DIFFERENTIAL CALCULUS

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Differential Calculus is a branch of Mathematics where we study the determination, properties, and applications of derivatives and differentials. This is also defined as the sub-field of calculus where we study the rate of change of quantities.

For example: If the distance changes with respect to time, then the differential calculus helps in finding the velocity i.e. the rate of change of distance with respect to time at any instant.

Or, If some balloon is being filled with air, then we can find the change in the volume of the inflating balloon with respect to time at any moment with the help of differential calculus.

Or, If water is leaking from a tank and we want to find out the rate at which level of water is getting down, the differential calculus will help us in determining it.

Basic Terminology

Function: If X and Y are two non-empty sets and to each element of X there corresponds a unique element of Y, then we say that there exists a function from X to Y and is written as $f : X \to Y$ or $f(x)=y$ where $x \in X$ and $y \in Y$.

Or, *A function f is a set of ordered pairs (x, y) such that no two of which have the same first member.*

Note: The word "***function***" was introduced into Mathematics by ***Leibniz***, who used this term primarily to refer to certain kinds of Mathematical Formulas. It was later realized that Leibniz's idea of *function* was much too limited in its scope, and the meaning of the word has since undergone many stages of generalization.

Domain and Range:

The set of values taken by ***X*** is called the **domain** and is denoted by ***Df***.

The set of values taken by ***Y*** is called the **range** and is denoted by ***Rf***.

Or, ***Domain*** is the value of ***x*** for which the function remains defined. Hence ***domain*** is also named as ***domain of definition***.

Facts Relating To Domain

(a) Domain of expressions under even roots i.e. square root, fourth root etc., is always ≥ 0.

(b) Domain of $f(x) \pm g(x)$ or $f(x).g(x)$ or $f(x)/g(x)$ is given by $D_1 \cap D_2$ or $D_1 \cap D_2 - \{g(x)=0\}$ respectively.

Facts Relating To Range

(a) If $R_f \subseteq Y$, then the function is called **Into** Function.
(b) If $R_f = Y$, then the function is called **Onto** Function.
(c) Each of the two ***Into*** and ***Onto*** (**Surjective**) functions is either ***One-One*** or ***Many–One*** function.
(d) ***One-one Onto*** function is also called ***Bijective*** (or ***Injective and Surjective***) function.

Types of Function

There are a number of functions. Some of them, that are useful for the students of class XII and equivalents are being discussed here with their properties.

Inverse Function: If $f: X \to Y$ is a function defined, then f^{-1}, the inverse of f is defined as $f^{-1}: Y \to X$. Inverse of a function is defined only when it is one-one-onto. Domain is Y and range is X.

Constant Function: If $f(x) = c$ for all $x \in X$, then $f : X \to Y$ is called a constant function. Domain of constant function is ***X*** and range is {c}, a singleton set.

Identity Function: If $f(x) = x$ for all $x \in X$, then $f : X \to Y$ is called an identity function.

Domain and Range of this function is X or, $(-\infty, \infty)$, if it is defined on set of Real Numbers.

Power Function: If $f(x) = x^a$ for $a \in R$, then $f : R \to R$ is called Power function.

Domain of power function is $(-\infty,\infty)$ and the range is also $(-\infty, \infty)$.

Absolute value function or Modulus Function: Absolute value function is also called modulus function or numerical value function. It is defined as $f(x) = |x|$.

$|x|$ means it is + x if x > 0, -x if x < 0, and 0 if it is 0.

Domain of absolute value function is $(-\infty, \infty)$. The range of this function is $[0, \infty)$.

Signum function: It is defined as, $y = \text{sgn}(x) = \frac{|x|}{x}$.

Domain of signum function is R.

The range is {-1,1}.

Sometimes, it is defined as,

$$y = \begin{cases} \frac{|x|}{x} \; i.e., \\ +1 \; when \; x > 0 \\ -1 \; when \; x < 0 \\ 0 \; when \; x = 0 \end{cases}$$

In this case the domain is R, and the range is {-1,0,1}

Greatest Integer Function: The greatest integer function is also called the step up function. It is written as ***f(x) = [x].***

It means, f(0) = [0] = 0; f(3/2) = [3/2] = 1;

f(-1/2) = [-1/2] = -1.

