SETS, FUNCTIONS, AND RELATIONS

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Set

Set is defined as a collection of well-defined objects.

It is represented in two forms:

(a) **Roster Form:** In this way of representation, all the elements are represented in List form. For example, if one is collecting vowels of English Alphabet, it will be written as {***a,e,i,o,u***}. The order of elements does not matter in writing the set.

(b) **Set builder Form:** In this way of expressing the set, we choose a property among all the elements that is common and single property and is not possessed by any element outside the set. For example: The set of vowels of English Alphabet is written as {***x: x is a vowel of English Alphabet***}

Note:

(a) Absolute properties do not form sets.

For example: We cannot collect the intelligent persons. We cannot have the set of beautiful or handsome persons. The reason behind this is simple: no one can decide the parameter of these virtues.

(b) A set does not change if one or more elements of it are repeated.
(c) Georg Cantor (1845-1918) is considered the originator of the modern set theory.
(d) The set is always represented through ***braces*** {}. It is also called the curly brackets.
(e) The things written in curly brackets are called ***elements*** of the set.

Important Terminology

Cardinal Number: It is the number of elements in a finite Set. The number of elements means the distinct elements in the set.

(a) Cardinal number of a Power Set = 2^n if the number of elements in the set is n.
(b) If cardinal number of set A is n, then total number of subsets of P(A)= 2^{2^n} and the total number of proper subsets = 2^{2^n} -1. P(A) represents the ***power set of A***.

Comparable Sets: Two sets A and B are said to be comparable if $A \subset B$ or $B \subset A$ or ***A=B***, otherwise they are called incomparable sets.

Empty Set: The set that contains no element is called an empty set. It is denoted by ϕ = { }.

(a) {0} or {ϕ } is not an empty set.
(b) Empty set is also called the ***Null set*** or the ***Void set***.
(c) Empty set is a finite set.

Singleton set: It is a set that contains only one element.

For example: {2}, {*a*}.

The set whose cardinal number is 1, is called a singleton set.

Finite Set: Finite set is a set that contains countable number of elements. For example: A={1,2,3,4,5,6}

The set whose cardinal number is a whole number, is called finite set.

Infinite Set: Infinite set is a set that contains uncountable number of elements. For example: N={1,2,3,…}, set of Natural numbers because there is no end of the elements of this set i.e. there is no largest natural number in this set.

Equal Sets: Equal sets are the sets that contains the same elements. For example: A={1,2,3,4} and B={2,4,1,3}. A and B are equal as their elements are same.

Equivalent Sets: Equivalent sets are the sets whose cardinal numbers are same but the elements may differ. For example: A={a,b,c,d,e} and B={1,2,3,4,5}. Both the sets have 5 elements but their elements are different, hence A and B are equivalent sets.

Note:

Equal sets are always equivalent but vice-versa not true.

Subset: Sub set is a set that has at least one element less than its super set (main set). The number of subsets in a finite set A, whose cardinal number is n is 2^n. This is also the number of distinct subsets of A.

Note:

(a) Every non-empty set has two improper subsets.
(b) Proper subset is the set that has lesser cardinal number than its superset. The number of proper subsets in a finite set of cardinal number n is 2^n-2.
(c) The null set has only one subset (i.e. null set ifself) that is improper.
(d) Improper subsets are null set and the set itself.

Power Set: Power set is the set of all possible subsets of the set.

Note:

(a) Set itself and the empty set are the members of this set.
(b) If $A \subseteq B$ then $P(A) \subseteq P(B)$.
(c) $P(A \cap B)=P(A) \cap P(B)$, and $P(A) \cup P(B) \subseteq P(A \cup B)$.

Universal Set : Universal set is the set that is the superset of all sets.

Note:

Universal sets differ as the point of reference differ. For example: the super set of all students of a city may differ from the super set of all students of other city. Super set of reading materials may differ from the super set of kitchen materials.

Union of Sets: Union of sets A and B is given by $A \cup B=\{x : x \in A \text{ or } x \in B\}$.

Note:

$x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B$.

Intersection of Sets: Intersection of sets A and B is given by $A \cap B=\{x : x \in A \text{ and } x \in B\}$.

Note:

(a) $x \notin (A \cap B) \Rightarrow x \notin A \text{ or } x \notin B$
(b) $n(A \cup B) = n(A)+n(B)-n(A \cap B)$

Disjoint Sets: Disjoint sets are the sets that have no common elements i.e., $(A \cap B) = \phi$.

