MATRICES AND DETERMINANTS

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Matrix is a simple method to solve equations involving many variables. Matrix is a method to write data or symbols in rows and columns. It has no numerical value. It is simply an arrangement of data. A British Mathematician Cayley is the Inventor of Matrix Algebra.

Way of writing a matrix

If we have some data ***a,b,c,d,e,f,g,h,i***, we can arrange them as below in matrix form:

$$A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

The matrix is denoted by capital letters as ***A*** and (a b c), (d e f), (g h i) i.e. the horizontals are called ***rows*** and verticals $\begin{pmatrix} a \\ d \\ g \end{pmatrix}, \begin{pmatrix} b \\ e \\ h \end{pmatrix}, \begin{pmatrix} c \\ f \\ i \end{pmatrix}$ are called the ***columns***.

Order of a Matrix

If in a matrix there are m rows and n columns then its order is $m \times n$ and is read as ***m by n***.

i.e. ***#Rows × #Columns*** (where # means number of). The value of the product ***(m × n)*** is equal to the number of elements in the matrix.

For example, the above matrix A has 4 rows and 4 columns and has total 4 × 4 i.e. 16 elements.

Notes:

(i) Matrix has no arithmetical value; it is simply an arrangement of elements. The elements may be arithmetical numbers but matrix as whole will remain an arrangement.

(ii) If the matrix has all its elements as 0 and is of order 4 × 4, the number of elements shall be 16.

Equality of Two Matrices

(i) Two matrices are said to be equal if they are of the same order and have the same corresponding elements for example:

(ii) If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ are two matrices and **A=B** then **a=1, b=2, c=3** and **d=4**

(iii) Different orders matrices cannot be equal.

Types of Matrix

(i) **Row matrix:** Matrix having only one row as $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix}$

Or, $A=\lfloor a_{ij} \rfloor_{m\times n}$ is a row matrix if m=1

(ii) **Column matrix**: Matrix having only one column as $B=\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$ Or, $A=\lfloor a_{ij} \rfloor_{m\times n}$ is a column matrix if n=1

(iii) **Horizontal Matrix**: Matrix where # rows < # columns as

$$C=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

(iv) **Vertical Matrix**: Matrix where # columns < # rows

$$D=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}$$

(v) **Singleton Matrix**:

Matrix having only one element i.e. $E=(a_{11})$ or a matrix of order 1×1.

(vi) **Null or Zero Matrix**: Matrix whose all elements are zero i.e. $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

A null matrix may be square or non-square i.e. it may be of any order.

(vii) **Square Matrix**: The matrix that has equal number of rows and columns.

i.e. 1×1, 2×2, 3×3, 4×4 etc.

(viii) **Upper Triangular Matrix**: A square matrix that has non - zero elements in principal diagonal and above it and all other elements are zero i.e.

$$F = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

(ix) **Lower Triangular Matrix**: A square matrix that has all zero elements above principal diagonal i.e. $G = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

(x) **Diagonal Matrix**: A square matrix in which all the elements are zero except the principal diagonal elements i.e.

$$H = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

(xi) **Scalar Matrix**: A diagonal matrix whose all elements are equal to some scalar k i.e. $J = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$

(xii) **Identity Matrix**: A scalar matrix that has all its elements as 1 i.e. $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

It is always denoted by I. It is always a square matrix.

Important Terms relating to Matrix

(a) **Principal Diagonal**: Only a square matrix has principal diagonal. It is the diagonal stretched from top left to down right. For example in matrix H elements $(a_{11} \quad a_{22} \quad a_{33})$ form principal diagonal.

(b) **Trace of Matrix**: The arithmetical sum of the elements of the principal diagonal is called the trace of a matrix and is denoted by Trace of A.

For example:

The trace of the matrix A is written as **Tr (A)=** $a_{11} + a_{22} + a_{33} + a_{44}$.

(1) The trace may be any data.

(2) Trace (A+B) = Tr A + Tr B

(3) Tr (kA)= k (Tr A)

(4) Tr A'= Tr A

(5) Tr I_n = n

(6) Tr O = 0.

(7) Tr (AB) ≠ (Tr A). (Tr B) (in general)

Operation on Matrix

(i) **Addition of Matrices:** The matrices whose sum is being done must be of same order. The sum is the sum of corresponding places elements. For example: if

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then the sum of the two matrices shall be

$$A + B = \begin{pmatrix} a+1 & b+2 \\ c+3 & d+4 \end{pmatrix}$$

Properties of Addition:

(a)A+B=B+A i.e. commutative

(b) A+ (B+C) = (A+B) +C i.e. associative

(ii) **Difference of two matrices:** Same as in addition where corresponding elements are added, here the corresponding elements are subtracted from A's to B if A-B is required i.e.

$$A-B=\begin{pmatrix} a-1 & b-2 \\ c-3 & d-4 \end{pmatrix}$$

(iii) **Multiplication of a matrix by a scalar:** If k is a scalar then kB of the above matrix will be $kB=\begin{pmatrix} 1k & 2k \\ 3k & 4k \end{pmatrix}$ i.e. all the elements of the original matrix are multiplied by the same scalar k.

(iv) **Multiplication of a matrix by another matrix**: If A is a matrix of order m×n and B is a matrix of order n×p then product AB is possible i.e. the product is possible only when the #columns in 1^{st} matrix=# rows in 2^{nd} matrix and the resultant i.e. product is of order m×p i.e. #row of 1^{st} × # columns of 2^{nd}. It is done as follows:

$A=\begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix}$ And $B=\begin{pmatrix} x & y \\ z & u \\ v & w \end{pmatrix}$ then

AB's order will be 2×2 and AB is possible as A is of order 2×3 and B is of order 3×2 hence the products order will be 2×2.

$$AB=\begin{pmatrix} ax+bz+cv & ay+bu+cw \\ ex+fz+gv & ey+fu+gw \end{pmatrix}$$

Note:

In matrix multiplication,

(a) if AB is possible then BA may or may not be possible.

