MATHEMATICAL INDUCTION

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Mathematical induction is a method to prove the truthfulness of mathematical formulae. Sometimes, when it becomes cumbersome to prove easily the existence of some formulae, the method of Induction gives easier way to prove it.

Note:

(a) Induction means the generalisation of particular cases or facts.

(b) The principle of mathematical induction is a tool that is used to prove mathematical statements.

(c) The statement to be proved is assumed as **P(*n*)** associated with positive integer ***n***.

(d) The correctness for the case ***n* = 1** is tested first, then **P(*k*)** for some positive integer ***k*** is assumed to be true and on the basis of these two, the truth of **P(*k*+1)** is established.

Example

If we have to prove by using Principle of Mathematical Induction that

1+2+3+…+n = $\frac{n(n+1)}{2}$

then, we follow three steps as below:

First Step:

Put n=1 into the expressions of both sides and observe that it gives correct result, i.e, LHS=RHZ, i.e., 1=1 (in the above example)

Second Step:

We assume that the given expression is true for n =r {i.e, 1+2+3+….+r= $\frac{r(r+1)}{2}$ }.

Third Step:

We prove that the given expression is true for n= (r+1).{i.e, 1+2+3+…+(r+1)= $\frac{(r+1)(r+2)}{2}$ on the basis of the second step.

In the end, we conclude that it is true for **n=1**, true for **n=r** (a random natural number) and also true for the next **n=r+1**, hence it is true for **all natural numbers**.

Note:

(a) Proof by mathematical induction is not the invention of a particular individual.

(b) Mathematical induction was known to the Pythagoreans.

(c) The French mathematician Blaise Pascal is credited with the origin of the principle of mathematical induction.

(d) The name induction was used by the English mathematician John Wallis. Later the principle was employed to provide a proof of the binomial theorem.

(e) De Morgan contributed many accomplishments in the field of mathematics on many different subjects. He was the first person to define and name "**mathematical induction**" and developed De Morgan's rule to determine the convergence of a mathematical series.

(f) G. Peano undertook the task of deducing the properties of natural numbers from a set of explicitly stated assumptions, now known as Peano's axioms. The principle of mathematical induction is a restatement of one of the Peano's axioms.

Some Solved Examples

1. Prove that $7^{2n}+2^{3n-3}.3^{n-1}$ is divisible by 25, $n \in N$.

Solution:

Let us write the statement

P(n): $7^{2n}+2^{3n-3}.3^{n-1}$ is divisible by 25.

Step I

Put n=1 and test the Truthfulness.

P(n=1): LHS=$7^2+2^0.3^0$=49+1=50

Obviously, 50 is divisible by 25.

Step II

Assume that the given statement is true for n=r, i.e.,

P(n=r): $7^{2r}+2^{3r-3}.3^{r-1}$ is divisible by 25.

$\Rightarrow 7^{2r}+2^{3r-3}.3^{r-1}$ =25 k, where k is a natural number

Step III

Let us prove that P(n=r+1) is true on the basis of step II.

P(n=r+1): $7^{2r+2}+2^{3r}.3^r$ $=49.7^{2r}+24.2^{3r-3}.3^{r-1}$

$=(50-1).7^{2r}+(25-1).2^{3r-3}.3^{r-1}$

$=25[2.7^{2r}+2^{3r-3}.3^{r-1}]-[7^{2r}+2^{3r-3}.3^{r-1}]$

=25 (a natural number)-(a number divisible by 25)

= divisible by 25.

2. ***For all positive integer n, prove that (1/7) n^7 +(1/5) n^5 + (2/3)n^3 –(1/105) n is an integer.***

Hint for Solution:

Assume the statement

P(n): (1/7) n^7 +(1/5) n^5 + (2/3)n^3 –(1/105)n

is an integer

Put n=1 and test the truthfulness.

