PROBABILITY

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Measure of various phenomenon generates the theory of probability. If we do some experiment, then there are two possibilities: either the result is known or the result is not known. The experiment whose result is known is called ***deterministic experiment****. All science experiments under the similar conditions may be named so. The experiment that does not produce the same result if performed under the same identical conditions is called the* ***random experiment****. The* ***theory of probability*** *starts from the random experiment.*

Tossing of a fair coin having Head on one side and Tail on the other side may result in any one (i.e. head or tail), but the double headed coin i.e. having heads on both sides only will always result in Head. First one is deterministic while the other case is of the random experiment.

Note:

(a) Probability or the theory of probability is a branch of mathematics that deals with the measurement of the likelihood of an event or experiment to have a particular outcome quantitatively.

(b) Probability is based on the study of permutations and combinations as they are used to count the number of possible arrangements of a system.

(c) Probability of an event may be any real number between 0 and 1.

(d) In 1654, a gambler Chevalier de Mere approached the well-known French philosopher and mathematician Blaise Pascal (1623-1662 AD) for certain dice problems. Pascal took interest in it and discussed with famous French mathematician Pierre de Fermet (1601-1665 AD) and together they solved the problems.

Important Terms

Sample Space

It is the set of all possible outcomes of random experiment.

Examples:

(a) If a coin is tossed, the Sample space = {H,T}
(b) If Two coins are tossed the Sample space = {HH,HT,TH,TT}
(c) If a Dice is thrown, the Sample space = {1,2,3,4,5,6}
(d) If two dice are thrown, the Sample Space= {(1,1),1,2),......(6,6)}
(e) If a dice of m faced is thrown n times, the Sample space = m^n.

Outcome:

It is the element of the sample space.

Equally likely outcomes

The outcomes that have the equal probability.

Trial

A random experiment repeated under identical conditions.

Simple event or Elementary event

Subset of sample space, Or each outcome of the random experiment, Or if an event can have only one sample point.

Occurrence of an event

An event E of the sample space S is said to have occurred if the outcome ω of the experiment is such that $\omega \in E$. and contrary to this if $\omega \notin E$, then the event E is said to not occur.

Types of Events

(a) Impossible event

The empty set, Or the event whose probability is 0, i.e, if P(E)=0 then E is an impossible event.

Example:

In toss of a coin coming up of neither head not tail is zero.

(b) Sure event

Happening of the whole sample space, Or whose probability is 1,

Example:

In tossing a coin, coming up of head or tail is sure.

(c) Compound event

A subset of the sample space is called compound event if it is disjoint union of single element subsets of the sample space.

Note:

Suppose there are n points in a sample space. So there will be n simple events. There will be 2^n subsets of these n elements. Then $2^n-(n+1)$ will be compound events excluding null set.

(d) Complementary event

The set A' or S-A. This event is also called "not event".

(e) Mutually exclusive events

A and B are mutually exclusively iff $A \cap B = \phi$ i.e, a null set.

(f) Exhaustive events

If $E_1 \cup E_2 \cup E_3 \cup ...... \cup E_n = S$ i.e. the sample space then all these are called exhaustive events.

(g) Odds in favour and Odds against

If an event can occur in m ways and cannot occur in n ways then **odds in favour** is given by m:n and **odds against** is given by n:m.

(h) The Probability of happening (odds in favour) is given by $\dfrac{m}{m+n}$

(i) The Probability of not happening (odds against) is given by $\dfrac{n}{m+n}$

Mathematical Definition

(a) For a finite sample space with equally likely outcomes, probability is defined as

$$P(Event) = \frac{number\ of\ elements\ in\ E}{number\ of\ elements\ in\ S}$$

(b) Probability of the event A or B:

$$P(AorB) = P(A \cup B)$$
$$= P(A) + P(B) - P(A \cap B)$$

(c) Probability of an event "not A":

$$P(notA) = 1 - P(A)$$

(d) Probability "not A" is denoted by $P(A')$ or $P(\overline{A})$ or $P(A^c)$

(c) $P(A - B) = P(A \cap B^C)$
$= P(A) - P(A \cap B)$

Conditional Probability

If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred, is represented by

$$P(E/F) = \frac{P(E \cap F)}{P(F)}, P(F) \neq 0$$

Properties of Conditional Probability

(i) If E and F be events of a sample space S of an experiment, then $P\left(\frac{S}{F}\right) = P\left(\frac{S}{E}\right) = 1$.