(b) If AB and BA are possible, even then AB and BA may or may not be equal.

(c) If AB=0 then it is possible that either A=0 or B=0 or both may not be zero separately.

(v) Since matrix has no arithmetical value, hence division of one matrix by another matrix is meaningless.

Some Special matrices

1. **Transpose of a Matrix :**A transpose matrix is a matrix obtained by changing the rows into columns of the original matrix. It is denoted by $A' or A^T$.

For example: the transpose of $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}_{2\times 3}$ is $A' or A^T=\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}_{3\times 2}$

Properties of a Transpose Matrix:

(i) (A')'= A

(ii) (kA)'=kA'

(iii) (A+B)'=A'+B'

(iv) (AB)'=B'A'

2. **Adjoint (or Adjugate) of a Matrix:** Adjoint of a matrix is the matrix obtained by the transpose of the matrix of the cofactors of the original matrix. It is always a square matrix.

If $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $AdjA=\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

where, C_{ij} are the cofactors of a_{ij}.

Example:

If $A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then $C_{11}=4, C_{12}=-3, C_{21}=-2$, $C_{22}=1$

The matrix with cofactors

$C=\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}=\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$ Then transpose of C is the Adjoint of A i.e.

$$Adjoint(A)=C^T=\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

3. ***Properties of Adjoint matrix:***

(a) $|AdjA|=|A|^{n-1}$ Where n is the order of the matrix.

(b) $|Adj(AdjA)|=|A|^{(n-1)^2}$

(c) Adj (AB) = (AdjB) (AdjA)

(d) $AdjA^m = (AdjA)^m$

(e) $Adj(kA) = k^{n-1}(AdjA)$

(f) Adj I=I

(g) If A is singular then $|AdjA| = 0$

(h) $Adj(AdjA) = |A|^{n-2} A$

(i) $AdjA' = (AdjA)'$

4. **Singular Matrix:** The square matrix, whose determinant is zero, is called Singular matrix.

For example: $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is a singular matrix as $|A| = 0$

5. **Non-singular Matrix:** The square matrix whose determinant is not zero is called non-singular matrix. I.e. if $|B| \neq 0$

6. **Symmetric Matrix:** A square matrix is said to be a symmetric matrix if A'=A.

7. **Skew Symmetric Matrix**

A square matrix is said to be skew-symmetric if $A' = -A$

(a) Diagonal elements of a skew-symmetric matrix are zero.

(b) Every square matrix can be written as a sum of a symmetric and a skew-symmetric matrix. i.e. $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

(c) If A, B are symmetric matrices of same order then AB is also a symmetric matrix.

(d) B'AB is a symmetric or skew-symmetric as A is symmetric or skew-symmetric.

(e) All integral powers of a symmetric matrix are symmetric.

8. **Conjugate of a Matrix:** A conjugate of a matrix is a matrix formed with the conjugates of the corresponding elements of the original matrix. For example,

If $A = \begin{pmatrix} a - ib & 3i \\ a & -4i \end{pmatrix}$,then

$\overline{A} = \begin{pmatrix} a + ib & -3i \\ a & +4i \end{pmatrix}$

9. **Conjugate Transpose of a Mtrix:** Is a matrix obtained by the transpose of the conjugate of the original matrix i.e?

$\overline{A}' = \begin{pmatrix} a + ib & a \\ -3i & 4i \end{pmatrix}$. It is also denoted by A^θ.

10. **Hermitian Matrix:** If $A^\theta = A$, the matrix A is called a Hermitian matrix.

11. **Skew-Hermitian Matrix:** If $A^\theta = -A$, the matrix A is called a skew-Hermitian matrix.

Note: Every square matrix is uniquely expressible as the sum of a Hermitian and a skew-Hermitian matrix.

12. **Orthogonal Matrix:** If $AA' = I$ then the matrix A is called an orthogonal matrix.

13. **Idempotent Matrix:** If $A^2 = A$, the matrix A is called an Idempotent matrix.

14. **Involuntary Matrix:** If $A^2 = I$ then the matrix A is called an involuntary matrix

15. **Nilpotent Matrix:** If $A^p = 0$ (but $A^{p-1} \neq 0$) then the square matrix A is called the Nilpotent matrix of nil potency p (or order p).The order of nilpotency is different from the order of the matrix A.

16. **Unitary Matrix:** If $A^\theta A = I$, the matrix A is called a Unitary matrix.

17. **Periodic Matrix:** If $A^{k+1} = A$ then the matrix A is called the periodic matrix of period k.

18. **Inverse of a Matrix:** Inverse of a matrix is possible only when it is a square matrix and

non-singular i.e. determinant of the matrix is not zero. It is given by $A^{-1} = \frac{AdjA}{|A|}$

(a) Inverse is unique, i.e. if A is an inverse of B then B is the inverse of A.

(b) If A and B are invertible matrices of same order then $(AB)^{-1} = B^{-1}A^{-1}$

(c) If A is invertible then transpose of A is also invertible i.e. $(A^T) = (A^{-1})^T$

19. Rank of a matrix

(a) Rank of a matrix is the order of the matrix or the sub matrix formed with the elements of the original matrix that is non-singular.

(b) It cannot be less than 1. i.e. it is always positive integer.

(c) Order of the null matrix is not defined as there exists no sub matrix which is non-singular. But some writer presume it to be 0 which is wrong.