For example:

(a) The set of positive integers $\boldsymbol{Z}^+$ and set of negative integers $\boldsymbol{Z}^-$ have no common elements, hence they are disjoint sets.
(b) A family of various sets is called pairwise disjoint if no two members of its family have a common element.
For example:
If $A_1, A_2, A_3, \ldots, A_n$ are pairwise disjoint then $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n = \phi$. But if $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n = \phi$, then $A_1, A_2, A_3, \ldots, A_n$ may not be pairwise disjoint.
(c) If $A_1, A_2, A_3, \ldots, A_n$ are pairwise disjoint sets then
$n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)=n(A_1)+n(A_2)+n(A_3)+\ldots$

$.+n(A_n)$.

Difference of two sets: Difference of two sets A and B is given by A-B= $\{x :\in A$ and $x \notin B\}$.

Symmetric difference of two sets: Symmetric difference of two sets A and B is given by (A-B)∪(B-A). It is denoted by AΔB.

Complement of a set: Complement of a set is given by ***A'= U-A***, that is, set of all elements of the Universal set other than the elements of set A.

Note:

(a) Complement is the difference with Universal set.
(b) Complement of Universal set is an Empty set and vice- versa.ie (U)'=ϕ and (ϕ)'= U.
(c) Complement of a complement is the set itself.i.e. (A')'=A.

Idempotent Law:

(a) A∪A=A
(b) A∩A=A

Identity Law:

(a) A∪ϕ = A
(b) A∩U=A

Commutative Law :

(a) A∪B=B∪A
(b) A∩B=B∩A.

Associative Law :

(a) A∪(B∪C)= (A∪B)∪C
(b) A∩(B∩C) = (A∩B)∩C

Distributive Law :

(a) A∪(B∩C)=(A∪B)∩(A∪C)
(b) A∩(B∪C) = (A∩B)∪(A∩C)

De-Morgan Law :

(a) (A∪B)'=A'∩B'
(b) (A∩B)'=A'∪B'

Ordered pair: The ordered pairs are said to be equal if and only if the corresponding first elements are equal and the second elements are also equal.

For example:

(a) $(a,b) \neq (b,a)$
(b) $(a,b) = (c,d)$ iff $a = c$ and $b = d$

Cartesian product of two sets: Cartesian Product of two sets A and B is given by A × B = Set of all ordered pairs starting from A to B= { (a,b): a∈A and b∈B}.

(a) Graph of A × B represents points on the Cartesian plane.
(b) The number of elements in the Cartesian Product = n(A). n(B)
(c) If either of A or B is an infinite set then A × B is also an infinite set.
(d) n(A × B)= n(A). n(B) and n(P(A × B))= $2^{n(A).n(B)}$.
(e) If A and B are two non-empty sets having n elements common then A × B and B × A have n^2 elements common.
(f) If A= ϕ or B= ϕ then A × B= ϕ .
(g) If at least one of the A and B is infinite set then A × B is also an infinite set provided that the other is non-empty set.
(h) A × A × A = {(a,b,c): a,b,c∈ A }. (a,b,c) is called an ordered triplet. It represents points in 3-D space.

Regarding Number of elements:

(a) Number of elements belonging ***to exactly one of the sets A, B and C***=n(A)+n(B)+n(C)-2n(A∩B)-2n(B∩C)-2n(C∩A)+3n(A∩B∩C)

(b) Number of elements belonging to ***exactly two of the sets A, B and C*** = n(A∩B)+ n(B∩C)+n(C∩A)-3 n(A∩B∩C)

(c) Number of elements belonging to ***at least two of the sets A, B and C***=n(A∩B)+ n(B∩C)+n(C∩A)-2 n(A∩B∩C)

(d) Number of elements belonging to ***at most two of the sets A, B and C***=n(A∪B∪C)-n(A∩B∩C)

Sets containing:

(a) ***Only A occurs*** out of *A, B and C* $\left(A\cap\overline{B}\cap\overline{C}\right)$

(b) ***Both A and B, but not C occurs*** $\left(A\cap B\cap\overline{C}\right)$

(c) ***All the three occur*** $\left(A\cap B\cap C\right)$

(d) ***At least one occurs*** $\left(A\cup B\cup C\right)$

(e) ***At least two occur*** $\left(A\cap B\right)\cup\left(B\cap C\right)\cup\left(A\cap C\right)$