Now assume true for n=r

and test true for n=r+1

To prove true for n=r+1

P(r+1):

(1/7) $(r+1)^7$ +(1/5) $(r+1)^5$ + (2/3)$(r+1)^3$ –(1/105) (r+1)

=[(1/7) r^7 +(1/5) r^5 + (2/3)r^3 –(1/105) r] + (1/7)(multiple of 7)+ (1/7)+(1/5)[multiple of 5] + (1/5)+(2/3) [multiple of 3]+ (2/3)- (1/105)

= an integer + an integer + an integer + (1/5) + (2/3)- (1/105)= an integer

3. ***Prove that the product of two consecutive natural numbers is an even number.***

Hint for solution:

Let the statement be

P(n): n(n+1) is even.

Put n=1, and test the truthfulness.

Now assume the statement is true for n=r

i.e. P(r): r(r+1) is even and prove that P(r+1) is even.

Obviously,

P(r): r(r+1)= 2k (say)

Hence

P(r+1): (r+1)(r+2) =r(r+1)+2(r+1) =2k+ 2(r+1)

=2(some quantity) = even number

Hence true for all n.

4. ***Use Principle of Mathematical Induction to prove that $1^2+2^2+3^2+\ldots+n^2 > n^3/3$, n $\in$ N.***

Hint for Solution:

Assume the statement

P(n): $1^2+2^2+3^2+\ldots+n^2 > n^3/3$, n $\in$ N

Test the truthfulness for n=1

Obviously it is true.

Assume true for n=r i.e.

P(m): $1^2+2^2+3^2+...+r^2 > r^3/3$ is true

i.e. $1^2+2^2+3^2+...+r^2 = k+ (r^3/3)$.

Then

P(r+1): $1^2+2^2+3^2+...+(r+1)^2$

$= k+ (r^3/3)+ (r+1)^2$

$=k+\{(r+1)^3/3\} + \{r+(2/3)\} > (k+1)^3/3$

Hence true for all n.

5. ***If $x \neq y$, then prove that $x^n - y^n$ is divisible by x-y for every natural number n.***

Hint for solution:

Assume the statement

P(n): x^n-y^n is divisible by x-y.

Test the truthfulness for n=1. It is true.

Assume true for n=r i.e.

P(m): x^r-y^r is divisible by x-y.

i.e. $x^r- y^r = k (x-y)$.

Hence P(r+1): $x^{r+1} - y^{r+1} = x. x^r - y^{r+1}$

$=(x-y)(y^r+kx)$

i.e. True.

BINOMIAL THEOREM

Binomial theorem is a theorem by which any binomial of the form (x+a) can be raised to any desired positive integral power.

The expansion of $(x+a)^n$ is written as $(x+a)^n={}^nC_0x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+......+{}^nC_rx^{n-r}a^r+....+{}^nC_na^n$

Or, $(x+a)^n = C_0x^n + C_1x^{n-1}a + C_2x^{n-2}a^2 + ... + C_rx^{n-r}a^r + ... + C_na^n$, Where, $C_0, C_1, C_2, ...C_r, ...C_n$ are called Binomial Coefficients. In case of $(x-a)^n = C_0x^n - C_1x^{n-1}a + C_2x^{n-2}a^2 - ......$

Important Facts

1. The difference between the expansion of $(x+a)^n$ and $(x-a)^n$ is the alternate + and – signs of the odd and even terms. Numerically the place values of both the expansions are the same.
2. The **$(r+1)^{th}$ term** in the expansion of $(x+a)^n$ is called the **General Term** of the expansion and this is written as, $T_{r+1}={}^nC_rx^{n-r}a^r$
3. The simple form of this Theorem is represented as follows: $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_rx^r + ... + C_nx^n$
4. In case of Index n being a Positive Integer, the number of terms in the expansion is (n+1).
5. In any term of the expansion, the sum of the exponents is always equal to the index.
6. The expansion $(x+a)^n$ is a polynomial of n^{th} degree in ***x*** and ***a***.
7. The first term is x^n and the last term is a^n.
8. The power (i.e. index) of ***x*** decreases by 1 and that of ***a*** increases by 1 from left to right.
9. The number of terms in the expansion of $(x+y+z)^n$= ${}^{n+2}C_2 = \frac{(n+2)(n+1)}{2}$.
10. The Binomial Coefficients of terms equidistant from the beginning and the end are numerically equal.