(d) It is denoted by ρ(A).

(e) If ρ(A)= ρ(B)=n then ρ(AB)=n where A and B are square matrices of order n.

(f) Rank of a matrix whose all elements are unity is 1.

(g) Every skew symmetric matrix of odd order has rank less than its order.

(h) The rank of a non-null matrix is always greater than or equal to 1.

(i) Elementary transformations do not change its rank.

(j) **Echelon Form of a Matrix:** A non-zero matrix A is said to be in Echelon form if A performs good in following tests:

(a) All the non-zero rows of A, if any precede the zero – row.

(b) The number of zeros preceding the first non-zero element in a row is less than the number of such zeros in the succeeding row.

(c) The first non-zero element in a row is unity

Note1: The number of non-zero rows of a matrix given in the Echelon form is its rank.

For example: The matrix , $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

by the definition it is obvious there are 3 non-zero rows that precede the 4th row.

Number of zeros in R2, R3, and R4 are 1, 2, and 4 that are in ascending order.

The first non-zero element is unity.

Rank= three non-zero rows

Note2: The matrix B obtained from matrix A by a finite number of elementary row (or column) operations is called a matrix **equivalent** to the matrix A and is written as B ~ A.

Note3:A matrix obtained by the application of any one of the elementary row (or column) operation to the identity matrix is called an **elementary** row (or column) matrix.

(d) Solution of system of linear equations by Rank Method: If AX=B is a system of linear equations in n variables. Then the following working rule is adopted to find solutions:

(e) If ρ(A) ≠ρ(A B), then the system of equations is inconsistent.

(f) If ρ(A) = ρ(A B) = the number of unknowns, then system of equations is consistent and has a unique solution.

(g) If ρ(A) = ρ(A B) < the number of unknowns, then the system of equations is consistent and has infinitely many solutions.

(h) In case of homogeneous system of equations AX=O

(i) If ρ(A) = number of variables, then it has a trivial solution.

(j) If ρ(A) < number of variables, then it has a non-trivial solution, i.e. infinitely many solutions.

(k) Uses of matrix in solving linear equations: homogeneous and non-homogeneous

Uses of Matrix in solving Linear equations: Linear Equations are of two types –

i. Homogeneous

ii. Non-homogeneous

Let us consider the set of equations:

$$a_{11}x + a_{12}y = b_{11}$$

$$a_{21}x + a_{22}y = b_{21}$$

1. If all of b_{11}, b_{21} are zero, the system of equations is called **homogeneous**.

$$a_{11}x + a_{12}y = 0$$

$$a_{21}x + a_{22}y = 0$$

For homogeneous system of equations D must be zero because D1, D2 are already zero.

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, D_1 = \begin{vmatrix} 0 & a_{12} \\ 0 & a_{22} \end{vmatrix} = 0,$$

$$D_2 = \begin{vmatrix} a_{11} & 0 \\ a_{21} & 0 \end{vmatrix} = 0$$

2. Matrix use-Solution of the equation is written as

AX=B where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the matrix formed by the coefficients of x and y.

$X = \begin{pmatrix} x \\ y \end{pmatrix}$ is a matrix formed with variables.

And $B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a matrix of numeral on right sides of equations.

3. **Consistency and Inconsistency**

(a) The solution of the equations is given by

$$X = \frac{AdjA}{|A|}B$$

(b) The system of equations is consistent and has infinitely many solutions if $(AdjA)B = 0$ and $|A| = 0$

(c) The system of equations is inconsistent if $(AdjA)B \neq 0$ and $|A| = 0$

(d) If $|A| \neq 0$, then the homogeneous system (B=O always), AX=O has only the trivial solution i.e. x=0, y=0.

(e) For homogeneous system AX=O, to have a non-trivial solution i.e. not all zero, $|A| = 0$ (must)

(f) For non-homogeneous system of equations i.e. AX=B, (B≠0) has a unique solution if $|A| \neq 0$ and the solution is given by $X = A^{-1}B$.

Notes:

(i) **Consistent**: if solution exists whether unique or many

(ii) **Inconsistent**: if the solution does not exist.

(g) Non-homogeneous system of equations:

If at least one of b_{11}, b_{21} is not zero, then it is called **non-homogeneous** equations.

4. **Characteristic equation of a matrix**

(a) If A is a square matrix then |A-x **I**|=0 is the characteristic equation of A.

For example:

If $x^3 - 4x^2 - 5x - 7 = 0$ is the characteristic equation of A then $A^3 - 4A^2 - 5A - 7I = O$.

(b) The roots of this equation are called the characteristic roots or characteristic values or Eigen values or latent roots of A.

(c) The set of the Eigen values of the matrix A is called the spectrum of the matrix A.

(d) Any matrix A and its transpose both have the same eigen values.

(e) The trace of the matrix is always equal to the sum of the eigen values of a matrix.

(f) The determinant of the matrix A is equal to the product of the eigen values of A.

(g) If $\lambda_1, \lambda_2, \lambda_3, \ldots\lambda_n$ are the n-eigen values of A, then eigen values of kA are $k\lambda_1, k\lambda_2, k\lambda_3, \ldots, k\lambda_n$.

(h) If $\lambda_1, \lambda_2, \lambda_3, \ldots\lambda_n$ are the n-eigen values of A, then eigen values of A^m are $\lambda^m{}_1, \lambda^m{}_2, \lambda^m{}_3, \ldots, \lambda^m{}_n$.

(i) If A and P are square matrices and if P is invertible then matrices A and $P^{-1}AP$ both have the same characteristic roots.

(j) 0 is the characteristic root of a matrix if and only if the matrix is singular.

(k) If A and B are two square invertible matrices then AB and BA have the same characteristic roots.