It consists of two parts:

The ***Integral part*** and the ***fractional part***.

The fractional part is always greater than or equal to zero but less than 1.

It is also denoted as ***f(x)=[x]=x+{x}***.

Its Domain is R and the Range is the set of all integers.

Note:

(a) [x+k]=[x]+k, if k is any positive integer.
(b) [-x]=-[x], for all $x \in Z$.
(c) [-x]=-[x]-1, $x \notin Z$.
(d) $[x_1+x_2] \geq [x_1]+[x_2]$
(e) $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right], \forall n \in N$
(f) $[x] + \left[x + \frac{1}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx]$

The fractional part function: The fractional part function

{x}	***= x,***	if $0 \leq x < 1$, and
	= 0	if $x \in Z$
{-x}	***= -1-{x}***	if $x \notin Z$

Domain is R, Range is [0,1).

Exponential function: Exponential function is defined as $f(x)= a^x$.

It increases if x > 0 and decreases if x < 0 in case of *a>0*.

Domain of exponential function $f(x) = a^x$, $a > 0$ and $a \neq 1$ is set of Real Numbers.

Logarithmic function: Logarithmic function is defined as $f(x)= \log_a x$.

It is defined only when ***a>0***. It is not defined when $a \leq 0$. It increases if ***a > 1*** and decreases if ***0<a<1***.

Domain of logarithmic function

$f(x) = \log_a x$; (x, a >0) and $a \neq 1$ is all real positive numbers.

Even function: A function f(x) is called an even function if ***f(-x) = f(x)*** for all x.

Odd function: A function f(x) is called an odd function if ***f(-x) = - f(x)*** for all x.

Note:

(a) The product of two even or two odd functions is always even function.
(b) The product of even and odd is always an odd function.
(c) Every function can be expressed as the sum of an even and an odd function.

For example

$$f(x) = \frac{1}{2}\left(f(x) + f(-x)\right) + \frac{1}{2}\left(f(x) - f(-x)\right)$$

Periodic function: A function f(x) is said to be periodic if there exists such an ***T > 0*** for which ***f(x + T) = f(x - T) = f(x)*** for all $x \in X$.

Note:

(a) There are infinitely many ***T*** satisfying the equality but the ***least positive*** is said to be ***the period***.
(b) Period of ***sinx, cosx, cosecx, secx is 2π*** and the period of ***tanx, cotx is π.***

Monotonic Function: Monotonic functions are defined on interval. Monotonic functions are of following types:

(a) Monotonic increasing if for all $x_1, x_2 \in [a,b]$ and $x_1 < x_2$ there exists $f(x_1) \leq f(x_2)$.
(b) Monotonic decreasing if for all $x_1, x_2 \in [a,b]$ and $x_1 < x_2$ there exists $f(x_1) \geq f(x_2)$.

(c) Strictly monotonic increasing if for all $x_1, x_2 \in [a,b]$ and $x_1 < x_2$ there exists $f(x_1) < f(x_2)$.
(d) Strictly monotonic decreasing if for all $x_1, x_2 \in [a,b]$ and $x_1 < x_2$ there exists $f(x_1) > f(x_2)$.

Composite Function: If ***f(x) = y*** and ***g(y) = z*** then ***fog(y)*** is defined if the range of ***g*** $\subseteq$ ***domain of f*** ; and ***gof(x)*** is defined if range of ***f*** $\subseteq$ ***domain of g***.

$fog \neq gof$ (in general).

Note:

(a) If ***f*** is one-one, ***gof*** is one-one.
(b) If ***f*** is onto, ***gof*** is onto.
(c) If ***f, g*** are one-one onto, ***gof*** is one-one onto.

