BASIC CALCULUS

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Calculus is the study of how things change. The development of calculus and its applications to physics and engineering is probably the most significant factor in the development of modern science beyond where it was in the days of Archimedes. In this article, we shall restrict ourselves to the knowing of the Basics of the Calculus. We shall elaborate its uses in day to day life in some coming issue.

The present day's calculus has been divided into two parts:

(a) Differential Calculus, and

(b) Integral Calculus.

In differential calculus, students are taught Functions, their nature, domain, range, continuity, discontinuity, value of functions, limits, differentiability, uses of derivatives to find the nature of curves etc.

In integral calculus, students learn how to find the anti-derivatives, their nature, their applications in evaluating the area, volume etc., of the bounded region.

Let us have a bird's eye view on the facts one by one.

FUNCTION

Definition:

If X and Y are two non-empty sets, we say that *f* is a function from X to Y if for every element *x* of X, there exists one and only one element *f(x)* in Y. The set X is called the domain. The set Y is called the codomain of *f*. The set of values of *f(x)* is called the range of *f*. The Range is always a subset of Codomain. Domain is represented by D*f* and Range is represented by R*f*.

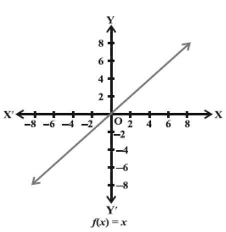
It is represented as $f : X \to Y$ or $y=f(x)$ where $x\in$X and $y\in$Y. The function *f* is called a real-valued function if for every real $x\in$X, there exists a real $y\in$Y.

TYPES OF FUNCTION

Identity Function

Let R be the set of real numbers. Identity function is defined as f : R$\to$ R by y = f(x) = x for each x$\in$R.

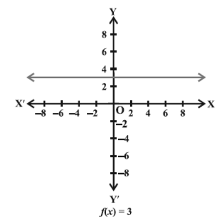
Domain of this function is R and the Range is also R.



Constant Function

Constant function is defined as $f : R\to R$ by $y = f(x) = c$, where c is a constant, for each $x \in R$.

Its domain is R and its range is only {c}.



Domain of the function in the graph is R and the Range is {3}.

Polynomial Function

It is defined as $f : R\to R$ where $y = f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, n is a non-negative integer and $a_0, a_1, a_2, ... a_n$ are Real Numbers.

Examples:

(a) $f(x) = 2 + 3x + 4x^2$

(b) $f(x) = 5 + \sqrt{6}x + 9x^4$

Linear function

A function *f* defined for all real *x* by a formula of the form $f(x) = ax + b$, is called a linear function because its graph is *a straight line*.

In other words, we may say that all the polynomial functions whose highest degree is equal to 1 are called the Linear functions.

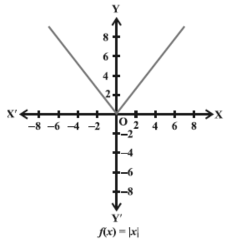
Rational Function

$y = h(x) = \dfrac{f(x)}{g(x)}$, where *f(x)* and *g(x)* are polynomial functions. This function is defined only when g(x) ≠ 0.

Domain for this function is the domain common for both *f(x)* and *g(x)* for which *y* is defined.

Modulus Function

It is defined as $f(x) = |x|$, x $\in$R.



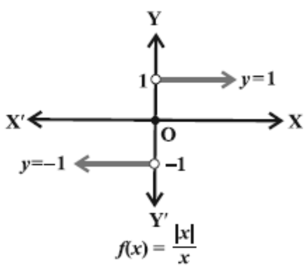
Its domain is R and its Range is [0,∞)

Signum Function

It is defined as $f(x) = \dfrac{|x|}{x}$ or

$f(x) = \dfrac{x}{|x|}$, x ∈R-{0}.

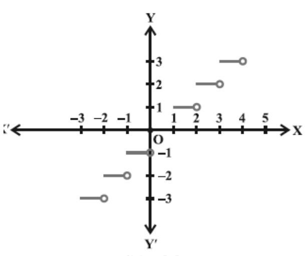
It is also defined as $f(x) = \begin{cases} 1, x > 0 \\ -1, x < 0 \\ 0, x = 0 \end{cases}$



The domain is R and the range is {-1,0,1}.