(l) The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

(m) All the characteristic roots of a Hermitian matrix are real.

(n) Characteristic roots of a real symmetric matrix are all real.

(o) Characteristic roots of a skew Hermitian matrix is either zero or a pure imaginary number.

(p) If λ is an eigen value of an orthogonal matrix, then $(1/\lambda)$ is also an eigen value.

(q) Every square matrix satisfies its characteristic equation. (**Cauchy-Hamilton Theorem**)

(r) If λ is a root of A then λ is also root of A^{-1}.

(s) If λ is a root of A then λ^{-1} is characteristic root of A^{-1}.

Note 1: There is no name of the matrix: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ nether it is unit nor unitary.

Note 2: If A is a square matrix of order n, then Maximum number of distinct entries if A is a triangular matrix= {n(n+1)/2} +1.

Minimum number of zeros if A is triangular matrix=n(n-1)/2

Maximum number of distinct entries if A is a diagonal matrix=n+1

Minimum number of zeros if A is a diagonal matrix=n(n-1)

DETERMINANTS

(1) Gottfied Wilhelm Leibnitz is treated as the Inventor of Determinant.

(2) Determinant is a method of writing n x n quantities in an array form. It has a fixed value.

(3) It is represented by a Capital letter of English Alphabet.

(4) If ax + by =0, cx + dy =0 then after elimination of x and y or writing the coefficients of x and y in the following form is called a determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

(5) Order of a determinant= (number of rows) x (number of columns). For example if it has 2 rows and 2 columns then the order = 2 x 2 .

(6) The elements forming parallel lines are called Rows and the vertical lines are called the columns.

(7) The determinant of a lesser order is called the Minor. Minor is a determinant formed

by excluding the row and column of the element in which it exists.

(8) The Principal diagonal is the line joining the elements a_{11} to a_{nn} if the determinant has n rows and n columns.

(9) The value of the determinant remains unchanged if the rows are changed into columns or vice-versa.

(10) If the two rows or columns are interchanged, the determinant is multiplied by (-). If the same process is repeated again, the determinant is multiplied by (-) (-) i.e. $(-)^2$.

(11) If two rows or columns of a determinant are identical i.e. all constituents are same, the value of that determinant is zero.

(12) Determinant of a skew symmetric matrix of odd order is always zero.

(13) If all the constituents of a row or column of a determinant are zero then the value of that determinant is zero.

(14) If all the constituents of a determinant above or below the principal diagonal are zero then the value of the determinant is equal to the product of the principal diagonal constituents.

(15) Determinant of a diagonal matrix is product of their diagonal elements.

(16) $|I_n| = 1$

(17) $|O_n| = 0$

(18) Non- square matrix has no determinant.

(19) Conjugate of a determinant is the determinant formed by the conjugates of all constituents.

(20) If A and B are square matrices of same order then $|AB| = |A|\,|B|$ and $\left|A^n\right| = \left|A\right|^n$

(21) If one row or one column is multiplied by some scalar quantity k, the value of the determinant gets multiplied by k i.e. if each row or column of a matrix of order 3 x 3 is multiplied by a scalar k then the value shall be multiplied by k.k.k i.e. k^3.

(22) If one row or column is sum of difference of two terms then the determinant can be represented as the sum or difference of two determinants as follows:

$$\begin{vmatrix} a_{11} \pm \alpha & a_{12} \pm \beta \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \pm \begin{vmatrix} \alpha & \beta \\ a_{21} & a_{22} \end{vmatrix}$$

(23) The determinant may be of any order but the number of rows must be equal to the number of columns.

(24) The determinant has a fixed order of + and − signs as below: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ i.e. alternate (+) and (-) starting from the upper extreme left.

(25) The determinant is written between two parallel lines.

(26) The value of a determinant is calculated by expanding it along a row or column according to Laplace Law as below: if $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then

$$A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$.

(27) C_{11}, C_{12}, C_{13} are called the Cofactors and A_{11}, A_{12}, A_{13} are called the Minors. Minors are all with plus sign while cofactors are always with proper corresponding signs of the elements.

(28) A common factor of any row (or column) may be taken outside of the determinant. In other words if all the elements of one row (or column) is multiplied by a non-

zero number; 'K' then the value of new determinant is K times the value of original determinant.

(29) If determinant becomes zero on putting x = α, then we say that (x-α) is a factor of the determinant.

(30) The value of determinant is unaffected when any row (or column) is multiplied by a number or any expression and then added or subtracted from any other row (or column).

(31) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

(32) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$= (a-b)(b-c)(c-a)(a+b+c)$

(33) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)$

$= \{a^2+b^2+c^2+ab+bc+ac\}$

(34) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$.

$= (a-b)(b-c)(c-a)(ab+bc+ac)$

(35) $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$

$= (\alpha+\beta+\gamma)$

$\{(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2\}$

(36) To differentiate a determinant, we differentiate one row (or column) at a time, keeping other unchanged.

Multiplication of two determinants can be done in 4 ways.

(i) Row by row multiplication

(ii) Row by column multiplication

(iii) Column by column multiplication

(iv) Column by row multiplication

(37) To express a determinant as product of two determinants, you have to require a lot of practice. This can be done only by inspection.

(38) System of linear equations is said to be consistent if it has at least one solution.

(39) System of linear equations is in consistent if it has no solution.

(40) System of linear equation in two variable x and y

$a_1x+b_1x+c_1=0$,

$a_2x+b_2x+c_2=0$,

$a_3x+b_3x+c_3=0$

is consistent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(41) System of linear equations in 3 variables x, y, & z are -

$a_1x+b_1y+c_1z=d_1$,

$a_2x+b_2y+c_2z=d_2$,

$a_3x+b_3y+c_3z=d_3$,

if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$,

$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$, if $\Delta \neq 0$, then, $x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta}$ and system of equations is called consistent and has unique solution.