(ii) For P(F)≠ 0, $P\left(\frac{A \cup B}{F}\right) = P\left(\frac{A}{F}\right) + P\left(\frac{B}{F}\right) - P\left(\frac{A \cap B}{F}\right)$ where A and B are two events associated with S.

Multiplication Theorem on Probability

For Two events

If E and F are two events associated with a sample space S, then

(a) $P(E \cap F) = P(E).P\left(\frac{F}{E}\right)$,P(E)≠0

(b) $P(E \cap F) = P(F).P\left(\frac{E}{F}\right)$,P(F)≠0

For Three events

If *E, F, G* are three events of sample space, then

(a) $P(E \cap F \cap G) = P(E).P\left(\frac{F}{E}\right)P\left(\frac{G}{E \cap F}\right)$; P(F)≠0

Independent Events

Two events *E* and *F* are said to be independent if the probability of happening of one of them is not affected by the occurrence of the other.

$$P\left(\frac{E}{F}\right) = P(E), \text{ P(F)≠0}$$

$$P(E \cap F) = P(E).P(F)$$

Independent and Mutually Exclusive Events

(a) If E and F are independent events associated with a random experiment then $\overline{A}$ and B; A and $\overline{B}$; or $\overline{A}$ and $\overline{B}$ are also independent events.

(b) Independent events are always taken from different experiments, while mutually exclusive events are taken from a single experiment.

(c) Independent events can happen together while mutually exclusive events cannot happen together.

(d) Independent events are connected by the word "and " but mutually exclusive events arr connected by the word "or".

Total Probability

If $\{E_1,E_2,E_3,...E_n\}$be the partition of the sample space S, and A be any event associated with S, then

$$P(A)=\sum_{i=1}^{n}P(E_i)P\left(\frac{A}{E_i}\right)$$

Bayes' Theorem

If $E_1,E_2,E_3,...E_n$ are mutually exclusive and exhaustive events associated with a sample space, and A be any event of non-zero probability, then

$$P\left(\frac{E_i}{A}\right)=\frac{P(E_i).P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^{n}P(E_i).P\left(\frac{A}{E_i}\right)}$$

Note:

$$P(A)+P(B)-1\le P(A\cap B)\le P(A)$$

Note

(a) Coin

A coin has a head side and a tail side. If an experiment consists of more than a coin, then coins are considered to be distinct if not otherwise stated.

(b) Dice

A die (cubical) has six faces marked 1, 2, 3, 4, 5, 6. We may have tetrahedral (having four faces 1, 2, 3, 4) or pentagonal (having five faces 1, 2, 3, 4, 5) die. Number of exhaustive cases of throwing *n* dice simultaneously (or throwing one dice *n* times)=6^n.

(c) Playing cards

A pack of playing cards contains 52 cards. There are 4 suits: *Spade, Heart, Diamond, Club*). Each has 13 cards. *Spade* and *Club* are of black colour cards and *Heart* and *Diamond* are of Red Colour, i.e. there are 26 Black cards and 26 Red cards. In 13 cards of each suit there are *Ace, 2,3,4,5,6,7,8,9,10, Jack, Queen, King*. ***Jack, Queen*** and ***King*** are called Face cards. *Ace, King, Queen* and *jack* are called honour cards.