Domains of some functions:

(a) $\sin^{-1} x$ = [-1,1]
(b) $\cos^{-1} x$ = [-1,1]
(c) $\tan^{-1} x$ = R
(d) $\cot^{-1} x$ = R
(e) $\text{cosec}^{-1} x$ = R-(-1,1)
(f) $\sec^{-1} x$ = R-(-1,1)
(g) $\sqrt{(a^2 - x^2)}$ =[-a,a]
(h) $\sqrt{(x^2 - a^2)}$ =$(-\infty,a]\cup[a,\infty)$
(i) $1/\sqrt{(a^2 - x^2)}$ =(-a,a)
(j) $1/\sqrt{(x^2 - a^2)}$ = $(-\infty,a)\cup(a,\infty)$
(k) $\sqrt{(x-a)(b-x)}$ = [a,b] iff a<b
(l) $1/\sqrt{(x-a)(b-x)}$ = (a,b) iff a<b
(m) $\sqrt{(x-a)(x-b)}$ = $(-\infty,a]\cup[b,\infty)$ if a<b
(n) $1/\sqrt{(x-a)(x-b)}$ = $(-\infty,a)\cup(b,\infty)$ if a<b
(o) $\sqrt{\dfrac{x-a}{x-b}}$ = $(-\infty,a]\cup(b,\infty)$ if a<b
(p) $\sqrt{\dfrac{x-a}{x-b}}$ = $(-\infty,b)\cup[a,b]$ if a>b
(q) $\sqrt{\dfrac{a-x}{b-x}}$ = [a,b] if a<b
(r) $\sqrt{\dfrac{a-x}{b-x}}$ = [b,a] if a>b
(s) log (x-a)(b-x) = (a,b) if a<b
(t) log (x-a)(x-b) = $(-\infty,a)\cup(b,\infty)$ if a<b

Ranges of some functions:

(a) $\sec^{-1}x$ = $[0,\pi]-\{\pi\}$
(b) $\text{cosec}^{-1}x$ = $[-\pi/2,\pi/2]-\{0\}$
(c) $\cot^{-1}x$ = $(0,\pi)$
(d) $\tan^{-1}x$ = $(-\pi/2,\pi/2)$
(e) $\sin^{-1}x$ = $[-\pi/2,\pi/2]$
(f) $\cos^{-1}x$ = $[0,\pi]$

Dirichlet function

$\lambda(x)$ = 1, if x is rational

= 0, if x is irrational.

It is discontinuous at each point.

Examples:

(a) $\lambda(0)$ =1
(b) $\lambda(-1/2)$ =1
(c) $\lambda(\sqrt{2})$ =0
(d) $\lambda(\pi)$ =0

Note:

(a) If ***f(x)*** is polynomial function satisfying ***f(x)f(1/x)=f(x)+f(1/x)*** then the function should be assumed as $f(x) = 1 \pm x^n$.
(b) $(x-1) < [x] \leq x$, $2x-1 < [2x] \leq 2x$.

The factorial function: This is defined as ***f(n)=*** n! = 1.2.3….. *n. for all positive integers.*

The domain of this function is the set of positive integers. The range increases so rapidly that it is more convenient to display this function in tabular form rather than as a graph. This is listed as the pairs *(n, n!)*.

Equal Functions: ***Two functions*** *f and g are equal if* and only if

(i) *f and g have the same domain, and*
(ii) *f(x) = g(x) for every x in the domain.*

Linear function: A function f defined for all real x by a formula of the form *f(x) = ax + b* , is called a linear function because its graph is *a straight line*.

Lattice Point: A point (x, y) in the plane is called a ***lattice point*** if both the co-ordinates x and y are integers.

Determinate Form: When a unique value of an expression ***f(x)*** at ***x = a*** is possible, it is said that it is of the determinate form.

For example: The expression f(x)= $\frac{x^2-4}{x-2}$ is determinate at ***x = 1*** as it has a unique value 3 at x=1.

Indeterminate Form: When a unique value of an expression ***f(x)*** at *x=a* is not possible, it is said that it is indeterminate.

For example: The expression f(x)= $\frac{x^2-4}{x-2}$ at x=2 becomes $\frac{0}{0}$ which will give no unique value at x=2, hence it is in Indeterminate form.

Some other Indeterminate forms

$\frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 1^{\infty}, 0^0, \infty^0$ **etc.**

Note:

(a) $\log_a 0$ is not defined but $\log_a 0 \to -\infty$ for a > 1 and $+\infty$ for 0 < a < 1.

(b) In case of $1^{\infty}, 0^0, \infty^0$ take logarithm and then use the appropriate method to evaluate the limit.