(42) System of linear equations in two variables x and y

$$a_1x + b_1x + c_1 = 0,$$

$$a_2x + b_2x + c_2 = 0,$$

$$a_3x + b_3x + c_3 = 0$$

is consistent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(43) System of linear equations in 3 variables x, y, & z is -

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

$$a_3x + b_3y + c_3z = d_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

(i) if $\Delta \neq 0$,) then, $x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta}$ is called consistent and has unique solution

(ii) if $\Delta = 0$ but at least $\Delta_x, \Delta_y, \Delta_z \neq 0.$ Then the system of equations has no solution, (Inconsistent solution).

(iv) if $\Delta = 0 = \Delta_x = \Delta_y = \Delta_z.$ then the system of equations is consistent and has infinitely many solutions.

Value of Determinant:

(i) The value of determinant made by the matrix of order 1 x 1, A= [a] is denoted by

|A|= a.

(ii) The value of determinant made by the matrix of order 2 x 2 is-

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Cofactors and Minors

Minors: In a determinant like A given under

$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, the value is given by -

$$A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$

then

$$A = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$0 = a_{11}A_{21} - a_{12}A_{22} + a_{13}A_{23}$$

$$= a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$$

i.e. the sum of the products of the elements of a row and the minors or cofactors of another row is always zero.

Note 1: A11, A12,A13,..are called Minors and C11,C12,C13 ...are called Cofactors.Cofactors are the minors with proper signs.

Note 2: cofactor $C_{ij} = (-1)^{i+j} A_{ij}$ where A_{ij} is the minor of the element of i^{th} row and j^{th} column.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,and $\Delta' = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$ is a determinant of corresponding cofactors then $\Delta\Delta' = \Delta^3$ i.e. $\Delta' = \Delta^2$

Note: $\Delta' = \Delta^{n-1}$ **where n is the order of the determinant i.e. if the determinant is of order 2 x 2 , then** $\Delta' = \Delta$ **and if it is of order 3 x 3, then** $\Delta' = \Delta^2$**.**

Uses of determinant

(1) For solution of simultaneous linear equations (Cramer's rule)

Consider the set of Linear equations

$$a_{11}x + a_{12}y = b_{11}$$

$$a_{21}x + a_{22}y = b_{21}$$

If all of b_{11}, b_{21} are zero, the system of equations is called homogeneous. i.e.

$$a_{11}x + a_{12}y = 0$$

$$a_{21}x + a_{22}y = 0$$

For homogeneous system of equations where D_1, D_2 are zero.

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad D_1 = \begin{vmatrix} 0 & a_{12} \\ 0 & a_{22} \end{vmatrix} = 0,$$

$$D_2 = \begin{vmatrix} a_{11} & 0 \\ a_{21} & 0 \end{vmatrix} = 0$$

(a) The solution is trivial i.e. x=y=z=0 if D≠0

(b) and non-trivial i.e. infinite solutions if D=0

(2) Area of triangles having vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$

Is given by $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

(3) The three points are collinear if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(4) Equation of a line passing through two given points (x1, y1) and (x2, y2) is given $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(5) Differentiation and Integration of a Determinant

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ d & e & f \end{vmatrix}$ is a determinant

where f, g, h are functions or x and a, b, c, d, e, f are all constants then the derivative of F(x) is given by

$$\frac{d}{dx}F(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ a & b & c \\ d & e & f \end{vmatrix}$$

And if the other rows are also functions of x then the required determinant will be the sum of the differentiated determinants of the rows keeping other constants. As under:

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ \phi(x) & \varphi(x) & \gamma(x) \\ d & e & f \end{vmatrix}$ then derivative of F(x) is given by,

$$\frac{d}{dx}F(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ \phi(x) & \varphi(x) & \gamma(x) \\ d & e & f \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \phi'(x) & \varphi'(x) & \gamma'(x) \\ d & e & f \end{vmatrix}$$

(6) Similarly, the Integration of F(x) of a single row variable is given by

$$\int F(x)dx = \begin{vmatrix} \int f(x)dx & \int g(x)dx & \int h(x)dx \\ a & b & c \\ d & e & f \end{vmatrix}$$

(7) Summation of determinant

If $F(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ d & e & f \end{vmatrix}$ then

$$\sum_{r=1}^{n} F(x)$$
$$=\begin{vmatrix} \sum_{r=1}^{n} f(x) & \sum_{r=1}^{n} g(x) & \sum_{r=1}^{n} h(x) \\ a & b & c \\ d & e & f \end{vmatrix}$$

(8) **Symmetric determinant:** In which the elements situated at equal distance from the principal diagonal are equal in sign and magnitude. For example:

$$A=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(9) Skew-symmetric determinant

Whose principal diagonal elements are all zero and elements at equal distance from the principal diagonal are equal in magnitude but different in sign.

Like, $A=\begin{vmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{vmatrix}$

(10) **Circulant**

$$A=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

(11) Non-homogeneous system of equations

D≠0 ⇒ unique solution, consistent

D=0, $D_1=D_2=D_3=0$ ⇒ many solutions, consistent

D=0,at least one of D_1, D_2, $D_3 \neq 0$ ⇒ no solution, inconsistent

(12) **Homogeneous system of equations**:

$D_1=D_2=D_3=0$ (always)

D=0 ⇒many solutions, consistent

D≠0 ⇒ unique solution, trivial solution, consistent

Solved Examples

1. Let a,b,c be positive and not all equal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.

Solution:

Simplify the determinant

$=(a+b+c)(ab+bc+ca-a^2-b^2-c^2)$

$= -(a+b+c)[(a-b)^2 + (b-c)^2+(c-a)^2]$

Since the expression in 2nd bracket is always positive being square and the first bracket is also positive hence the sign of the expression will be negative.