11. The middle term is one $\left(\frac{n+1}{2}\right)th$ if the index is Even and are two $\left(\frac{n+1}{2}\right)th$ and $\left(\frac{n+3}{2}\right)th$ if the index is odd.
12. Value of ${}^{n}C_{r}$ is defined as $\frac{n!}{r!.(n-r)!}$.
13. If the term independent of x is to be determined in the expansion, then **First calculate the (r+1)th** term and **then put the index of x equal to zero** and **find the value of r**. The **(r+1)th term will be the required term**.
14. The greatest term is given by using the following working rule:
    (i) Find out T_{r+1} and T_r .
    (ii) Calculate $\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{n-r+1}{r}.\frac{2^{nd}Term}{1^{st}Term}\right| = \lambda$.
    (iii) If **λ is a positive integer** and equal to **m**, the **greatest term will be the m^th and (m+1)^th terms.**
    (iv) If **λ is a fraction** and equal to **(m+k)** where **m** is the integral part and **k is the fractional part**, then the greatest term will be **(m+1)^th term**.
15. The Sum of the binomial coefficients is *always* 2^n .i.e.
    $(2)^n = C_0 + C_1 + C_2 + ...+ C_r + ...+ C_n$
16. The sum of the binomial coefficients of the Odd terms is equal to the sum of the binomial coefficients of the even terms in the expansion i.e. $C_0 + C_2 + C_4 + ...$
    $= C_1 + C_3 + C_5 + ... = 2^{n-1}$
17. If n is a (+) positive integer and $a_1, a_2, ..., a_n \in R$ then $(a_1 + a_2 + ... + a_m)^n$
    $$= \sum \frac{n!}{n_1! \cdot n_2! \cdot n_3! \;......n_m!} a_1^{n_1} a_2^{n_2} ....a_m^{n_m}$$
    where $n_1, n_2, ... , n_m$ are all non-negative integers such that
    $n_1 + n_2 + n_3 + ... + n_m = n.$
18. The number of distinct terms in the expansion of $(a_1 + a_2 + ... + a_m)^n = {}^{n+m-1}C_{m-1}$.
19. Coefficient of $a_1^{n_1} a_2^{n_2} ....a_m^{n_m}$ in the expansion of $(a_1 + a_2 + ... + a_m)^n$ is $\frac{n!}{n_1! \cdot n_2! \cdot n_3! \;......n_m!}$.
20. Coefficient will be zero if $n_1+n_2+n_3+....+n_m$ is either greater than(>) or less than(<) n.
21. The greatest coefficient in the expansion of $(a_1 + a_2 + ... + a_m)^n$ is given by $\frac{n!}{(q!)^{m-r}.\{(q+1)!\}^r}$ where m= number of terms and(n= m.q+r) ; i.e. q is the quotient when n is divided by m and r is the remainder.
22. Number of different terms in the expansion of $(x+y+z)^n = (n+1)(n+2)/2$.
23. Number of different terms in the expansion of $\left(x + \frac{1}{x} + 1\right)^n$ is not ${}^{n+2-1}C_{2-1} = n+1$ but it will be (2n+1) as the expression is of the form $(x^2+x+1)^n$ or $(x+1)^{2n}$ i.e. some terms in the expansion will merge with similar terms.
24. The **rth term from the end** in the expansion of $(x+a)^n$ is given by=**T_r (from end)**
    =**T_{n-r+2}(from beginning)**.
25. Greatest binomial coefficient in the expansion of $(1 + x)^n$ is the coefficient of Middle term:
    (i) If *n* is even, then greatest coefficient = ${}^{n}C_{n/2}$.
    (ii) If *n* is odd, then greatest coefficient is ${}^{n}C_{\frac{n-1}{2}}$ or ${}^{n}C_{\frac{n+1}{2}}$.
26. If the greatest terms in the expansion of $(1 + x)^{2n}$ has the greatest coefficient then $\frac{n}{n+1} < x < \frac{n+1}{n}$.
27. If $(1 + x + x^2)^n = 1 + a_1x + a_2x^2 + a_3x^3 + ... + a_{2n}x^{2n}$, then Sum of all the coefficients (Not Binomial Coefficients but including it) in the expansion will be got by putting *x* = 1 on both the sides.