(f) ***One and no more occur*** $\left(A\cap\overline{B}\cap\overline{C}\right)\cup\left(\overline{A}\cap B\cap\overline{C}\right)\cup\left(\overline{A}\cap\overline{B}\cap C\right)$

(g) ***Exactly two of A, B and C occur*** $\left(A\cap B\cap\overline{C}\right)\cup\left(\overline{A}\cap B\cap C\right)\cup\left(A\cap\overline{B}\cap C\right)$

(h) ***None occurs*** $\left(\overline{A}\cap\overline{B}\cap\overline{C}\right)=\overline{A\cup B\cup C}$

(i) ***Not more than two occurs*** $\left(A\cap B\right)\cup\left(B\cap C\right)\cup\left(A\cap C\right)-\left(A\cap B\cap C\right)$

(j) ***Exactly one of A and B occurs*** $\left(A\cap\overline{B}\right)\cup\left(\overline{A}\cap B\right)$.

Notes:

(a) Some of the infinite sets cannot be expressed in the Roster form or Listing form.
For example:
The set of Real Numbers cannot be written in Roster form as there are always an infinite number of real numbers between any two numbers.

(b) The real numbers are very well depicted on a Real Number Line.

(c) The Real Numbers are also expressible in forms of Intervals: Closed Interval [3,5], Open Interval (3,5), Open-Closed Interval (3,5] and the Closed-Open Interval [3,5).

(d) Closed Interval [3,5] means { x: $3 \le x \le 5$}

(e) Open Interval (3,5) means { x: $3 < x < 5$}

(f) Open-Closed Interval means { x: $3 < x \le 5$}

(g) Closed-Open Interval means {x: $3 \le x < 5$}

Relation:

If A and B are two non-empty sets then ***a relation from A to B*** is defined as ***a subset of A × B***. This subset may be an empty as well as A × B.

If *(a,b)* belongs to R, then *a* is related to *b*, and written as *a R b*. If *(a,b)* does not belong to *R* then it is written as *a* ~~*R*~~ *b*.

Note:

(a) The set of all first element of the ordered pairs in a relation R from a set A to set B is called the domain of the relation R and the second element of ordered pair is called the range. The whole set B is called the codomain of the relation R.
(b) Range ⊆ Codomain (always)

Types of Relations

Void Relation: A relation R in a set A is called an Empty relation, if no element of A is related to any element of A, i.e. R= $\phi \subset A \times A$. Empty set is called the void relation as it is a subset of A × A.

Universal Relation: A relation R in a set A is called universal relation, if each element of A is related to every element of A, i.e. R= A × A. A × A is called the Universal relation.

The Empty Relation: R on a non-empty set X (i.e. a R b is never true) is not an equivalence relation, because although it is vacuously symmetric and transitive, it is not reflexive (except when X is also empty)

Identity Relation: Consider a set A= {1,2,3}. Identity relation of set A from A to A is given by $I_A=\{(1,1),(2,2),(3,3)\}$.

Every Identity relation is reflexive but every reflexive relation need not be an Identity relation, i.e., *Reflexive relation* has always some more elements than an *Identity relation*.

Inverse Relation: R^{-1}= { (1,3),(2,4),(5,6)} is said to be inverse relation of R ={(3,1),(4,2),(6,5)}.

The domain of R becomes the range of R^{-1} and vice-versa.

Reflexive Relation: If all the elements of A are related to itself, then the relation is said to be reflexive.

For example:

If A={1,2,3} then the relation R_1={(1,1),(2,2),(3,3)} is a reflexive relation or R_2={(1,1),(2,2),(3,3),(1,2),(1,3)} is reflexive relation but R_3={(1,1),(2,2)} is not a reflexive relation as (3,3) is not therein.

Note:

(a) ***A reflexive relation on A is not necessarily the identity relation on A.***
(b) ***The Universal relation on a non-void set A is reflexive***.
(c) The cardinal number of Reflexive relations 2^A where A is the number of elements in set A.

Symmetric Relation: If $(x,y)\in R \Rightarrow (y,x)\in R$. ***R is symmetric if and only if*** $R^{-1}=R$.

Note:

(a) ***The Identity and the Universal relations on a non-void set are symmetric relations.***
(b) ***A reflexive relation on a set A in not necessarily symmetric***.

Transitive Relation: R is called a transitive relation if and only if $(a,b)\in R,(b,c)\in R \Rightarrow (a,c)\in R$.