Continuity: A function is said to be continuous when the value of the function is equal to its limit, i.e.

Value = Right Hand limit = Left Hand limit.

Facts regarding Continuity:

(a) A function is said to be continuous at a point x=c if f(c)= $\lim_{x\to c+} f(x) = \lim_{x\to c-} f(x)$

(b) If *RH Limit=LH Limit=value of the function*, the function is continuous otherwise discontinuous.

(c) If f(x) and g(x) are continuous then cf(x) is also continuous.

(d) f(x)±g(x), f(x).g(x), f(x)/g(x) are also continuous.

(e) If f(x) is defined on [a,b] then f(x) is said to be continuous at end points at x=a if $f(a) = \lim_{x\to a^+} f(x)$ and at x=b if $f(b) = \lim_{x\to b^-} f(x)$. At x=a, LHL and at x=b, RHL cannot be checked.

(f) A function is said to be continuous on its domain if it is continuous at the end points and at all points lying between a and b.

(g) If f(x) is defined on (a,b) then the function cannot be checked for continuity at end points as they are not included in the domain at all. In this case only at the interior point, the continuity may be checked.

(h) If f(x) is continuous on [a,b] such that f(a) and f(b) are of opposite signs, then there exists at least one solution of f(x)=0 in the open interval (a,b).

(i) Every polynomial is continuous at every point of the real line.
For example
$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$
is continuous on R.

(j) Every Rational function is continuous at every point where the denominator is not zero.

(k) Logarithmic, Exponential, Trigonometric, Inverse Trigonometric, Modulus functions are continuous in their domain of definition.

(l) Point Function (i.e. domain and range containing only one point) is a discontinuous function.

Cauchy's Definition of Continuity: A real valued function f defined on an open interval I is said to be continuous at a $\in$ I iff for any arbitrarily chosen positive number ε, however small, we get a corresponding number δ > 0 such that | f(x) - f(a) | < ε for all values of x for which | x – a | < δ .

Heine's Definition of Continuity: let a function f be defined on some neighbourhood of a point a, then f is said to be continuous at a iff for every sequence ***a_1, a_2, a_3,…., a_n,…***of real numbers for which

$\lim_{n\to\infty} a_n = a$, we have

$\lim_{n\to\infty} f(a_n) = \lim f(a)$

Discontinuity: The function is said to be discontinuous if either the limit does not exist or value is not equal to its limit.

Note:

(a) The discontinuity is said to be of ***first kind*** if both the limits (Right Hand Limit and Left Hand Limit) exist and are not equal. This is also called non-removable discontinuity of first kind.

(b) The discontinuity is said to be ***removable*** if $\lim_{x\to c+} f(x) = \lim_{x\to c-} f(x) \neq f(c)$

(c) The discontinuity is said to be of ***second kind*** if at least one of the limit does not exist. **Remember!** The limits are said to be existing if they are finite and definite. This is also called infinite discontinuity.

(d) The difference between RHL and LHL is called the ***jump*** discontinuity.

Properties of Limits

(a) If $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = m$, then $\lim_{x\to a}\{f(x) \pm g(x)\} = l \pm m$; if l and m exist

(b) $\lim_{x\to a}\{f(x).g(x)\} = l.m$

(c) $\lim_{x\to a}\{f(x)\}^{g(x)} = l^m$

(d) $\lim_{x\to a}\{fog(x)\} = f\left(\lim_{x\to a} g(x)\right) = f(m)$

(e) In particular, $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)} = e^l$

(f) $\lim_{x\to a} \frac{1}{f(x)} = 0$ if $\lim_{x\to a} f(x)$ is + ∞ or - ∞.

(g) $\lim_{x\to 0^+}\left[\frac{\tan x}{x}\right] = 1^+$

(h) $\lim_{x\to 0^+}\left[\frac{\sin x}{x}\right] = 0$, as $\frac{\sin x}{x} < 1$.

(i) $\lim_{x\to 0^-}\left[\frac{\sin x}{x}\right] = 0$, as $\frac{\sin x}{x} < 1$

(j) $\lim_{x\to 0^+} \frac{\{x\}}{\tan\{x\}} = 1$, as {x} →0 when x→ 0.