Greatest Integer Function

It is defined as $f(x) = [x]$



Domain of this function is R and the range is set of all Integers.

Examples:

$$\left[\frac{1}{2}\right] = 0, \left[-\frac{1}{2}\right] = -1, \left[3\frac{1}{2}\right] = 3, \left[-4\frac{1}{2}\right] = -5$$

Composite function

Let *f* and *g* be real valued functions of a real variable. The composite function of *g* and *f* is denoted by *gof* or *(gof)(x)=g(f(x))* for all $x \in dom\, f$ such that $f(x) \in dom\, g$.

Periodic function: A function *f(x)* is said to be periodic if there exists such a T>0 for which *f(x+T)=f(x-T)=f(x)* for all $x \in X$.

Note:

There are infinitely many T satisfying the equality but the least positive is said to be the period.

All six basic trigonometric functions are periodic functions. Period of *sinx, cosx, cosecx, secx is 2π* and the period of *tanx, cotx is π*.

Factorial function

This is defined as *f(n)*= n! = 1.2.3….. *n. for all positive integers.*

The domain of this function is the set of positive integers. The value of the function i.e., the Range increases so rapidly that it is more convenient to display this function in tabular form rather than as a graph. This is listed as the pairs *(n, n!)*.

VALUE OF A FUNCTION

Example:

Let $f(x) = \dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$, then find the value of $f\left(\dfrac{1}{2}\right)$. Find the domain of *f* also.

Solution:

To find $f\left(\dfrac{1}{2}\right)$, put $x = \dfrac{1}{2}$ in the expression.

$$f\left(x=\frac{1}{2}\right) = \frac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}} = \frac{1}{1+\dfrac{1}{1+2}} = \frac{1}{1+\dfrac{1}{3}} = \frac{1}{1+\dfrac{1}{3}} = \frac{3}{4}$$

To find the domain of this function, we must keep in mind the fact that "the domain is the set of values of x for which f(x) has a unique value", and according to this, x cannot be 0 and -1 as these values make f(x) undefined. So the domain will be R-{-1,0}.

Determinate Form

When a unique value of an expression *f(x)* at *x = a* is possible, it is said to be in the

determinate form.

Example:

The expression $f(x)=\dfrac{x+1}{x+3}$ has unique value for each x when it is 0, 1, 2, 3 etc.

Indeterminate Form

When a unique value of an expression *f(x)* at *x=a* is not possible, it is said that it is in Indeterminate form.

Example:

The expression $f(x)=\dfrac{x-1}{x^2-1}$ becomes $\dfrac{0}{0}$, and cannot have any value at *x=1*. Hence it is called to have indeterminate form at *x=1*.

Note:

1. If R*f* ⊆ Y, it is called **Into** Function.
2. If R*f* = Y , it is called **Onto** Function.
3. One-one function is called Injective.
4. Onto function is called Surjective.
5. One-one Onto is called Bijective.
6. **Inverse** of a function is defined only when it is one-one onto.
7. If *f'(x)* is possibly negative or positive then it is many-one.
8. If *f'(x)* >0, for all real x then it is One-one onto.
9. If *f(-x)=f(x)* for all x, it is called an Even function.
10. If *f(-x)= - f(x)* for all x, it is called an Odd function.
11. The product of two even or two odd functions is always even function.
12. The product of even and odd is always an odd function.
13. Every function can be expressed as the sum of an even and an odd function.
    For example:
    $$f(x)=\frac{1}{2}\big(f(x)+f(-x)\big)+\frac{1}{2}\big(f(x)-f(-x)\big)$$
14. Two functions *f* and *g* are equal if and only if

    (i) *f* and *g* have the same domain, and

    (ii) *f(x) = g(x)* for every x in the domain.

LIMITS

Definition

Let *f* be a function defined in a domain which we take to be an interval *(a,b)*. Let also *h* be a quantity between *a* and *b*.

We say $\lim_{x\to h-} f(x)$ is the expected value of *f* at *x=h* given the value of *f* near to the left of *h*. This value is called the Left hand limit of *f* at *h*.