Some Results for Quick References

(i) ${}^{n}C_{r} = 0$ if $r < 0$

(ii) ${}^{n}C_{r} = {}^{n}C_{n-r}$

(iii) ${}^{n}C_{x} = {}^{n}C_{y} \Rightarrow x = y$ or $x + y = n$

(iv) $\dfrac{{}^{n}C_{r}}{{}^{n}C_{r-1}} = \dfrac{n-r+1}{r}$

(v) ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{(n+1)}C_{r}$

(vi) ${}^{n}C_{r} = \dfrac{n}{r} . {}^{(n-1)}C_{r-1}$

(vii) $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots\ldots + C_{n-r}C_n = {}^{2n}C_{n-r}$

${}^{m}C_{r} + {}^{m}C_{r-1} . {}^{n}C_{1} + {}^{m}C_{r-2} . {}^{n}C_{2} + \ldots\ldots {}^{n}C_{r} = {}^{(m+n)}C_{r}$

(viii) $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \ldots\ldots\ldots C_n^{\,2} = {}^{2n}C_n$

$= 2^n . \dfrac{1.3.5.....(2n-1)}{n!}$

(ix) $C_0^{\,2} - C_1^{\,2} + C_2^{\,2} - \ldots\ldots\ldots + (-)^n C_n^{\,2} = 0$, if n is odd.

(x) $C_0^{\,2} - C_1^{\,2} + C_2^{\,2} - \ldots\ldots\ldots + (-)^n C_n^{\,2} = (-1)^{\frac{n}{2}} . {}^{n}C_{n/2}$, if n is even.

(xi) $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \ldots\ldots + C_n^{\,2} = \dfrac{2n!}{n!.n!}$

(xii) $C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n+1)C_n = 2^{n-1}(n+2)$

(xiii) $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots.. + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

(xiv) $C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \ldots.. + \dfrac{(-1)^n C_n}{n+1} = \dfrac{1}{n+1}$

(xv) ${}^{n}C_{n} + {}^{n+1}C_{n} + {}^{n+2}C_{n} + \ldots.. + {}^{2n-1}C_{n} = {}^{2n}C_{n+1}$

(xvi) ${}^{n}C_{k}$ is divisible by n if n is prime and $1 \le k \le (n-1)$.

(xvii) ${}^{n}C_0 < {}^{n}C_1 < {}^{n}C_2 < \ldots.. < {}^{n}C_{\left[\frac{n-1}{2}\right]} = {}^{n}C_{\left[\frac{n}{2}\right]}$.

(xviii) **Pascal Law:** ${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$.

(xix) **Reciprocal Pascal Law**:

$$\frac{1}{{}^{n+1}C_r} + \frac{1}{{}^{n+1}C_{r+1}} = \left(\frac{n+2}{n+1}\right) . \frac{1}{{}^{n}C_r}.$$

(xx) If $\left(\sqrt{A} + B\right)^n = I + f$, where I and n are odd and $0 \le f < 1$ then $(I+f) = k^n$, where $A - B^2 = k > 0, \sqrt{A} + B < 1$.

(xxi) If n is even then $(1+f)(1-f) = k^n$.

(xxii) When the index n is not a positive integer, and is either a fraction or a negative integer, the Binomial Coefficient ${}^{n}C_{r}$ looses meaning and gives no value. The terms go on increasing and increasing and do not terminate i.e. the number of terms in the expansion of $(1+x)^n$ is not countable, but infinite. The expansion is meaningful only when $|x| < 1$.

Uses of Binomial of any Index Series

When the number of terms are infinite

(i) Assume the series $(1+x)^n$, if all the terms are positive and coefficients are of nature n(n-1)(n-2).......

(ii) Assume the series $(1-x)^n$, if all the terms are alternate (+)positive and (-) negative and the coefficients are of nature
n(n-1)(n-2).......