Equivalence Relation: A relation R in a set A is said to be an **equivalence relation** if R is reflexive, symmetric and transitive.

Ordered Relation

R is called to be an ordered relation if it is transitive but not equivalence relation.

Partial Ordered Relation

R is called a Partial Ordered relation if it is reflexive, transitive but not symmetric relation.

Note:

(a) If R is reflexive, then R^{-1} is also reflexive.

(b) If R is symmetric, then R^{-1} is also symmetric.

(c) If R is transitive, then R^{-1} is also transitive.

(d) If R is an equivalence relation, then R^{-1} is also an equivalence relation.

(e) Total ***number of relations*** from A to B is the number of possible subsets of A × B. If *n(A)=p* and *n(B)=q* then total number of relations = 2^{pq} as each subset of A×B defines relation from A to B.

(f) Among the total number of relations 2^{pq}, the void relation ϕ and the universal relation A × B are trivial relations from A to B.

Equivalence Class [a] : Let R be an equivalence relation in A (non-empty set). Let $a \in X$. Then the equivalence class of a, (denoted by [a]) is defined as the set of all those points of A that are related to a under the relation R.

It is expressed as $[a]=[x \in A: x R a]$.

Note:

(a) $b \in [a] \Rightarrow a \in [b]$
(b) $b \in [a] \Rightarrow [a]=[b]$
(c) ***Two equivalence classes are either disjoint or identical***.

Congruence Modulo (m) :

Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m if a-b is divisible by m. It is written as

$a \equiv b \pmod{m}$.

i.e. $a \equiv b \pmod{m} \Leftrightarrow$ a-b is divisible by m.

For example:

$18 \equiv 3 \pmod{5}$ as 18-3 is divisible by 5.

Also, $5 \equiv 10 \pmod{5}$ as 5-10 is divisible by 5.

Note:

The relation "congruence modulo m" is an equivalence relation.

Composition of Relations : Let R and S be two relations from sets A to B and B to C respectively.

Then the relation *SoR* from A to C such that $(a,c) \in SoR \Leftrightarrow \exists\, b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. This relation is called the composition of R and S.

In general $RoS \neq SoR$.

Also if ***SoR*** exists then ***RoS*** may not exist. But $(SoR)^{-1} = R^{-1} o\, S^{-1}$.

Remembering Facts

(a) Let A and B be two non-empty sets having n elements common, then $A \times B$ and $B \times A$ will have n^2 elements common.

(b) The universal relation on a set A containing at least 2 elements is not anti-symmetric, because if $a \neq b$ are in A, then a is related to b and b is related to a under the universal relation will imply that a=b but $a \neq b$.

(c) The set $D=\{(a,a): a \in A\}$ is called the diagonal line of $A \times A$. The relation R in set A is anti-symmetric iff $R \cap R^{-1} \subseteq D$.

(d) The relation "is congruent to" on the set T of all triangles in a plane is a transitive relation.

(e) The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

(f) If a set contains $2n+1$ elements. Then the number of subsets of this set that contains more than n elements is equal to 2^{2n}.

Function

A RELATION from set A to set B is called a function if every element of set A has one and only one image in set B.

In other words, a function ***f*** is a relation from a non-empty set A to a non-empty set B such that the domain of ***f*** is A and no two distinct ordered pairs in ***f*** have the same first element.

Or,

If a perpendicular dropped on the axis of X, cuts the curve at ***one*** point only, then the graph represents a function *otherwise not*.

Note:

The word "*function*" was introduced into Mathematics by Leibniz, who used this term primarily to refer to certain kinds of Mathematical Formulas. It was later realized that Leibniz's idea of *function* was much too limited in its scope, and the meaning of the word has since undergone many stages of generalization.

Number of Functions : If *n(A)=a* and *n(B)=b* then ***total number of functions*** from *A to B*=b^a.

(a) Total ***number of one-one functions*** from A to B = $^{b}P_{a}$ if $b \geq a$ otherwise it is *0* where *n(A)=a* and *n(B)=b*.

(b) Total ***number of Onto function s***(surjections) from A to B=$\sum_{r=1}^{b}(-1)^{b-r}.^{b}C_{r}.r^{a}$

(c) The ***number of Onto functions*** defined from a finite set A containing *a* elements onto a finite set B containing 2 elements = 2^a-2.

