarrow (ab+bc+ca) \geq -1/2$

Also we know that

$(b-c)^2 + (c-a)^2 + (a-b)^2 \geq 0$

Or,

$(ab + bc + ca) \leq 1$

Hence the limit is [-1/2, 1].

4. ***Prove for every integer > 1, the inequality***

   $(n!)^{1/n} < n + ½$ holds.

Hint:

Assume n terms as n,(n-1),(n-2),…., 3.2.1
And use AM > GM

5. ***Prove for all x, cos(sinx) > sin (cosx).***

Hint:

Change the expression into:
cos(sinx)-sin(cosx)>0
cos(sinx)-cos{π/2 – cosx}>0
use cos C – cos D formula to express in product form and simplify.

6. ***Find the solution set contained in R of the inequation $3^x + 3^{1-x} - 4 < 0$.***

Hint:

Assume $3^x = y$ and then make quadratic as $y^2 - 4y + 3 < 0$ and then solve.

7. ***Find the value of $5^{\log_7 11} - 11^{\log_7 5}$.***

   Hint: Assume a= $5^{\log_7 11}$ and b=$11^{\log_7 5}$

Take log of a at base 5 and of b at base 7 and rewrite as below:

$$\log_5 a \times \log_7 5 = \log_7 11 \times \log_{11} b$$
$$\Rightarrow \log_7 a = \log_7 b \Rightarrow a = b.$$

And hence a-b=0 is the required answer.

8. ***Find interval in which x lie so that*** $\log_{0.3}(x-1) < \log_{0.09}(x-1)$.

Hint: Simplify to

$$\log_{0.3}(x-1) < \log_{0.09}(x-1)$$

$$\Rightarrow \log_{0.3}(x-1) < \log_{(0.3)^2}(x-1)$$
$$= \frac{\log_{0.3}(x-1)}{\log_{0.3}(0.3)^2} = \frac{\log_{0.3}(x-1)}{2}$$

$$\Rightarrow 2\log_{0.3}(x-1) < \log_{0.3}(x-1)$$

Since base of log is less than 1, the sign of inequality will change from < to > i.e.

$\Rightarrow (x-1)^2 > (x-1)$

$\Rightarrow$ either x < 1 or x >2

But when x<1, the logarithm becomes undefined hence the required interval is (2,∞)

9. ***Find the value of $\log_{12}54$ in terms of x if $x=\log_{12}24$***.

Hint: Rewrite,

x= $\dfrac{\log_3 24}{\log_3 12} = \dfrac{1+3\log_3 2}{1+2\log_3 2}$

$$\Rightarrow \log_3 2 = \frac{1-x}{2x-3}.$$

Also assume,

y= $\log_{12} 54 = \dfrac{\log_3 54}{\log_3 12} = \dfrac{3+\log_3 2}{1+2\log_3 2} \Rightarrow \log_3 2 = \dfrac{3-y}{2y-1}$.

And hence y = 8-5x

10. ***Solve: $\log_{(2x+3)}(6x^2+23x+21)=4-\log_{(3x+7)}(4x^2+12x+9)$***

Hint: Rewrite the given equation as

$\log_{(2x+3)}(2x+3)(3x+7)=4-\log_{(3x+7)}(2x+3)^2$

change all to base (2x+3) and simplify to

$\Rightarrow [\log_{(2x+3)}(3x+7)]^2-3.\log_{(2x+3)}(3x+7)+2=0$

$\Rightarrow \log_{(2x+3)}(3x+7)$= 1 or 2

$\Rightarrow$2x+3= 3x+7 or $(2x+3)^2=(3x+7)$

$\Rightarrow$x= -4 or x=-2, -1/4

But the required conditions are

2x+3>0, and $\neq$ 1

$\Rightarrow$ x > (-3/2) and x $\neq$ - 1

Also 3x+7>0, and $\neq$ 1

$\Rightarrow$ x > (-7/3) and x $\neq$ - 2

Obviously x= -1/4

fulfills all requirements hence the solution.