We say $\lim_{x\to h+} f(x)$ is the expected value of *f* at *x=h* given the value of *f* near to the right of *h*. This value is called the Right hand limit of *f* at *h*.

When the **Left Hand Limit** is equal to the **Right Hand Limit**, we say that the **Limit** of the function **exists** at *x=h* and it is represented by $\lim_{x\to h} f(x)$.

Some Properties of Limits

(a) $\lim_{x\to a}\{f(x)+g(x)\}=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$

(b) $\lim_{x\to a}\{f(x)-g(x)\}=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$

(c) $\lim_{x\to a}\{cf(x)\}=c\lim_{x\to a} f(x)$

(d) $\lim_{x\to a}\{f(x).g(x)\}=\lim_{x\to a} f(x).\lim_{x\to a} g(x)$

(e) $\lim_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided g(x)≠0

Important methods to evaluate Limits

(a) Factorization: In this method, we factorize the Numerator and the Denominator, cancel the common factors and then put the value of the variable.

Example:

If $\lim_{x\to 3}\dfrac{x^3-13x^2+51x-63}{x^3-4x^2-3x+18}=\dfrac{a}{10}$, then find the value of *a*.

Solution:

By putting *x=3*, we get left hand expression to be 0/0.

We shall use method of factorization to eliminate the factor (x-3) from Numerator and Denominator.

$$\lim_{x\to 3}\frac{x^3-13x^2+51x-63}{x^3-4x^2-3x+18}=\frac{a}{10}$$

$$\Rightarrow \lim_{x\to 3}\frac{(x-3)(x-3)(x-7)}{(x-3)(x-3)(x+2)}=\frac{a}{10}$$

$$\Rightarrow \lim_{x\to 3}\frac{(x-7)}{(x+2)}=\frac{a}{10}\Rightarrow\frac{(3-7)}{(3+2)}=\frac{a}{10}\Rightarrow\frac{-4}{5}=\frac{a}{10}$$

$$\Rightarrow a=-8$$

(b) L'Hospital Rule:

When the expression is of the form 0/0 or ∞/∞ and both the Numerator and the Denominator are differentiable, we differentiate them separately and check whether they are in indeterminate form. When this form eliminates, we put the value of the variable.

Example:

If the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$ for all x>0, and $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}=\frac{1}{a}$ then find the value of *a*.

Solution:

On putting x=9 in the Left hand expression, it becomes $\frac{0}{0}$. It is given that function is differentiable. We shall use L'Hospital Rule to evaluate this limit, i.e., differentiate Numerator and Denominator separately w.r.t x,

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}=\frac{1}{a}\Rightarrow \lim_{x\to 9}\frac{\frac{1}{2\sqrt{x}}}{1}=\frac{1}{a}\Rightarrow \frac{1}{2\sqrt{9}}=\frac{1}{a}\Rightarrow a=6$$

Example:

Evaluate: $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

Solution:

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right)$$

It is of the form 0/0. Let us apply L'Hospital Rule.

Differentiate Numerator and Denominator separately w.r.t x

$=\lim_{x\to 0}\left(\frac{1-\cos x}{x\cos x+\sin x}\right)$ on putting x=0, again it is of the form 0/0. Let us again differentiate Nr and Dr w.r.t x

$$=\lim_{x\to 0}\left(\frac{0+\sin x}{-x\sin x+\cos x+\cos x}\right)=\lim_{x\to 0}\left(\frac{0}{1+1}\right)=0$$

Some Important Limits

(a) Trigonometric Limits

$$\lim_{x\to 0}\frac{\sin mx}{mx}=1$$

$$\lim_{x\to 0}\frac{\tan mx}{mx}=1$$

Example:

Find the value of $\lim_{x\to 0}\frac{\sin 2x}{5x^2+7x}$.

Solution:

On putting x=0, the expression becomes 0/0, an indeterminate form.