(d) Total ***number of one-one onto*** (bijection) from A to B = *a! or b!* (factorial a or factorial b) as the number of elements in A = number of elements in B.

Special Functions

One-one function or Injective function : A function f : X → Y is defined to be **one-one** (or **injective**) if for every $x_1, x_2 \in X$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Otherwise, f is called many-one.

Onto Function or Surjective:A function f : X → Y is said to be **onto** (or **surjective**) if for every y ∈ Y, there exists an element x in X such that f (x) = y.

One-one onto or Bijective :A function f : X → Y is said to be **one-one** and **onto** (or **bijective**), if f is both one-one and onto.

Remembering Facts

(a) For ***an onto function range = co domain***.

(b) A one- one function defined from a finite set to itself is always onto but if the set is infinite then it is not the case.

(c) Linear polynomial functions (ax+b), x, e^x , log x are always one-one function.

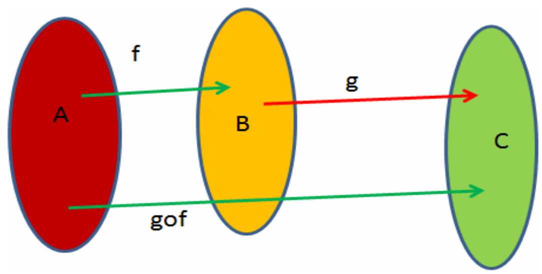
(d) If (dy/dx) > 0, or, (dy/dx) < 0, then ***y=f(x)*** is one-one, iff ***f*** is a continuous function.

(e) All even functions, modulus functions, periodic functions are always many-one functions.

(f) Square functions and Trigonometric functions are many-one functions in their domain.

(g) sin√x and sin x^2 are not periodic as they cannot be expressed in ***f(x+T)=f(x)***.

Composite Functions : If ***f : A → B*** and ***g : B → C*** are two functions, then the **composition** of *f* and *g*, denoted by *gof*, is defined as the function ***gof** : A → C* given by *gof (x) = g(f (x)),* ∀ x ∈ A.



Composition of *f* and *g* is written as *gof* and not *fog*. *gof* is defined if the range of
f ⊆ *domain of f* and *fog* is defined if *range of g* ⊆*domain of f*.

Remembering Facts:

(a) A function f : X → Y is defined to be **invertible**, if there exists a function g : Y → X such that gof = I_X and fog = I_Y. The function g is called the inverse of f and is denoted by f^{-1}

(b) If f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible.

(c) In general fog and gof are not equal, i.e. composition of functions is not commutative but it is associative. i.e. if f: X→ Y, g: Y→ Z, and h: Z →S then h o (g o f)=(h o g) o f

(d) If either of the two f and g, one is an Identity function then, fog and gof are identity functions.

(e) If f: A →B and g: B →C are one-one and onto then gof : A→ C is also one-one and onto. But if gof is one-one then only f is one-one g may or may not be one-one. If gof is onto then g is onto, f may or may not be onto.

(f) Given a finite set X, a function f : X → X is one-one (respectively onto) if and only if f is onto (respectively one-one). This is the characteristic property of a finite set. This is not true for infinite set.

(g) Let f : X → Y and g : Y → Z be two invertible functions. Then gof is also invertible with $(gof)^{-1} = f^{-1} o\, g^{-1}$.

Facts Related to the Domain and the Range:

(a) The set of values taken by X is called the **domain** and is denoted by Df and the set of values taken by Y is called the **range** and is denoted by Rf.

(b) Domain of y= f(x) ± g(x) or f(x).g(x) or f(x)/g(x) is given by $D_1 \cap D_2$ or $D_1 \cap D_2 - \{g(x)=0\}$.

(c) Domain of **absolute value of function** or **Modulus function**, y = |x| is (-∞ ,∞).

(d) Domain of **exponential function** $f(x) = a^x$, a > 0 and a ≠ 1 is set of Real Number.

(e) Domain of **logarithmic function** $f(x)=\log_a x$; (x , a >0) and a ≠ 1 is all real positive numbers.

(f) If f(x)=c for all c∈ X ,and Y is singleton set of c, f: X →Y is called the **constant function**.

(g) If f(x)=x for all x∈ X , f: X →Y is called the **identity function**.

(h) If $f(x)= x^a$ for a∈ R , then f:R→ R is called the **Power function**.