(iii) Assume the series $(1+x)^{-n}$, if all the terms are alternate (+)positive and (-) negative and the coefficients are of nature n(n+1)(n+2).......

(iv) Assume the series $(1-x)^{-n}$, if all the terms are positive and coefficients are of nature n(n+1)(n+2).......

(v) If by mistake we assume either way the result may not differ.

Expansions of Binomial for any Index

1. $(1+x)^n =$

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots. + \frac{n(n-1)(n-2)\ldots.(n-r+1)}{r!}x^r + \ldots$$

2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...$

3. $(1+x)^{-1} = 1 - x + x^2 - x^3 + ...$

4. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...$

5. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ...$

6. $(1-x)^{-3} = 1 + 3x + 6x^2 + ...$

Binomial Fallacy

In the expansion of

$(1-x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r + ...$

If we put x=2 then we get

$(-1)^{-1} = 1 + 2 + 2^2 + 2^3 + ... + 2^r + ...,$

which is impossible. Hence Binomial Theorem is true under some specific conditions.

Some Solved Examples

1. ***There are two bags each of which contains m balls. A man has to select an equal number of balls from both the bags. Find the number of ways in which the man can choose at least one ball from each bag.***

Solution:

We know that r balls out of m from one bag can be chosen in ${}^mC_r$ ways. So from the two bags we may choose in $({}^mC_r)^2$ ways.

So for at least one ball from the two bags, it will be

$$({}^mC_1)^2 + ({}^mC_2)^2 + ({}^mC_3)^2 + ... + ({}^mC_m)^2$$

This can be calculated using binomial expansions of $(1+x)^n$ and that of $\left(1+\frac{1}{x}\right)^n$ as below:

${}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n x^n$ ....(i)

$\left(1+\frac{1}{x}\right)^n = {}^nC_0 + {}^nC_1 \frac{1}{x} + {}^nC_2 \frac{1}{x^2} + ... + {}^nC_n \frac{1}{x^n}$ ...(ii)

On multiplication of (i) and (ii), the constant terms from both sides generate the required result as

$$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + ... + ({}^nC_n)^2 = {}^{2n}C_n$$

2. ***The sum of the coefficients in the expansion of $(x+y)^n$ is 4096, then show that the greatest coefficient in the expansion is 924.***

Hint for solution:

Use the result that sum of the coefficients= 2^n.

And if 2^n = 4096, then n = 12.

So the greatest binomial coefficient= ${}^{12}C_6$.

3. ***Find the coefficient of x^k ($0 \leq k \leq n$)in the expansion of $E=1+(1+x)+(1+x)^2+...+(1+x)^n$.***

Hint for solution:

Observe that the expression on the RHS is a geometric series of (n+1) terms.

Calculate their sum $\frac{((1+x)^{n+1}-1)}{(1+x)-1}$ using the formula

$\frac{a(r^n-1)}{r-1}$ where a is first term and r is the common ratio.

Find the coefficient of x^k.

The result is ${}^{n+1}C_{k+1}$.

4. ***If the coefficient of $(2r+4)^{th}$ and $(r-2)^{th}$ terms in the expansion of $(1+x)^{18}$ are equal, find r.***

Hint for solution:

${}^{18}C_{(2r+4)-1} = {}^{18}C_{(r-2)-1}$. Calculate r.

5. ***Find the value of*** $C_2 + 2C_3 + 3C_4 + ... + 14C_{15}$.

Hint for solution:

Rewrite the given expression as:

$$(2-1)C_2 + (3-1)C_3 + (4-1)C_4 + ... + (15-1)C_{15}.$$

= $2C_2 + 3C_3 + 4C_4 + ... + 15C_{15} - (C_2 + C_3 + C_4 + ... + C_{15})$

= $15(C_1 + C_2 + C_3 + ... + C_{14}) - (C_2 + C_3 + C_4 + ... + C_{15})$.

Simplify to get the required answer.

6. ***Show that*** $\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6 = 198$. ***Hence show that the Integral part of*** $\left(\sqrt{2}+1\right)^6$ ***is 197.***

Hint for solution:

Expand using binomial theorem and prove the first part.