Let us use trigonometric limit $\lim_{x\to 0}\frac{\sin mx}{mx}=1$

$$\lim_{x\to 0}\frac{\sin 2x}{5x^2+7x}=\lim_{x\to 0}\frac{2\sin x\cos x}{x(5x+7)}=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\frac{2\cos x}{(5x+7)}$$

$$=1.\frac{2}{(7)}=\frac{2}{7}$$

(b)Exponential Limits

$$\lim_{x\to 0}(1+mx)^{\frac{1}{mx}}=e$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$$

Example:

Find the value of $\lim_{x\to\infty}\left(1-\frac{5}{2x}\right)^{4x}$.

Solution:

Let us use the special limit $\lim_{x\to 0}(1+mx)^{\frac{1}{mx}}=e$

$$\lim_{x\to\infty}\left(1-\frac{5}{2x}\right)^{4x}=\lim_{x\to\infty}\left\{\left(1-\frac{5}{2x}\right)^{\left(-\frac{2x}{5}\right)}\right\}^{-\frac{5(4x)}{2x}}$$

$$=e^{\lim_{x\to\infty}-\frac{5(4x)}{2x}}=e^{-10}$$

(c)The limits when x tends to infinity

Example: $\lim_{x\to\infty}\frac{1}{x}=0$

Example:

Evaluate $\lim_{x\to\infty}\frac{x^3-13x^2+51x-63}{x^3-4x^2-3x+18}$

Solution:

The expression becomes of the form $\frac{\infty}{\infty}$ after replacing x by infinity. In this case we divide the Numerator and the denominator by the highest power of x i.e., x^3.

$$\lim_{x\to\infty}\frac{x^3-13x^2+51x-63}{x^3-4x^2-3x+18}=\lim_{x\to\infty}\frac{1-\frac{13}{x}+\frac{51}{x^2}-\frac{63}{x^3}}{1-\frac{4}{x}-\frac{3}{x^2}+\frac{18}{x^3}}$$

$$=\frac{1-0+0-0}{1-0-0+0}=1$$

CONTINUITY

A function *f* is said to be a continuous function at *x=c* if the left hand limit, the right hand limit and its value at *c*, coincide i.e.,

Limit = value,

Or,

Right hand limit= Left hand limit= Value at the point

A function *f* is said to be continuous in an open interval *(a,b)* if it is continuous at every point in this interval.

A function *f* is said to be continuous in the closed interval *[a,b]* if

(a) *f* is continuous in *(a,b)*

(b) $\lim_{x\to a+} f(x)=f(a)$

(c) $\lim_{x\to b-} f(x)=f(b)$

DISCONTINUITY

A function *f* is said to be discontinuous at *x = a* if

(a) $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$ exist but are not equal.

(b) $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$ exist and are equal but are not equal to *f(a)*.

(c)f(a) is not defined.

Example:

If $f(x)=\frac{x^3-2x^2-2x-3}{x^3-4x^2+4x-3}$ for all x $\neq$3. Explain how the function *f* should be defined at x = 3 so that the function becomes a continuous function on all of R.

Solution:

$$\lim_{x\to3}\frac{x^3-2x^2-2x-3}{x^3-4x^2+4x-3}=\lim_{x\to3}\frac{(x-3)(x^2+x+1)}{(x-3)(x^2-x+1)}$$

$$=\lim_{x\to3}\frac{(x^2+x+1)}{(x^2-x+1)}=\frac{13}{7}$$

So, if the value of the function is defined to be 13/7, equal to the limit, the function will become continuous.

DERIVATIVE AND DIFFERENTIATION

Definition of derivative

A function *f(x)* is said to have a derivative at *x=h* if the Right hand derivative is equal to the Left hand derivative. It is denoted by $\frac{d}{dx}f(x)$ or $f'(x)$.

Right hand derivative

$$\frac{d}{dx}f(x)=f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

Left hand derivative

$$\frac{d}{dx}f(x)=f'(x)=\lim_{h\to0}\frac{f(x-h)-f(x)}{-h}$$

When

Right Hand Derivative = Left Hand Derivative = a finite value,

The function is said to have a derivative. The finite value is called the derivative of *f(x) at x=h*.

The process to evaluate the derivative is called **Differentiation.**

DERIVATIVE OF A COMPOSITE FUNCTION

Example:

Suppose *g* is a differentiable function and that *f(x)=g(x+5)* for all *x*. If *g'(1)=3*, and *f'(a)=3*, then find the value of *a*.