(i) Absolute value function is also called modulus function or **numerical value function**. It is defined as |x| = + x if x>0 and |x| = -x if x <0

(j) **Signum function** y = sgn (x)= |x|/x .

(k) Domain of signum function is R.

(l) The **greatest integer function**: it is also called the **step up function**. It is written as f(x)= [x]. It means f(0)=[0]=0, f(3/2)=[3/2]= 1, f(-1/2)=[-1/2]=-1. It consists of two parts: integral part and the fractional part. The fractional part is always greater than or equal to zero but less than 1. It is also denoted as f(x)=[x]=x+{x}.

The Fractional Part Function

(a) {x}=x, if 0 ≤x < 1 and 0 if x∈ Z.
(b) {-x}=-1-{x} if x ∉Z .

Exponential Function: Exponential function is defined as $f(x)= a^x$. It increases if x > 0 and decreases if x < 0.

Logarithmic Function : Logarithmic function is defined as $f(x)= \log_a x$. It is defined only when a>0, it is not defined when a≤ 0. It increases if a > 1 and decreases if 0<a<1.

Even function : If f(-x)=f(x) for all x.

Odd function: If f(-x)= **-** f(x) for all x.

Facts relating to EVEN and ODD functions

(x) The product of two even or two odd functions is always even function.

(y) The product of even and odd is always an odd function.

(z) Every function can be expressed as the sum of an even and an odd function.for example: $f(x)=\frac{1}{2}\left(f(x)+f(-x)\right)+\frac{1}{2}\left(f(x)-f(-x)\right)$

Periodic Function: A function f(x) is said to be periodic if there exists such an T>0 for which f(x+T)=f(x-T)=f(x) for all x∈ X. note: there are infinitely many T satisfying the equality but the least positive is said to be the period.

Period of sinx, cosx,cosecx, secx = 2π and the period of tanx, cotx = π,

Monotonic Functions : Monotonic functions are defined on interval.

These are of nature: monotonic increasing if for all $x_1,x_2 \in[a,b]$ and $x_1 < x_2$ there exists $f(x_1)\le f(x_2)$.

Note:

(a) Monotonic decreasing if for all $x_1,x_2 \in[a,b]$ and $x_1 < x_2$ there exists $f(x_1)\ge f(x_2)$.

(b) Strictly monotonic increasing if for all $x_1,x_2 \in[a,b]$ and $x_1 < x_2$ there exists $f(x_1)<f(x_2)$.

(c) Strictly monotonic decreasing if for all $x_1,x_2 \in[a,b]$ and $x_1 < x_2$ there exists $f(x_1)>f(x_2)$.

Special type of function:

Dirichlet function

λ(x) = 1, if x is rational

= 0, if x is irrational.

It is discontinuous at each point.

Example:

λ(0) =1

λ(-1/2) =1

λ(√2) =0

λ(π) =0

Facts to Remember for Solving Questions:

(a) If f(x) is polynomial function satisfying $f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$ then the function should be assumed as $f(x)=1\pm x^n$

(b) (x-1) < [x]≤ x, 2x-1 < [2x] ≤ 2x.

(c) $\left[\frac{n+1}{2}\right]+\left[\frac{n+2}{4}\right]+\left[\frac{n+4}{8}\right]+\left[\frac{n+8}{16}\right]+....=n, n\in N.$

(d) $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\left[x+\frac{3}{n}\right]+....$
$=[nx], n\in N.$

The Factorial Function: This is defined as ***f(n)=*** n! = l.2.3….. *n. for all positive integers.*
The domain of this function is the set of positive integers.
The function value (Range) increases so rapidly that it is more convenient to display this function in tabular form rather than as a graph. This is listed as the pairs *(n, n!)* .

Equal Functions : Two functions f and g are equal if and only if
f and g have the same domain, and
f(x) = g(x) for every x in the domain.

Linear **Function :** A function f defined for all real x by a formula of the form *f(x) = ax + b* , is called a linear function because its graph is *a straight line*.

Lattice Point : A point (x, y) in the plane is called a ***lattice point*** if both the co-ordinates x and y are integers.

BINARY OPERATION

A **binary operation** * on a set A is a function * : A × A → A. Notation: * (a, b) by a * b.