Observe that $\sqrt{2} - 1 = 1.4 - 1 = 0.4 < 1 \Rightarrow (0.4)^6 < 1$.

So 198- something less than unity= 197+ fractional part.

7. ***If the coefficients of second, third and fourth terms in the expansion of*** $(1+x)^{2n}$ ***are in AP then show that*** $2n^2 - 9n + 7 = 0$.

Hint for solution:

Use binomial coefficients of T_2, T_3 and T_4 terms to make an AP i.e. $2T_3 = T_2 + T_4$

8. ***If*** a_1, a_2, a_3, a_4 ***are the coefficients of any four consecutive terms in the expansion of*** $(1+x)^n$, ***prove that*** $\dfrac{a_1}{a_1+a_2} + \dfrac{a_3}{a_3+a_4} = \dfrac{2a_2}{a_2+a_3}$.

Hint for solution:

Assume coefficients of $(r+1)^{th}$, $(r+2)^{th}$, $(r+3)^{th}$ and $(r+4)^{th}$ terms of the expansion as a_1, a_2 , a_3, a_4 , i.e. $a_1 + a_2 = T_{r+1} + T_{r+2}$

Use the formula: ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$ (Pascal law) and simplify as per requirement.

9. ***Given that*** $S_n = 1 + q + q^2 + ... + q^n$ ***and***

$\sigma_n = 1 + \dfrac{q+1}{2} + \left(\dfrac{q+1}{2}\right)^2 + ... + \left(\dfrac{q+1}{2}\right)^n$ ***and*** $q \neq 1$, ***then prove that***

$${}^{n+1}C_1 + {}^{n+1}C_2.S_1 + {}^{n+1}C_3.S_2 + ... + {}^{n+1}C_{n+1}.S_n = 2^n \sigma_n$$

Hint for solution:

Calculate S_n and σ_n using Geometric series sum to n terms.

Put values of S_1, S_2, S_3, ... etc and $\sigma_1, \sigma_2, \sigma_3, ... \sigma_n$ etc in the expression ${}^{n+1}C_1 + {}^{n+1}C_2.S_1 + {}^{n+1}C_3.S_2 + ... + {}^{n+1}C_{n+1}.S_n$

And breaking into two parts solve as under:

$\left(\dfrac{1}{1-q}\right)\left({}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + ... + {}^{n+1}C_{n+1}\right)$- $\left(\dfrac{1}{1-q}\right)\left({}^{n+1}C_1 q + {}^{n+1}C_2.q^2 + {}^{n+1}C_3.q^3 + ... + {}^{n+1}C_{n+1}.q^{n+1}\right)$ Simplify and get the required result.

10. ***Find the sum of the series***

$$\sum (-1)^r . {}^nC_r \left(\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + ...m..terms\right)$$

Hint for solution:

Solve all terms separately and get the given expression

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + ...m..terms$$

$=\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n+\left(\frac{1}{8}\right)^n+....m..terms$

Use formula for sum to m terms of a GP.

11. Show that $3^{2n+2} - 8n - 9$ is divisible by 64, $n \in N$.

Hint:

We know that $(1+x)^n$-nx-1 is divisible by x^2.

Hence if we choose x 8 ; the required expression is achieved, i.e. $(1+8)^n$-n8-1 is divisible by 8^2.

Or 9^n-8n-1 is divisible by 64.

12. What is the sum of the coefficients of the polynomial $(1+x-3x^2)^{2143}$?

Hint:

Put x=1 in the given expression to get the required sum.