Solution:

$f(x)=g(x+5)$

$\Rightarrow f'(x)=g'(x+5)$ on differentiating both sides w.r.t x

$\Rightarrow f'(-4)=g'(1)$, putting *x+5=1 i.e., x = -4*

$\Rightarrow f'(-4)=g'(1)=3$

$\Rightarrow a=3$

Some Important Derivatives

(a) Power Rule

$$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$$

(b) Exponential Rule

$$\frac{d}{dx}\left(a^x\right)=a^x\log_e a\,;$$

$$\frac{d}{dx}\left(e^x\right)=e^x$$

(c) Logarithmic Rule

$$\frac{d}{dx}\left(\log_a x\right)=\frac{1}{x\log_e a}\,;\quad \frac{d}{dx}\left(\log_e x\right)=\frac{1}{x}$$

Example:

Let $y=\log_3\left(x^2+1\right)^{1/3}$, and $\frac{dy}{dx}=\frac{2x}{a\left(x^2+1\right)}$, then find the value of a.

Solution:

$$y=\log_3\left(x^2+1\right)^{1/3}\Rightarrow y=\frac{1}{3}\log_3\left(x^2+1\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{3}.\frac{2x}{(x^2+1)\log_e 3}$$

On comparing the given expression with this expression,

$$a=3\log_e 3$$

(d) Derivative of the product of a constant and a function

$$\frac{d}{dx}\left(cf(x)\right)=c\frac{d}{dx}f(x)$$

(e) Sum and difference rule

$$\frac{d}{dx}\left(f(x)\pm g(x)\right)=\frac{d}{dx}f(x)\pm\frac{d}{dx}g(x)$$

(f) Product Rule

$$\frac{d}{dx}\left(f(x)g(x)\right)=f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)$$

(g) Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{\left(g(x)\right)^2},$$

if $g(x)\neq 0$

(h) Chain Rule

$$\frac{d}{dx}\left(f(g(x)\right)=f'(g(x))\frac{d}{dx}g(x)$$

(i) Implicit Differentiation

Example:

Differentiate *sin(xy)=xy* w.r.t *x*

Solution:

$$\sin(xy)=xy\Rightarrow\cos(xy)\left(x\frac{dy}{dx}+y\right)=\left(x\frac{dy}{dx}+y\right)$$

$$\Rightarrow\left(\cos xy-1\right)\left(x\frac{dy}{dx}+y\right)=0\Rightarrow\frac{dy}{dx}=-\frac{y}{x}$$

(j) Trigonometric functions derivatives

$$\frac{d}{dx}\sin x=\cos x$$

$$\frac{d}{dx}\cos x=-\sin x$$

$$\frac{d}{dx}\tan x=\sec^2 x$$

$$\frac{d}{dx}\cos ecx=-\cos ecx\cot x$$

$$\frac{d}{dx}\sec x=\sec x\tan x$$

$$\frac{d}{dx}\cot x=-\cos ec^2 x$$

$$\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}x=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot^{-1}x=-\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x=\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cos ec^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$$

Application of derivative

(a) Stationary point

A point *a* in the domain of a function *f* is called a stationary point of *f* if *f'(a)=0*. It is called a critical point of *f* if it is either a stationary point of *f* or if it is a point where the derivative of *f* does not exist.

(b) Slope of the tangent lines

$\frac{dy}{dx}$ or $\frac{d}{dx}f(x)$ is called the slope.

Example:

Let $f(x) = \frac{x - \frac{3}{2}}{x^2+2}$ and $g(x) = \frac{x^2+1}{x^2+2}$. At what values of *x* do the curves *y = f(x)* and *y = g(x)* have parallel tangent lines?