Note:

Regarding mappings

Let A and B be two finite sets. If a mapping is to be selected randomly from A into B, then the

(a)Probability for being one-one function(mapping) = $\frac{{}^{B}P_{A}}{B^{A}}$

(b) Probability for being many-one function = 1- $\frac{{}^{B}P_{A}}{B^{A}}$

(c)Probability for being a constant function = $\frac{n(B)}{B^{A}}$

(d) Probability for being one-one-onto function = $\frac{n(A)}{B^{A}}$; $n(B)=n(A)$

Probability for selecting squares on a Chess board

r squares are selected on a chess-board. The probability of them being along a diagonal= $\frac{4\left[{}^{7}C_{r}+{}^{6}C_{r}+.....+{}^{r}C_{r}\right]+{}^{8}C_{r}}{{}^{64}C_{r}},1\le r\le 7$

Probability for drawing shoes from cupboard

Let there be *n* pairs of Shoes. If *r* shoes are selected at random, then the

(a)Probability that there is no pair = p = $\frac{{}^{n}C_{r}.2^{r}}{{}^{2n}C_{r}}$

(b) Probability of being at least one pair = *1-p*.

Probability related to envelopes and letters

(a)Probability that all letters are in right envelopes= $\frac{1}{n!}$.

(b) Probability that all letters are not in right envelopes (or at least one is in wrong envelope)=$1-\frac{1}{n!}$.

(c)Probability that no letter is in right envelopes = p = $\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-....+(-)^{n}\frac{1}{n!}$.

(d) Probability that at least one letter is in right envelope = 1-p

(e)Probability that exactly *r* letters are in right envelopes

= $\frac{1}{r!}\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-....+(-)^{n-r}\frac{1}{(n-r)!}\right)$.

Measure of central tendency:

It gives an idea where data points are centered. It consists of Mean, Median and Mode.

Mean

It is denoted by $\bar{x}$. If the data are $x_1, x_2, x_3,\ldots x_n$ then mean is denoted by

(i) $\bar{x} = \dfrac{x_1 + x_2 + x_3 + \ldots\ldots + x_n}{n} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$

(ii) $\bar{x} = \dfrac{f_1x_1 + f_2x_2 + f_3x_3 + \ldots\ldots + f_3x_n}{f_1 + f_2 + \ldots\ldots + f_n}$;

$= \dfrac{1}{N}\sum_{i=1}^{n} f_ix_i$;

f_i is corresponding frequencies of x_i

(iii) $\bar{x} = \dfrac{f_1d_1 + f_2d_2 + f_3d_3 + \ldots\ldots + f_3d_n}{f_1 + f_2 + \ldots\ldots + f_n}$

$= \dfrac{1}{N}\sum_{i=1}^{n} f_id_i, d_i = x_i - a,$

where a= assumed mean.

Median

It is the value of the middle term or terms when the data are arranged in increasing or decreasing order. It is denoted by

(a) M=$\left(\dfrac{n+1}{2}\right)th$ if the number of terms is odd.

(b) M=mean of $\left(\dfrac{n}{2}\right)th$ and $\left(\dfrac{n}{2}+1\right)th$ if the number of terms is even.

(c) For class interval data,

M= $L + \dfrac{\frac{N}{2} - F}{f} \times i$, where N= total number of frequency; F= cumulative frequency before median class; f= frequency of the median class; L= Lower limit of the median class; i=class interval of the median class.

Mode

It is the observation that occurs maximum number of times. It is denoted by

$$Mode = L + \frac{f - f_1}{2f - f_1 - f_2} \times i$$

where L= lower limit of Modal class, f= frequency of the modal class, f_1= frequency of the class preceding the modal class, f_2= frequency of the class following the modal class, i= class interval of the modal class.

Note: Mode = 3 Median – 2 Mean

Measure of Dispersion

Dispersion means scattered ness of data. It is measured on the basis of the measure of central tendency i.e, mean, mode or median. It is mainly of 4 types:

Range

It does not give any idea about the dispersion of data from a measure of central tendency as no central tendency is considered here.

Range = (Maximum value of the data)- (Minimum Value of the data)

Quartile deviation

Q_1= $L + \dfrac{\frac{N}{4} - F}{f} \times i$

Mean Deviation

(a) Mean deviation about mean

$= \dfrac{\textit{sum of absolute deviations from mean}}{\textit{number of observations}}$

(b) Mean deviation (about assumed mean $\boldsymbol{a}$)

$= \dfrac{\sum_{i=1}^{n} |x_i - a|}{n}$

(c) Mean deviation in case of grouped data

$= \dfrac{\sum_{i=1}^{n} f_i|x_i - a|}{N}$ where

N = sum of all frequencies.