13. Find the last three digits of 17^{256}.

Hint:

Note that 172 = 289 = 290- 1

$\Rightarrow 17^{256} = (17^2)^{128} = (290-1)^{128}$

$=\left({}^{128}C_0(290)^{128}+{}^{128}C_1.(290)^{127}(-1)^1+.....+{}^{128}C_{128}.(-1)^{128}\right)$=an integer as all the terms are integers.

$$= 1000n+\frac{128\times127}{2}\times(290)^2-128\times290+1$$

=1000 n+683527680 +1

=1000(k+683527)+681

14. What is the remainder when $32^{32^{32}}$ is divided by 7 ?

Hint:

Note $32=2^5$

$\Rightarrow (32)^{32} = (2^5)^{32} = 2^{160} = (3-1)^{160}$

$=\left({}^{160}C_0.3^{160}-{}^{160}C_1.3^{159}+{}^{160}C_2.3^{158}-.....-{}^{160}C_{159}.3\right)+1.$

= (3k+1) form , where k is a natural number

Now the required expression becomes

$32^{3k+1} = (2^5)^{3k+1} = (2)^{15k+5} = 2^2.\ 2^{3(5k+1)}$

$=4.(8)^{5k+1} = 4.(1+7)^{5k+1}$

$=4.\left({}^{5k+1}C_0+{}^{5k+1}C_1.7+{}^{5k+1}C_2.7^2+.....+{}^{5k+1}C_{5k+1}.7^{5k+1}\right)=4+4.7\left({}^{5k+1}C_1+{}^{5k+1}C_2.7+.....+{}^{5k+1}C_{5k+1}.7^{5k}\right).$

=4+28 (a natural number)

Hence the remainder is 4.

15. Prove that the square of any odd number is of the form (2n+1) or (4p+1) or (8m+1).

Hint:

Note that

$(2n+1)^2 = 4n^2 + 4n+1 = 4n(n+1) + 1$

=8m+1 as n(n+1) is always an even.

=4(2m)+1 =4p+1

16. Which is larger of $99^{50} + 100^{50}$ and 101^{50} ?

Hint:

Use expansions

$101^{50} = (100+1)^{50}$

$99^{50} = (100-1)^{50}$

And $101^{50}-99^{50} = 100^{50}$+terms containing only positive terms

Hence $101^{50} > 99^{50}$

17. Find the coefficient of x^{50} in the expansion

$(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}+...+1001x^{1000}$

Hint:

Observe the series ; it appears to be Arithmetico-geometric type.

Assume the series

$S=(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}+\ldots+1001x^{1000}$.... (i)

Multiply both sides by x/(1+x) and then subtract it from the (i) by using Method of difference.

Get

$S=(1+x)^{1001}+x(1+x)^{1000}+x^2(1+x)^{999}+\ldots+(1+x)x^{1000}-1001x^{1001}$

Obviously the series on the RHS is a geometric one and on simplification,

Get, $(1+x)^{1002} - x^{1002} - 1002\, x^{1000}$

Now coefficient of x^{50} in the RHS expression.

18. ***Find the coefficient of $x^{10}y^{12}z^8$ in the expansion of $(xy+yz+zx)^{15}$.***

Hint:

Write the general term as

$$T_{r+1} = \frac{15!}{p!q!r!}(xy)^p.(yz)^q.(zx)^r$$

Note the essential condition p+q+r= 15

Also $x^{p+r}\, y^{p+q}\, z^{q+r} = x^{10}y^{12}z^8$

i.e. p+r= 10, p+q=12 and q+r=8

Solve and find the values of p, q, and r and evaluate the required coefficient.

19. ***Show that $(1+x+x^2+x^3+\ldots)^2 = 1 + 2x + 3x^2 + 4x^3 +\ldots.$***

Hint:

LHS is an infinite geometric series. Use formula for the sum to infinity: a/(1-r) where r<1.

Now use Binomial theorem to expand the expression to get the RHS.

20. ***Find the sum of the following infinite series***

$$1+\frac{1}{6}+\frac{1.4}{6.12}+\frac{1.4.7}{6.12.18}+\frac{1.4.7.10}{6.12.18.24}+\ldots\ldots$$

Hint:

Assume the series to be expansion of $(1+x)^n$.

Equate the terms on the RHS to get the values of x and n.

21. ***A student puts n=0 in the statement***

$$(a+b)^n = a^n + nba^{n-1} + \frac{n(n-1)}{2!}b^2a^{n-2} + .$$

$..+ nab^{n-1} + b^n$ ***of binomial theorem and proved 1 =2. Do you agree with this? If not, why?***

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