Facts:

(a) Given a binary operation $* : A \times A \to A$, an element $e \in A$, if it exists, is called **identity** for the operation *, if $a * e = a = e * a$, $\forall a \in A$. Identity element is unique.
(b) An element $a \in X$ is invertible for binary operation $* : X \times X \to X$, if there exists $b \in X$ such that $a * b = e = b * a$ where, e is the identity for the binary operation *, then the element b is called **inverse of a** and is denoted by a^{-1}.
(c) An operation * on X is **commutative** if $a * b = b * a$ $\forall$ a, b in X.
(d) An operation * on X is **associative** if $(a * b) * c = a * (b * c)$ $\forall$ a, b, c in X.
(e) If the operation table is symmetric about the diagonal line then, the operation is commutative.
(f) Addition (+) and multiplication (.) on N , the set of natural numbers are binary operations. But subtraction (-) and division (÷) are not since (4,5)= 4-5 = -1 does not belong to N and (4÷5) also does not belong to N.

(g) Binary operations are functions.

Number of Binary Operations : Number of binary operations on A = Number of functions from A x A to A = A^{AxA} , where A represents the Cardinal Number of the Set A. For example: if the cardinal number of A is 3 then the number of binary operations= $3^{3x3}= 3^9$.

Solved Questions

1. If f: R→ R and g:R →R are defined by f(x)= 2x+3, and g(x)=x² + 7, then find the values of x such that g(f(x))=8.

Solution:

$g(f(x))= g(2x+3)=(2x+3)^2+7=8$ (given).

Therefore, $4x^2+12x+8=0$

Or, $x^2+3x+2=0$

Or, x=-1,-2

2. If f: R→ R, and g: R→ R are given by f(x)= |x|, and g(x)= [x] for each x∈ R, then find { x ∈R:g(f(x)) ∈ f(g(x)).

Solution:

g(f(x))=g(| x|)= [| x |]

f(g(x))=f([x])= |[x]|

Two cases may arise

(i) x ≥ 0, i.e. [| x|] = [x]= |[x]| i.e. f(g(x))=g(f(x))
(ii) x <0, i.e. [x] ≤x <0 i.e. |[x]| ≥ |x| i.e. |[x]| ≥|x| ≥ [|x|] ,as [x]≤ x for all x f(g(x))≥ g(f(x)) for all x ∈R.

3. Let f: [-2,2]→ R be defined by

f(x) = -1, for -2≤ x≤ 0

=x-1, 0≤ x≤ 2

Then find { x∈ [-2,2]: x≤ 0 and f(|x|) =x }

Solution:

Obviously,

f(-1)=-1, f(1)=0,f(0)=-1,f(|-1/2|)=f(1/2)=1/2 -1=-1/2

Then the required result is {-1/2}

4. If $e^{f(x)} = \frac{10+x}{10-x}, x \in (-10,10)$ ***and***
$f(x) = k\left(\frac{200x}{100+x^2}\right)$ ***then find value of k***

Solution:

Using the definition of logarithm

$x^y = z \Rightarrow \log_x z= y$

Let us write the given expression as below:

$$f(x) = \log_e \frac{10+x}{10-x}$$

Also write $f\left(\frac{200x}{100+x^2}\right)$ using the above definition of the function as : $= \log_e \frac{10 + \frac{200x}{100+x^2}}{10 - \frac{200x}{100+x^2}}$

= 2 f(x)

i.e. f(x)= (1/2) $f\left(\frac{200x}{100+x^2}\right)$

Hence

k= 1/2

5. ***If x,y,z are positive real numbers, then find the relation between x,y,z when*** $\frac{x^3}{z} < \frac{x^3+y^3+z^3}{x+y+z} < \frac{z^3}{x}$

Solution:

Out of many possible cases, assume

If x < y < z then $x^3 < y^3 < z^3$

Or,

2x < x + y < 2z, $2x^3 < x^3 + y^3 < 2z^3$

Or

3x < x + y +z < 3 z, $3x^3 < x^3 + y^3 + z^3 < 3z^3$

Or

(1/3x) > 1/(x+y+z) > (1/3z),under the above condition

Or

(1/3z) < 1/(x+y+z) < (1/3x), under the above condition

Or

$3x^3/(3z) < (x^3 + y^3 + z^3)/(x+y+z) < 3z^3/(3x)$

6. ***Find the number of reflexive relations of a set with 3 elements.***

Hint: The number of reflexive relations on an n-element set is 2^{n^2-n}.

7. ***If A={(x, y): $x^2 + y^2 = 25$} and B={(x, y): $x^2 + 9y^2 = 144$} then evaluate A∩B.***

Hint: Note A ∩ B means the points of intersection of the two curves. Obviously it is 4.