(k) $\lim_{x\to\infty} x.\sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = 1$

(l) $\lim_{x\to\infty} \cos\frac{1}{x} = 1$

(m) $\lim_{x\to\infty} x.\tan\frac{1}{x} = 1$

(n) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ then $\lim_{x\to a}(1 + f(x))^{\frac{1}{g(x)}} = e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$

(o) If $\lim_{x\to a} f(x) = 1, \lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a}\left(f(x)\right)^{g(x)} = \lim_{x\to a}(1 + f(x) - 1)^{g(x)}$ $= e^{\lim_{x\to a}\left(f(x)-1\right)g(x)}$

(p) Particularly: $\lim_{x\to 0}(1 + x)^{\frac{1}{x}} = e; \lim_{x\to 0}(1 + \lambda x)^{\frac{1}{x}} = e^{\lambda}$

L'Hospital Rule: This is applied if the function is differentiable and is of the form (0/0) or (∞/∞).

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}; g'(x) \neq 0$$

Example: $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$. This is of the form $\infty - \infty$.

Let us change to (0/0) form by simplifying as $\lim_{x\to 0}\left(\frac{\sin x - x}{x\sin x}\right)$

Apply L'Hospital Rule i.e. differentiate Numerator and Denominator separately equal number of times and when it is not of 0/0 form , put x=0 to find the limit = 0.

Methods of finding Limits

Factorization Method: When the expression is of the form (0/0). First the Numerator and the Denominator are factorised. Common factors are cancelled and then the value of x is put in the expression left after cancellation of the factors to find the required limit.

Example: $\lim_{x\to 2}\frac{x^2-4}{x-2}$ is of the form (0/0) when x=2 is put in the Numerator and Denominator. Hence first Factoise the Numerator and then proceed as below:

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=2+2=4$$

It is the limit of f(x) at x=2.

Rationalisation Method: When the expression is one of the Indeterminate forms and is not factorisable, first it is multiplied by its conjugate of the irrational part, in the Numerator and the Denominator so that its value does not change. This transforms the expression into the Factorization method where a common factor is generated to be cancelled and then by putting the value it gives the limit.

Example:

$\lim_{x\to\infty}\left(\sqrt{x^2+kx}-\sqrt{x^2-kx}\right)$ is of the form $\infty-\infty$ It cannot be factorized so multiplication by conjugate is done to find the limit as below:

$$\lim_{x\to\infty}\left(\sqrt{x^2+kx}-\sqrt{x^2-kx}\right)$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+kx}-\sqrt{x^2-kx}\right)}{\left(\sqrt{x^2+kx}+\sqrt{x^2-kx}\right)}.\left(\sqrt{x^2+kx}+\sqrt{x^2-kx}\right)$$

$$=\lim_{x\to\infty}\frac{\left(x^2+kx\right)-\left(x^2-kx\right)}{\left(\sqrt{x^2+kx}+\sqrt{x^2-kx}\right)}$$

$$=\lim_{x\to\infty}\frac{2kx}{\left(\sqrt{x^2+kx}+\sqrt{x^2-kx}\right)},\left(\frac{\infty}{\infty}\right)\text{form.}$$

Divide the Numerator and the Denominator by the **highest power of x** i.e. x and evaluate the limit assuming $\lim_{x\to\infty}\frac{1}{x}\to 0$.

$$\lim_{x\to\infty}\frac{2kx}{\left(\sqrt{x^2+kx}+\sqrt{x^2-kx}\right)}$$

$$=\lim_{x\to\infty}\frac{2k}{\left(\sqrt{1+\frac{k}{x}}+\sqrt{1-\frac{k}{x}}\right)}=\frac{2k}{1+1}=k$$

Method for finding limit when x tends to infinity: When the expression is of the form $\left(\frac{\infty}{\infty}\right)$ then divide the Numerator and the Denominator by the highest power of the variable x present in the Numerator or denominator and then put x=∞ that leads to the required limit as (1/∞)→0.