Solution:

The point or points at which the curves will have parallel tangents is/are the solution points of the equation formed by equating *f'(x)=g'(x)*.

$$f'(x) = \frac{x^2+2-2x\left(x-\frac{3}{2}\right)}{\left(x^2+2\right)^2}$$ and

$$g'(x) = \frac{2x\left(x^2+2\right)-2x\left(x^2+1\right)}{\left(x^2+2\right)^2}$$

On solving

$$\frac{x^2+2-2x\left(x-\frac{3}{2}\right)}{\left(x^2+2\right)^2} = \frac{2x\left(x^2+2\right)-2x\left(x^2+1\right)}{\left(x^2+2\right)^2}$$

We get x=-1, x=2

(c) Point of Inflexion

The point at which $\frac{d^2y}{dx^2} = 0$ is called the point of inflexion.

(d) Monotonicity: Increasing and decreasing functions,

A function *y=f(x)* is said to be strictly increasing if $\frac{dy}{dx} > 0$ and strictly decreasing if $\frac{dy}{dx} < 0$.

A function *y=f(x)* is said to be increasing if $\frac{dy}{dx} \geq 0$ and decreasing if $\frac{dy}{dx} \leq 0$.

Example:

Suppose that the derivative of a function *f* is given by $f'(x)=(x-2)^2(x+4)$, then find the interval on which *f* is increasing.

Solution:

A function f(x) is said to be increasing if and only if $f'(x) \geq 0$.

Hence, $f'(x)=(x-2)^2(x+4) \geq 0$

$\Rightarrow (x+4) \geq 0$ *as* $(x-2)^2$ *is positive for all values of x.*

$\Rightarrow x \geq -4$ *or* $x \in [-4,\infty)$.

(e)Maxima and Minima: Local maxima and local minima, Absolute maxima and absolute minima, First derivative test, Second derivative test

FIRST DERIVATIVE TEST

If *f'(x)* changes sign from + to – as x increases through c, then c is a point of local maxima and *f(c)* is called the local maximum value of *f(x)*.

If *f'(x)* changes sign from - to + as x increases through c, then c is a point of local minima and *f(c)* is called the local minimum value of *f(x)*.

If *f'(x)* does not change the sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Such point is called a point of inflexion.

SECOND DERIVATIVE TEST

If the function *y=f(x)* is twice differentiable, then

(a) *x=c* is a point of local maxima if *f'(c)=0* and *f''(c)<0; f(c)* is the local maximum value of the function.

(b) *x=c* is a point of local minima if *f'(c)=0* and *f''(c)>0; f(c)* is the local minimum value of the function.

If *f'(c)=0* and *f''(c)=0*, then the test fails and we should go through the first derivative test.

RATE OF CHANGE

Let a function be *y=f(x)*.

$\frac{d}{dx}f(x)$ is called the rate of change of y with respect to x.

Thus, if *s* represents the distance and *t* represents the time, then $\frac{ds}{dt}$ represents the change of distance with respect to time i.e., the speed.

TANGENT AND NORMAL

A line that touches a curve *f(x)* at a point (x_1,y_1) is named as the tangent to the curve at that point. Its equation is given by

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} (x - x_1)$$

The normal to the curve is defined as the line that is perpendicular to the tangent and passes through the point of contact. Its equation is given by

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} (x - x_1)$$

APPROXIMATION

We know from the definition of derivative that

$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$. It means that the approximate value of *f'(x)* is $\frac{f(x+\delta x) - f(x)}{\delta x}$

Or, $f(x+\delta x) = f(x) + f'(x)\delta x$

δx is taken with proper sign. If the function is increasing, it is positive (+). If the function is decreasing, it is negative (-).

ROLLE'S THEOREM

Let *f* be a continuous function defined on closed interval *[a,b]*, differentiable on open interval *(a,b)* and *f(a)=f(b)*, where *a* and *b* are some real numbers, then there exists at least one point c in *(a,b)* such that *f'(c)=0*.

Example:

Verify the Rolle's Theorem for $f(x)=x(x-1)^2$ in [0,1].

Solution:

Given function is

(a) Continuous in the interval [0,1].

(b) It is differentiable in (0,1).

(c) f(0)=f(1)=0

hence

$$f'(x) = (x-1)^2 + 2(x-1)x = (x-1)(x-1+2x)$$
$$= (x-1)(3x-1)$$

On solving $(x-1)(3x-1) = 0$ we get x=1 and 1/3. Since x=1 does not lie between 0 and 1 hence it is neglected and x=1/3 is the required value.