8. ***Let A_1, A_2, ….,A_{30} be thirty sets each containing five elements and B_1, B_2,…, B_n are n sets each with three elements such that $\bigcup_{i=1}^{30} A_i = \bigcup_{i=1}^{n} B_i$ A_i =S. If each element of S belongs to exactly ten of the A_i's and exactly 9 of the a_j's, then find the value of n.***

Hint:

Each of A_i has 5 elements, hence $\sum_{i=1}^{30} n(A_i) =$ 5.30=150=10m

(as each element of S belongs to exactly 10 of A_i's if S has m distinct elements). i.e. m=15.

Similarly, B_j has 3 elements, hence $\sum_{j=1}^{30} n(B_j)$ =3n=9m i.e. n=45

9. ***A set contains (2n+1) elements. Find the number of subsets of this set containing more than n elements.***

Solution:

The required number= ${}^{2n+1}C_{n+1}$ + ${}^{2n+1}C_{n+2}$+….+ ${}^{2n+1}C_{2n+1}$= $(1/2)(2^{2n+1})$ by use of binomial theorem.

10. A survey shows that 63% people like cheese, 76% like apples. If x% like both cheese and apples, then find the value of x.

Solution:

Obviously, n(C)=63, n(A)=76, n(C ∩A)=x.

Use: n(C ∪A)= n(C)+n(A)-n(C ∩A) … (i)

And n(C ∪A) ≤ 100 … (ii)

i.e. n(C) + n(A)- n(C∩ A)≤ 100

⇒ 63+76-x ≤ 100

⇒ x ≥39

And n(C ∩A) ≤ n(C)

⇒ x ≤ 63

⇒ 39 ≤ x ≤ 63

11. Let A be a non-empty set and let * be a binary operation on P(A), the Power set of A defined by X * Y = (X- Y)∪ (Y-X) for X,Y ∈P(A). Show that

(i) ϕ is the identity element for * on P(A).

(ii) X is invertible for all X ∈ P(A) and X=X^{-1}.

Solution:

For any X∈ P(A),

X * ϕ = (X- ϕ) ∪(ϕ -X) by definition.

= X ∪ ϕ = X

Similarly,

ϕ * X = X

⇒ X * ϕ= ϕ* X = X,∀ X∈ P(A)

⇒ ϕ is an Identity element in P(A) for binary operation * on P(A).

Also, for any X ∈P(A), evaluate that

X * X= ϕ by definition

⇒ every element X of P(A) is invertible and is inverse of itself.

12. ***If A is finite, then find the number of distinct subsets of A***.

Hint:

Number of distinct subsets= ${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \ldots + {}^{n}C_n = 2^n$

13. Two finite sets A and B have m and n elements. Number of subsets of A is 56 more than that of B. Evaluate the values of m and n.

Hint:

Obviously, $2^m = 56+2^n$.

By trial, m=6 and n=3

14. Let P = {(x, y): y = a^x, x ∈ R}, Q = {(x, y): y = $\log_a x$, x > 0} and 0 < a < 1 then find the cardinal number of the set of solutions of P and Q.

Hint:

Obviously, both the graphs $y=a^x$ and $y=\log_a x$ are the reciprocal of each other that intersect at only one point. Hence number of solutions=1.

15. ***If two sets A and B are having 99 elements in common, then find the number of elements common to each of the sets A x B and B x A*** .

Hint:

The required number = 99^2

16. On the power set P of a non-empty set A, an operation Δ is defined as XΔ Y= (X ∩Y^c)∪ (X^c

∩Y). Then show that for (P, Δ), it is commutative and associative with an identity.

Solution:

Δ is obviously, symmetric difference i.e.

$X \Delta Y = (X-Y) \cup (Y-X)$

And

$X-Y = X \cap Y^c$

Δ is commutative, associative and ϕ is an identity element for this operator.

$A \Delta \phi = (A - \phi) \cup (\phi - A) = A \cup \phi = A$ for all $A \in P$.

17. If B is the set of numbers obtained by adding 1 to each of the even numbers, then write it in its set builder form.

Solution:

$B = \{t+1 : t = 2n, n \in Z\} = \{2n+1, n \in Z\} = \{x : x$ is odd and $x \in Z\}$

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—oo—

First they will ignore you,

then they laugh at you,

then they fight you,

then you win (truth follows…)

- Mahatma Gandhi