11. ***Solve :*** $\log_{\sin\frac{\pi}{4}} \sin x > 0, x \in [0,4\pi)$.

Hint: The given inequality can be rewritten as below:

$$\log_{\frac{1}{\sqrt{2}}} \sin x > 0 \Rightarrow (\sin x) < \left(\frac{1}{\sqrt{2}}\right)^0$$

Note: Sign of inequlity changes for log if the base is less than 1.

i.e. sin x < 1 but sin x should be greater than 0 (for logarithmic functions) ; hence x cannot take value 0.

$\Rightarrow$0 < sin x < 1

$$\Rightarrow x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \cup \left(2\pi, \frac{5\pi}{2}\right) \cup \left(\frac{5\pi}{2}, 3\pi\right)$$

12. ***Find the range of values of x that satisfies***

$\log_{\cos x}(\sin x) \geq 2$.

Hint: Obviously the domain of definition for x shall be

(a) $\cos x > 0$, $\cos x \neq 1$.

(b) $\sin x > 0$ i.e. $0 < \sin x \leq 1$

now rewrite the inequality as below:

$\sin x \leq (\cos x)^2 = 1 - \sin^2 x$

$\Rightarrow \sin^2 x + \sin x - 1 \leq 0$

$$\Rightarrow \left(\sin x + \frac{1}{2}\right)^2 \leq \frac{5}{4}$$

$$\Rightarrow \sin x \leq \frac{\sqrt{5}}{2} - \frac{1}{2} ...or... \geq \frac{\sqrt{5}}{2} - \frac{1}{2}$$

But $\sin x > 0$

Hence, $\sin x \leq \frac{\sqrt{5}}{2} - \frac{1}{2}$.

13. ***Solve the equation:*** $x^2 + \frac{x^2}{(x+1)^2} = 3$.

Hint: Write the given expression as

$$\left(x - \frac{x}{x+1}\right)^2 + \frac{2x^2}{(x+1)^2} = 3,$$

use the formula

$x^2 + y^2 = (x+y)^2 - 2xy$

Simplify to

$$\left(\frac{x^2}{x+1}\right)^2 + \frac{2x^2}{(x+1)} = 3$$

Put $x^2/(x+1) = y$

and solve treating it a quadratic and then values of x.

14. ***If k > 0, then solve the in-equation:*** $|x| \leq k$

Hint: The value of k depends upon x.

So consider two cases for x.

Case I

$x < 0$

by the definition of modulus function,

$-x \leq k$

$\Rightarrow x \geq -k$ (by the property of inequality)

$\Rightarrow -k \leq x < 0$

$\Rightarrow x \in [-k,0]$

Case II

$x \geq 0$

$\Rightarrow x \leq k$

$\Rightarrow 0 \leq x \leq k$

$\Rightarrow x \in [0,k]$

Combine the two results to get the required range of x as below:

$x \in [-k,0] \cup [0,k] \Rightarrow x \in [-k,k]$

15. ***Solve the in-equation:*** $\frac{-3x+10}{x+1} > 0$

Hint: Rewrite the given inequality by multiplying the Numerator and Denominator by (x+1) as below:

$(-3x+10)(x+1) > 0$ as the denominator $(x+1)^2$ will always be +ive.

Hence using Descartese Rule of sign,

$x \in (-1, 10/3)$

16. ***Solve the following system of linear in-equations: 3x + 9 ≤ 0, 7x - 2 < 0, 1 - x > 9***

Hint: Solve each inequality separately and combine the common value to get the required result.

From the first

$x \leq -3$

from second

$x < 2/7$

from the third

$x < -8$

on combining the all results

x∈(-∞,-3]∪(-∞,2/7)∪(-∞,-8)

⇒ x∈(-∞,-8) by depicting on Number line.

17. ***Solve the following system of in-equations:*** $\frac{x}{2x+1} \geq \frac{1}{4}, \frac{6x}{4x-1} < \frac{1}{2}$

Hint: Do by yourself to get x ∈ [-1/8, ¼).

18. A manufacturer has 600 liters of a 12% solution of acid. How many liters of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%?