LAGRANGE' MEAN VALUE THEOREM

Let *f* be a continuous function defined on closed interval *[a,b]*, differentiable on open interval *(a,b)*, where a and b are some real numbers, then there exists at least one point c in *(a,b)* such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Example:

Let $f(x) = x^4 + x + 3$ for 0 ≤ x ≤ 2. Find a point c whose existence is guaranteed by the mean value theorem.

Solution:

We shall search for the completion of Lagrange's Mean Value Theorem.

(i) $f(x) = x^4 + x + 3$ is continuous being a polynomial between [0,2]

(ii) $f(x) = x^4 + x + 3$ is differentiable between (0,2)

(iii) $f'(c) = 4c^3 + 1 = \frac{f(2) - f(0)}{2 - 0} = \frac{21-3}{2} = 9$

$\Rightarrow 4c^3 = 8 \Rightarrow c^3 = 2 \Rightarrow c = (2)^{\frac{1}{3}}$

(iv) Since $(2)^{\frac{1}{3}} \in (0,2)$, hence the conditions of Mean Value Theorem are satisfied.

INDEFINITE INTEGRATION

Integrals represent a family of curves. Integrals are also called as antiderivatives or primitives.

If two functions differ by a constant, then they have the same derivative. The process of finding integrals is called Integration.

If $\frac{d}{dx} f(x) = F(x)$, then we write $\int F(x)dx = f(x) + C$.

C is called the constant of integration and symbol ∫ is called the integration sign.

F(x) is called the derivative of *f(x)*.

f(x) is called the integral of *F(x)*.

Some Properties of Integral

$$\int (F(x) + G(x))dx = \int F(x)dx + \int G(x)dx$$

$$\int (F(x) - G(x))dx = \int F(x)dx - \int G(x)dx$$

$$\int (aF(x))dx = a\int F(x)dx$$

Some standard integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1$$

$$\int \frac{1}{x} dx = \log_e x$$

$$\int e^x dx = e^x$$

$$\int \sin x dx = -\cos x$$

$$\int \cos x dx = \sin x$$

$$\int \tan x dx = \log \sec x$$

$$\int \cos ec x dx = \log(\cos ec - \cot x)$$

$$\int \sec x dx = \log(\sec + \tan x)$$

$$\int \cot x dx = \log \sin x$$

METHODS TO EVALUATE INTEGRALS

(a) Using direct anti-derivatives

Example:

Integrate $\tan^2 x$ w.r.t.x

Solution:

$$\int \tan^2 x dx = \int (\sec^2 x - 1)dx = \int \sec^2 x dx - \int dx$$

$= \tan x - x + C$, C is a constant of Integration

(b) Integration by substitutions or Change of variables

Example:

Integrate $\frac{2x}{x^2 - 1}$ w.r.t.x.

Solution:

$\int \frac{2x}{x^2 - 1} dx = \int \frac{dt}{t}$ where $t = x^2 - 1$

$$= \log t + C$$

$$= \log(x^2 - 1) + C$$

(c) Trigonometric integrals

Example:

Integrate $\sin^2 x$ w.r.t. x

Solution:

We know that $\sin^2 x$ is not a derivative of any known function.

Let us change it to some known terms.

$$\cos 2x = 1 - 2\sin^2 x$$

Or,

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int \sin^2 x dx = \frac{1}{2}\int (1 - \cos 2x)dx$$

$$= \frac{1}{2}\left(\int dx - \int \cos 2x dx\right)$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

(d) Trigonometric substitutions

Example:

$$\int \frac{1}{\sqrt{4x - x^2}} dx$$

Solution:

$$\int \frac{1}{\sqrt{4x - x^2}} dx = \int \frac{1}{\sqrt{4 - (x-2)^2}} dx$$

Put $x - 2 = 2\sin t \Rightarrow dx = 2\cos t\, dt$

Therefore,

$$= \int \frac{1}{\sqrt{4 - (x-2)^2}} dx = \int \frac{2\cos t dt}{\sqrt{4 - 4\sin^2 t}}$$

$$= \int \frac{2\cos t dt}{2\cos t} = \int dt = t + C$$

$$= \sin^{-1} \frac{x-2}{2} + C$$

(e) Integration by parts

Rule: *II function must have integral*. ILATE Rule (I-Inverse, L-Lograrithmic, A-Algebraic, T-Trigonometric, E-Exponential function) is followed in selecting the first function.