(d) Mean deviation (about mean $\bar{x}$)

$= \dfrac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$

(e) In case of grouped data

$$= \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{N}$$

(f) Mean deviation about median M

$$= \frac{\sum_{i=1}^{n} |x_i - M|}{n}$$

(g) Mean deviation (grouped data)

$$= \frac{\sum_{i=1}^{n} f_i |x_i - M|}{N}$$

Standard Deviation

$$= \sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

Standard deviation for frequency distribution

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \bar{x})^2}$$

A short-cut method to avoid calculation of mean $\bar{x}$ is used

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{n} f_i x_i^2 - \left(\frac{1}{N} \sum_{i=1}^{n} f_i x_i \right)^2}$$

Variance= (standard deviation)^2 =σ^2

If the series have equal means, then the series with lesser standard deviation is more consistent or less scattered.

Coefficient of variance= $\frac{\sigma}{Mean} \times 100$

Random Variable and its Probability distribution

A random variable is a real valued function whose domain is the sample space of a random experiment.

The probability distribution of a random variable X is the system of numbers.

X	X_1	X_2	X_3	..	X_n
$P(X)$	p_1	p_2	p_3	..	p_n

where $p_i \neq 0$, $i=1, 2, 3, \ldots, n$ and $\sum_{i=1}^{n} p_i = 1$.

Mean and variance of a random variable

Mean = $\mu = E(X) = \sum_{i=1}^{n} p_i x_i$

Variance = $\sigma^2 = \sum_{i=1}^{n} p_i (x_i - \bar{x})^2$

Bernoulli Trials

Trials of a random experiment are called Bernoulli trails if they satisfy the following conditions:

(a) Number of trials are finite

(b) Trials are independent

(c) Each trial has only two outcomes: success and failure

(d) All trials have the same probability of success or failure

Binomial distribution

A random variable X is said to have a binomial distribution with parameters n and p if its probability distribution is given by

$$P(X = r) = {}^nC_r p^r q^{n-r}$$

where $q=1-p$ and $r=0,1,2,3,..n$

Solved Examples:

Ex. 1: An ordinary deck of cards contains 52 cards divided into four suits. If one card from the deck at random is picked up, then

(a) Find the sample space of the experiment.

(b) Find the event that the chosen card is a black face card.

Solution 1:

(a) The outcomes in the sample space S are 52 cards in the deck.

(b) Let E be the event that a black face card is chosen. The outcomes in E are Jack, Queen, King or spades or clubs.

Symbolically

E = {J, Q, K, of spades and clubs}, Or

E = {J♣, Q♣, K♣, J♠, Q♠, K♠}

Ex. 2: An experiment has four possible outcomes A, B, C and D, that are mutually

exclusive. Explain whether the following assignments of probabilities are permissible.

(a) $P(A) = 0.12$, $P(B) = 0.63$, $P(C) = 0.45$, $P(D) = -0.20$

(b) $P(A) = \frac{9}{120}$, $P(B) = \frac{45}{120}$, $P(C) = \frac{27}{120}$, $P(D) = \frac{46}{120}$

Solution 2:

(a) Since $P(D) = -0.20$, this is not possible as $0 \leq P(A) \leq 1$ for any event A.

(b) $P(S) = P(A \cup B \cup C \cup D) = \frac{9}{120} + \frac{45}{120} + \frac{27}{120} + \frac{46}{120} \neq 1$

This violates the condition that $P(S) = 1$.

Ex. 3: Three squares of chess board are selected at random. Find the probability of getting 2 squares of one colour and other of a different colour.

Solution 3:

In a chess board, there are 64 squares of which 32 are white and 32 are black. Since 2 of one colour and 1 of other can be

2W, 1B, or 1W, 2B,

The number of ways= ${}^{32}C_2 \times {}^{32}C_1 \times 2$ and also, the number of ways of choosing any 3 boxes is ${}^{64}C_3$.