2. Find the number of solutions of the system of equations

$2x + y - z = 7$,

$x - 3y + 2z = 1$

$x + 4y - 3z = 5$

Solution:

Note the value of the determinant made by the coefficients = $\begin{vmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{vmatrix}$ =0

And at least one of the $D_1=\begin{vmatrix} 7 & 1 & -1 \\ 1 & -3 & 2 \\ 5 & 4 & -3 \end{vmatrix} \neq 0$,

Hence the given system has no solution.

3. If p≠ a, q≠ b, r≠ c and $\begin{vmatrix} p & b & c \\ p+a & q+b & 2c \\ a & b & r \end{vmatrix}=0$,

then find the value of $\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}$.

Solution: After applying R_2 by R_2-R_1,

$$\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

Now apply R_1 by R_1-R_3,R_2 by R_2-R_3,

$$\begin{vmatrix} p-a & 0 & c-r \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

Taking (p-a)(q-b)(c-r) common from First, Second and Third columns respectively,

$$(p-a)(q-b)(c-r)\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \dfrac{a}{p-a} & \dfrac{b}{q-b} & \dfrac{r}{c-r} \end{vmatrix} = 0$$

Or, $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \dfrac{a}{p-a} & \dfrac{b}{q-b} & \dfrac{r}{c-r} \end{vmatrix} = 0$, on expanding, we get the required value 2.

4. If A=$\begin{pmatrix} 2 & 2 \\ -3 & 2 \end{pmatrix}$ and B=$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then find the value of $(B^{-1}A^{-1})^{-1}$.

***Hint for the Solution**:*

Note the given expression is AB as $(AB)^{-1}=B^{-1}A^{-1}$ and $(A^{-1})^{-1}=A$, so $(B^{-1}A^{-1})^{-1}=AB$.

So just write the product of A and B matrices.

5. If A is an Invertible matrix, then evaluate the value of det (A^{-1}).

Solution: We know that $AA^{-1}=A^{-1}A= I$

$\Rightarrow |AA^{-1}| = |A^{-1}A| = |I| =1 \Rightarrow |A||A^{-1}| =1$

$\Rightarrow |A^{-1}| = 1/(|A|)$

6. For what conditions we can conclude B=C from the matrix equation AB = AC ?

Solution: We know that AB=AC

$\Rightarrow A^{-1}(AB)=A^{-1}(AC)$

$\Rightarrow (A^{-1}A)B=(A^{-1}A)C$

$\Rightarrow IB=IC\Rightarrow B=C$

But this is possible only when A is Invertible i.e. A is also a non-singular matrix i.e. its determinant is not zero.

7. Find a root of the equation $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$

Solution:

Obviously, x=0 is a root as it makes the given determinant a skew-symmetric matrix of odd order and the determinant of a skew symmetric matrix of odd order is zero.

$\begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix}$. Hence $\begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

Note: A skew-symmetric matrix is a matrix whose all the principal diagonal elements are necessarily zero and A'=-A

8. If A is a n x n matrix with real entries, then prove that the value of det($A^2 + I_n$)≥0.

Hint:

$A^2+I_n = (A+iI_n)(A-iI_n)$

Assume $\lambda_1,\lambda_2,\lambda_3,..,\lambda_n$ Eigen values of A then $(A+I_n)$ will have its Eigen values as λ_1+i, λ_2+i, λ_3+i,....λ_n+i.

And similarly, Eigen values of $(A-I_n)$ will be λ_1-i, λ_2-i, λ_3-i,....,λ_n-i.

$\Rightarrow$det $(A + I_n)= (\lambda_1+i)(\lambda_2+i)(\lambda_3+i)....(\lambda_n+i)$ and det $(A - I_n)= (\lambda_1-i)(\lambda_2-i)(\lambda_3-i)....(\lambda_n-i)$

Since A has real entries, its roots (i.e. eigen values) come in conjugate pairs.

Obviously $(\lambda_1-i)(\lambda_1+i)= \lambda_1^2+1$= a real positive number. Hence the required value is always ≥ 0.

9. If $A = \begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix}$, then evaluate $\lim_{n\to\infty} \frac{1}{n} A^n$

Solution:

Rewrite the given matrix A as

$$A= \frac{1}{n}\begin{pmatrix} n & \alpha \\ -\alpha & n \end{pmatrix}$$

Assume for simplification purposes:

$n=r\cos\theta$, $\alpha= r\sin\theta$

$\Rightarrow n^2+\alpha^2= r^2$ and $\theta= \tan^{-1}(\alpha/n)$.

$\Rightarrow$A=

$$\frac{1}{n}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} = \frac{r}{n}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\Rightarrow A^n= \frac{r^n}{n^n}\begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}.$$

$$= \left(\frac{n^2+\alpha^2}{n^2}\right)^{n/2}\begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$$

$$\Rightarrow \frac{A^n}{n} = \left(1+\frac{\alpha^2}{n^2}\right)^{n/2}\begin{pmatrix} \frac{\cos n\theta}{n} & \frac{\sin n\theta}{n} \\ -\frac{\sin n\theta}{n} & \frac{\cos n\theta}{n} \end{pmatrix}.$$

Hence $\lim_{n\to\infty} \frac{1}{n} A^n =$

$$\lim_{n\to\infty}\left(1+\frac{\alpha^2}{n^2}\right)^{n/2} \lim_{n\to\infty}\begin{pmatrix} \frac{\cos n\theta}{n} & \frac{\sin n\theta}{n} \\ -\frac{\sin n\theta}{n} & \frac{\cos n\theta}{n} \end{pmatrix}.$$

$= 1.\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ =O, a null matrix.