$$\int (I.II)dx = I\int II dx - \int\left(\left(\frac{d}{dx}I\right)\int II\right)dx$$

Example:

Integrate *x sinx w.r.t.x*

Solution:

$$\int x\sin x dx = x\int \sin x dx - \int\left(\frac{d}{dx}x\right)(-\cos x)dx$$

$$= -x\cos x + \int \cos x dx$$

$$= -x\cos x + \sin x + C$$

(f) Partial fractions

When a fraction is decomposed into two or more fractions in such a way that their sums or differences together make the original fraction, then they are called the partial fractions of the original fraction.

For example:

$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$.

$\frac{1}{2}$ and $\frac{1}{4}$ are called the partial fractions of $\frac{3}{4}$.

WAYS TO FIND PARTIAL FRACTIONS

(a) $\frac{1}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$, the values of A and B can be calculated by equating both sides.

(b) $\frac{1}{(x-a)^2} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$

(c) $\frac{1}{(x-a)(x^2+b)} = \frac{A}{(x-a)} + \frac{Bx+C}{(x^2+b)}$

Example:

Integrate $\frac{x}{1-x^2}$ *w.r.t.x.*

Solution:

$$\frac{x}{1-x^2} = \frac{x}{(1-x)(1+x)} = \frac{A}{(1-x)} + \frac{B}{(1+x)}$$

Or, $x = A(1+x) + B(1-x)$

$\Rightarrow 1 = A - B$ and $0 = A + B$

$\Rightarrow A = \frac{1}{2}$ and $B = -\frac{1}{2}$

Hence, $\frac{x}{1-x^2} = \frac{x}{(1-x)(1+x)}$

$$= \frac{A}{(1-x)} + \frac{B}{(1+x)} = \frac{A}{2(1-x)} - \frac{B}{2(1+x)}$$

Or,

$$\int \frac{x}{1-x^2}dx = \int \frac{dx}{2(1-x)} - \int \frac{dx}{2(1+x)}$$

$$= -\frac{1}{2}\log_e(1-x) - \frac{1}{2}\log_e(1+x)$$

$$= -\frac{1}{2}\log_e(1-x^2) + C$$

DEFINITE INTEGRALS

It is denoted by $\int_{x=a}^{x=b} F(x)dx$.

a is called the lower limit and ***b*** is called the upper limit of the integral. It represents the area between the bounded regions.

It is evaluated by two ways:

(a) as the limit of the sum, and

(b) $\int_{x=a}^{x=b} F(x)dx = f(b) - f(a)$ where *f(x)* is the integral of *F(x).*

Properties of definite integrals

$$\int_{x=a}^{x=b} F(x)dx = \int_{t=a}^{t=b} F(t)dt$$

$$\int_{x=a}^{x=b} F(x)dx = -\int_{x=b}^{x=a} F(x)dx$$

$$\int_{x=a}^{x=b} F(x)dx = \int_{x=a}^{x=c} F(x)dx + \int_{x=c}^{x=b} F(x)dx \text{ provided } a < c < b$$

$$\int_{x=a}^{x=b} F(x)dx = \int_{x=a}^{x=b} F(a+b-x)dx$$

$$\int_{x=0}^{x=a} F(x)dx = \int_{x=0}^{x=a} F(a-x)dx$$

$$\int_{x=0}^{x=2a} F(x)dx = 2\int_{x=0}^{x=a} F(x)dx$$ if $F(2a-x)=F(x)$

$$\int_{x=0}^{x=2a} F(x)dx = 0$$ if $F(2a-x)=-F(x)$

$$\int_{x=-a}^{x=a} F(x)dx = 2\int_{x=0}^{x=a} F(x)dx$$ if $F(x)$ is an even function

$$\int_{x=-a}^{x=a} F(x)dx = 0$$ if $F(x)$ is an odd function

Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_{a}^{x} f(t)dt = f(x)$$

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