Hence, the required probability

$$= \frac{{}^{32}C_2 \times {}^{32}C_1 \times 2}{{}^{64}C_3} = \frac{16}{21}$$

Ex. 4: A and B are two candidates seeking admission in a college. The probability that A is selected is 0.7 and the probability that exactly one of them is selected is 0.6. Find the probability that B is selected.

Solution 4:

Let p be the probability that B gets selected.

P (Exactly one of A, B is selected) = 0.6 (given)

P (A is selected, B is not selected; B is selected, A is not selected) = 0.6

$P(A \cap B') + P(A' \cap B) = 0.6$

$P(A)\, P(B') + P(A')\, P(B) = 0.6$

$\Rightarrow (0.7)(1 - p) + (0.3)\, p = 0.6$ *Or* $p = 0.25$

Thus the probability that B gets selected is 0.25.

Ex. 5: 10% of the bulbs produced in a factory are of red colour and 2% are red

and defective. If one bulb is picked up at random, determine the probability of its being defective if it is red.

Solution 5:

Let A and B be the events that the bulb is red and defective, respectively.

$P(A) = \frac{10}{100}$, $P(A \cap B) = \frac{2}{100}$; $P(B \mid A) = P(A \cap B)/P(A)$

$$= \frac{\frac{2}{100}}{\frac{10}{100}} = \frac{2}{10} = \frac{1}{5}$$

Thus the probability of the picked up bulb of its being defective, if it is red, is1/5.

Ex. 6: Find the probability that in 10 throws of a fair die a score which is a

multiple of 3 will be obtained in at least 8 of the throws.

Solution 6:

Here success is a score which is a multiple of 3 i.e., 3 or 6.

Therefore, p (3 or 6) = $\frac{2}{6}$

The probability of r successes in 10 throws is given by

$$P(r) = {}^{10}C_r \left(\frac{4}{6}\right)^{10-r} \left(\frac{2}{6}\right)^r$$

P (at least 8 successes)

= P (8) + P (9) + P (10)

$$= {}^{10}C_8 \left(\frac{4}{6}\right)^2 \left(\frac{2}{6}\right)^8 + {}^{10}C_9 \left(\frac{4}{6}\right)^1 \left(\frac{2}{6}\right)^9 + {}^{10}C_{10} \left(\frac{2}{6}\right)^{10} = \frac{201}{3^{10}}.$$

Ex 7: A car manufacturing factory has two plants, X and Y. Plant X manufactures

70% of cars and plant Y manufactures 30%. 80% of the cars at plant X and 90% of the cars at plant Y are rated of standard quality. A car is chosen at random and is found to be of standard quality. What is the probability that it has come from plant X?

Solution

Let E be the event that the car is of standard quality. Let B1 and B2 be the

events that the car is manufactured in plants X and Y, respectively. Now

$P(B_1) = \frac{70}{100}$, $P(B_2) = \frac{30}{100}$

$P(E \mid B_1)$ = Probability that a standard quality car is manufactured in plant $= \frac{80}{100}$

$P(E \mid B2) = \frac{90}{100}$

P (B1 | E) = Probability that a standard quality car has come from plant X

$$= \frac{P(B_1)\times P(E/B_1)}{P(B_1)\times P(E/B_1)+P(B_2)\times P(E/B_2)}$$

$$= \frac{\frac{70}{100}\times\frac{80}{100}}{\frac{70}{100}\times\frac{80}{100}+\frac{30}{100}\times\frac{90}{100}} = \frac{56}{83}.$$

Dr S.B. Dhar, is **Editor of this Quartrerly e-Bulletin**. He is an eminent mentor, analyst and connoisseur of Mathematics from IIT for preparing aspirants of Competitive Examinations for Services & Admissions to different streams of study at Undergraduate and Graduate levels using formal methods of teaching shared with technological aids to keep learning at par with escalating standards of scholars and learners. He has authored numerous books – Handbook of Mathematics for IIT JEE, A Textbook on Engineering Mathematics, Reasoning Ability, Lateral Wisdom, Progress in Mathematics (series for Beginner to Class VIII), Target PSA (series for class VI to class XII) and many more.

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