Hint: Assume the amount of "to be added liters of 30% acid solution" to 600 liters = x.

New amount of mixture= (x+600) liters

Acid content in (x+600) liters mixture

= (30/100).x +(12/100).600

Given that, 15% of (x+600) <{(30/100).x +(12/100).600}

< 18% of (x+600)

⇒120<x<300

19. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If we have 640 liters of the 8% solution, how many liters of the 2% solution to be added?

Hint: Assume the "to be added solution of 2% boric acid"= x liters

The total quantity of mixture= (x+640) liters

From the given condition

8% of 640 + 2% of x > 4% of (x+640) and

8% of 640 + 2% of x < 6% of (x+640)

On combining the two results

x∈(320,1280)

20. If $\log_a$ (ab) = x, then find the value of $\log_b$ (ab).

Hint: Obviously, $\log_a$ (ab) = log ab/log a

⇒x= (loga+logb)/loga

⇒x= 1+ (logb/loga)

⇒(logb/loga)= (x-1)

And, $\log_b$ (ab) = (log a + log b)/log b

= 1 + (loga/logb)

= 1+ {1/(x-1)}=x/(x-1)

21. Find the value of x : $\log_2$(3x-2) = $\log_{1/2}$ x.

Hint: Rewrite the given equation at equal base as below:

log(3x-2)/(log2)

= (log x)/log(1/2)

=logx/(-log2)

i.e. log(3x-2)=-logx, or, 3x-2= (1/x)

x=1, -1/3, but log x will not be defined at x=-1/3 hence x=1

22. If $\log_{30}3$ = x, $\log_{30}5$ = y then find the value of $\log_{30}8$.

Hint: x= log3/(log10+log3)

=log3/(1+log3) , or log3= x/(1-x)

similarly, y=log5/(log10+log3)

=(log10-log2)/(log10+log3)

=(1-log2)/(1+log3)

or,

log2=(1-x-y)/(1-x)

after elimination of log 3 by its value x/(1-x).

now,

$\log_{30}8$ =3log2/(log10+log3)

=3log2/(1+log3)

=3(1-x-y) after putting values of log2 and log3.

23. ***Find the set of real values of x that satisfy*** $\log_{0.2}\left(\frac{x+2}{x}\right) \le 1$.

Hint: Note: when the base is less than 1, the sign of inequality changes and when the base is greater than 1, it remains unchanged.

$$\left(\frac{x+2}{x}\right) \ge (0.2)^1 = \frac{1}{5}$$

$\Rightarrow 5x(x+2) \ge x^2$

$\Rightarrow 4x^2+10x \ge 0$

$\Rightarrow 2x(2x+5) \ge 0$

Consider two cases:

Case I

$x \ge 0$ and $2x+5 \ge 0 \Rightarrow x \in [0, \infty)$

Case II

$x \le 0$ and $2x+5 \le 0 \Rightarrow x \in (-\infty, -5/2]$

Hence the required values are:

$x \in (-\infty, -5/2] \cup [0, \infty)$

24. ***If*** $\log_{\cos x} \sin x \ge 2$ ***and*** $0 \le x \le 3\pi$ ***then find the interval in which sin x lies***.

Hint:

Note: cos ≤ 1 hence the sign of inequality will change.

$\sin x \le (\cos x)^2$

$\Rightarrow \sin^2 x + \sin x - 1 \le 0$

$$\Rightarrow \left(\sin x + \frac{1}{2}\right)^2 \le \frac{5}{4} \Rightarrow \left(\sin x + \frac{1}{2}\right) \le +\frac{\sqrt{5}}{2}$$

as sin x should be always positive for defining log sin x

Hence,

$0 < \sin x \le (\sqrt{5} - 1)/2$

25. ***Find the set of real values of x satisfying*** $\log_{1/2} (x^2-6x+12) \ge -2$.

Hint:

Base is less than 1, hence the sign of inequality will change. i.e. $(x^2-6x+12) \le (1/2)^{-2} \Rightarrow x \in [2,4]$

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