As, $\lim_{n\to\infty}\left(1+\frac{\alpha^2}{n^2}\right)^{n/2} = 1$, and

$$\lim_{n\to\infty}\frac{\cos n\theta}{n} = \lim_{n\to\infty}\frac{\sin n\theta}{n} = 0.$$

10. If abc=p and $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$ and AA'=I then find the equation whose roots are a,b,c.

Solution:

Obviously,

$$AA' = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}\begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix} =$$

$$\begin{pmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{pmatrix} = I$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i.e. $a^2+b^2+c^2=1$, $ab+bc+ca=0$

since $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$

$\Rightarrow$ (a+b+c)=+ 1 or -1

also abc=p

so the required equation is

$x^3-(a+b+c)x^2+(ab+bc+ca)x-abc=0$

i.e. $x^3 \pm x^2+ 0.x - p = 0$

11. Find the coefficient of x in the expansion of

$$\begin{vmatrix} (1+x)^{22} & (1+x)^{44} & (1+x)^{66} \\ (1+x)^{33} & (1+x)^{66} & (1+x)^{99} \\ (1+x)^{44} & (1+x)^{88} & (1+x)^{144} \end{vmatrix}.$$

Hint:

Obviously the determinant is a polynomial of degree 22+66+144=232

i.e. $f(x)= A_0 + A_1x + A_2x^2 + A_3x^3 +....+ A_{232}x^{232}$.

and A_1= coefficient of x = f'(0), hence, f'(x)= $\begin{vmatrix} 22(1+x)^{21} & 44(1+x)^{43} & 66(1+x)^{65} \\ (1+x)^{33} & (1+x)^{66} & (1+x)^{99} \\ (1+x)^{44} & (1+x)^{88} & (1+x)^{144} \end{vmatrix}$ +

$$\begin{vmatrix} (1+x)^{22} & (1+x)^{44} & (1+x)^{66} \\ 33(1+x)^{32} & 66(1+x)^{65} & 99(1+x)^{98} \\ (1+x)^{44} & (1+x)^{88} & (1+x)^{144} \end{vmatrix} +$$

$$\begin{vmatrix} (1+x)^{22} & (1+x)^{44} & (1+x)^{66} \\ (1+x)^{33} & (1+x)^{66} & (1+x)^{99} \\ 44(1+x)^{43} & 88(1+x)^{87} & 144(1+x)^{143} \end{vmatrix}.$$

⇒f'(0)=

$$\begin{vmatrix} 22 & 44 & 66 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 33 & 66 & 199 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 44 & 88 & 144 \end{vmatrix} = 0$$

12. If, $\begin{vmatrix} a & \cot\frac{A}{2} & l \\ b & \cot\frac{B}{2} & m \\ c & \cot\frac{C}{2} & n \end{vmatrix} = 0$ where a,b,c, A,B,C are the elements of a triangle ABC with usual meaning, then find the value of a(m-n) + b(n-l)+c(l-m).

Hint:

Use cot (A/2)=s(s-a)/Δ, cot (B/2)= s(s-b)/Δ, and cot (C/2)=s(s-c)/Δ, 2s=a+b+c, r=Δ/s, the usual notations.

Simplify to $\begin{vmatrix} a & s-a & l \\ b & s-b & m \\ c & s-c & n \end{vmatrix} = 0 = \begin{vmatrix} a & s & l \\ b & s & m \\ c & s & n \end{vmatrix}$.

And the required value of the expression after the expansion=0.

13. If a,b,c are real then find the interval in which, f(x)= $\begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$ is decreasing.

Solution:

Simplify f(x) = $x^2(x+a^2+b^2+c^2)$

⇒f'(x)=$2x(x+a^2+b^2+c^2)+ x^2$

⇒f'(x) < 0 if x∈ $(-2/3(a^2+b^2+c^2),0)$

14. If f(x) satisfies the equation $\begin{vmatrix} f(x-3) & f(x+4) & f\left((x+1)(x+2)-(x-1)^2\right) \\ 5 & 4 & -5 \\ 5 & 6 & 15 \end{vmatrix} = 0$ for all x, then test whether it is a periodic function or not and if it is a periodic function , find its period.

Solution:

Note $(x+1)(x+2)-(x-1)^2=(x-3)$

Simplify the determinant to

100f(x-3)-100f(x+4)=0 ⇒f(x-3)=f(x+4)

Replace x by x+3

⇒f(x)=f(x+7) hence it is periodic with period 7

15. If g(x)= $\begin{vmatrix} a^{-x} & e^{x\log_e a} & x^2 \\ a^{-3x} & e^{3x\log_e a} & x^4 \\ a^{-5x} & e^{5x\log_e a} & 1 \end{vmatrix}$, then show that it is an odd function.

Hint:

Rewrite g(x)= $\begin{vmatrix} a^{x} & a^{x} & x^2 \\ a^{3x} & a^{3x} & x^4 \\ a^{5x} & a^{5x} & 1 \end{vmatrix}$.

Obviously g(-x)=-g(x) hence an odd function.

16. A skew symmetric matrics S satisfies the relation $S^2+I=0$, where I is an Identity matrics then show that S is an orthogonal matrix.

Hint: Note S is a skew symmetric matrix

⇒S'=-S, Also S^2 = - I ⇒S.S'=-I

⇒S.S.S'=-I.S'=I.S=S

⇒S^{-1}.S.S.S'=S^{-1}.S⇒(S^{-1}.S).S.S'=I

⇒I.S.S'=I ⇒S.S'=I⇒ S an orthogonal matrix.

17. If, f(x)= $\begin{vmatrix} x+a & x+b & x+a-c \\ x+b & x+c & x-1 \\ x+c & x+d & x-b+d \end{vmatrix}$ and $\int_0^2 f(x)dx = -16$, where a,b,c, d are in AP then find the common difference of the AP.

Hint: Obviously common difference is b-a=c-b=d-c= k (say)

Apply operators R3-R2,R2-R1, to find b-a,c-b,d-c

f(x)=

$$\begin{vmatrix} x+a & x+b & x+a-c \\ k & k & 2k-1 \\ k & k & 1+2k \end{vmatrix} = \begin{vmatrix} x+a & k & -c \\ k & 0 & k-1 \\ k & 0 & k+1 \end{vmatrix}.$$

$=-2k^2$

$$\Rightarrow \int_0^2 f(x)dx = -16 \Rightarrow$$

$$\int_0^2 -2k^2 dx = -16 \Rightarrow k = \pm 2$$

18. Express the determinant Δ= $\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$ as the product of two determinants and hence show that Δ=0.

Hint: Rewrite the given determinant as below:

Δ=

$$\begin{vmatrix} \cos^2\alpha+\sin^2\alpha & \cos\beta\cos\alpha+\sin\beta\sin\alpha & \cos\gamma\cos\alpha+\sin\gamma\sin\alpha \\ \cos\alpha\cos\beta+\sin\alpha\sin\beta & \cos^2\beta+\sin^2\beta & \cos\gamma\cos\beta+\sin\gamma\sin\beta \\ \cos\alpha\cos\gamma+\sin\alpha\sin\gamma & \cos\beta\cos\gamma+\sin\beta\sin\gamma & \cos^2\gamma+\sin^2\gamma \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} = 0$$

19. A triangle has its three sides equal to a,b,c. If the coordinates of its vertices are A(x_1,y_1), B(x_2,y_2)and C(x_3,y_3), then show that $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2$ = (a+b+c)(b+c-a)(c+a-b)(a+b-c).

Solution: Assume the area of the triangle =

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow 2\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \Rightarrow 4\Delta = 2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}$$

$$\Rightarrow 16\Delta^2 = \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2$$

Also note

Δ^2=s(s-a)(s-b)(s-c) and s=(a+b+c)/2= half of the perimeter. On simplification the required result can be evaluated.

20. If $\bar{a},\bar{b},\bar{c}$ are coplanar vectors, then show that $\begin{vmatrix} \bar{a} & \bar{b} & \bar{c} \\ \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \end{vmatrix} = \bar{0}$.

Hint:

$\bar{a},\bar{b},\bar{c}$ are coplanar vectors hence there exist non-zero x,y,z such that $x\bar{a} + y\bar{b} + z\bar{c} = 0$

On applying $xC_1+yC_2+zC_3$ to the given determinant, we may get

$$\frac{1}{x}\begin{vmatrix} x\bar{a}+y\bar{b}+z\bar{c} & \bar{b} & \bar{c} \\ x\bar{a}.\bar{a}+y\bar{a}.\bar{b}+z\bar{a}.\bar{c} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ x\bar{b}.\bar{a}+y\bar{b}.\bar{b}+z\bar{b}.\bar{c} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \end{vmatrix} =$$

$$\frac{1}{x}\begin{vmatrix} x\bar{a}+y\bar{b}+z\bar{c} & \bar{b} & \bar{c} \\ \bar{a}.(x\bar{a}+y\bar{b}+z\bar{c}) & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.(x\bar{a}+y\bar{b}+z\bar{c}) & \bar{b}.\bar{b} & \bar{b}.\bar{c} \end{vmatrix} =$$

$$\frac{1}{x}\begin{vmatrix} \bar{0} & \bar{b} & \bar{c} \\ \bar{0} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{0} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \end{vmatrix} = \bar{0}.$$

21. If l_1, m_1, n_1 ; l_2, m_2, n_2 *and* l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines, then prove that $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm 1$.

Solution:

We know that when the three lines are mutually perpendiculars, then

$l_1^2 + m_1^2 + n_1^2 = 1;$

$l_2^2 + m_2^2 + n_2^2 = 1;$

$l_3^2 + m_3^2 + n_3^2 = 1$

and

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$;

$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$;

$l_3 l_2 + m_3 m_2 + n_3 n_2 = 0$

Let us assume the given determinant = Δ

then,

$$\Delta^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} . \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} =$$

$$\begin{vmatrix} l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1l_3+m_1m_3+n_1n_3 \\ l_1l_2+m_1m_2+n_1n_2 & l_2^2+m_2^2+n_2^2 & l_3l_2+m_3m_2+n_3n_2 \\ l_1l_3+m_1m_3+n_1n_3 & l_3l_2+m_3m_2+n_3n_2 & l_3^2+m_3^2+n_3^2 \end{vmatrix}$$

$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$ and hence the required result is obtained after taking the square root of both sides.

Dr S.B. Dhar, is **Editor of this Quartrerly e-Bulletin**. He is an eminent mentor, analyst and connoisseur of Mathematics from IIT for preparing aspirants of Competitive Examinations for Services & Admissions to different streams of study at Undergraduate and Graduate levels using formal methods of teaching shared with technological aids to keep learning at par with escalating standards of scholars and learners. He has authored numerous books – Handbook of Mathematics for IIT JEE, A Textbook on Engineering Mathematics, Reasoning Ability, Lateral Wisdom, Progress in Mathematics (series for Beginner to Class VIII), Target PSA (series for class VI to class XII) and many more.

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