Example:

$$\lim_{x\to\infty}\frac{2kx}{\left(\sqrt{x^2+kx}+\sqrt{x^2-kx}\right)}$$

$$=\lim_{x\to\infty}\frac{2k}{\left(\sqrt{1+\frac{k}{x}}+\sqrt{1-\frac{k}{x}}\right)}=\frac{2k}{1+1}=k$$

Important Note: Sometime the evaluation of the limit appears of no form. Then the exponential form of rewriting it helps in its evaluation.

For example: $\lim_{x\to 0}|\cot x|^{\sin x}=e^{\lim_{x\to 0}\sin x\log_e|\cot x|}$,as it is obvious that the use of exponential writing helps and changes one of the known forms as $e^{\log_e z}=z$

Evaluation of Trigonometric Functions: Trigonometric Functions are expandable. So using method of expansions the nature of Indeterminate is firstly removed and then the limit is evaluated. Remember Expansions of sin x, cos x, tan x, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc.

Evaluation of Exponential Functions and Logarithmic functions are evaluated by the Method of expansion as they are also expandable. Remember the expansions of e^x, e^{-x}, a^x, log(1+x), log(1-x) etc.

Some limits that do not exist

$$\lim_{x\to 0}\left(\frac{1}{x}\right)\qquad\qquad \lim_{x\to 0}\cos\left(\frac{1}{x}\right)$$

$\lim_{x\to 0}\left(\frac{1}{x}\right)$ $\lim_{x\to 0} e^{\frac{1}{x}}$

$\lim_{x\to\infty}\cos x$ $\lim_{x\to\infty}\sec x$

$\lim_{x\to 0} x^{1/x}$ $\lim_{x\to a}\frac{|x-a|}{x-a}$

$\lim_{x\to\infty}\sin x$ $\lim_{x\to\infty}\cos ecx$

$\lim_{x\to\infty}\tan x$ $\lim_{x\to\infty}\cot x$

Useful Important Expansions for finding limits

(a) $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+......$ for x < 1

(b) $(1-x)^n = 1-nx+\frac{n(n-1)}{2!}x^2-......$

(c) $(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+......$

(d) $\frac{x^n-a^n}{x-a} = x^{n-1}+ax^{n-2}+a^2x^{n-3}+.....+a^{n-1}$

(e) $\frac{x^n-a^n}{x+a} = x^{n-1}-ax^{n-2}+a^2x^{n-3}-...+(-)^{n-1}a^{n-1}$

(f) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+......+\frac{x^r}{r!}+.......$

(g) $e^{-x} = 1-\frac{x}{1!}+\frac{x^2}{2!}-......+(-)^r\frac{x^r}{r!}+.......$

(h) $\log_e(1+x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3-...........$

(i) $\log_e(1-x) = -x-\frac{1}{2}x^2-\frac{1}{3}x^3-...........$

(j) $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-...........$

(k) $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-............$

(l) $\sec x = 1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}+...........$

(m) $x\cos ecx = 1+\frac{x^2}{6}+\frac{7x^4}{360}+\frac{31x^6}{15120}+........$

(n) $\sin^{-1}x = x+\frac{x^3}{3!}+\frac{1^2.3^2.x^5}{5!}+...........$

(o) $\cos^{-1}x = \frac{\pi}{2}-\left(x+\frac{x^3}{3!}+\frac{1^2.3^2.x^5}{5!}+.......\right)$

(p) $\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...........$

(q) $\sec^{-1}x = 1+\frac{x^2}{2!}+\frac{5x^4}{4!}+\frac{61x^6}{6!}+.........$

(r) $x\cot x = 1-\frac{x^3}{3}+\frac{x^4}{45}-\frac{2x^6}{945}+...........$

DIFFERENTIATION

A function f(x) is said to be differentiable if ***RH derivative = LH derivative = finite*** , otherwise it is said to be not differentiable or,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$$

The Right hand derivative is also called Progressive derivative and the Left hand derivative is called the Regressive derivative

Properties:

(a) The derivatives should be finite.

(b) A function defined on open interval (a, b) is said to be differentiable in an open interval (a, b) if it is differentiable at each point of (a, b).

(c) A function defined on closed interval [a, b] is said to be differentiable at end points a and b, if it is differentiable from the right at a i.e.

$\lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$ exists and is finite and is differentiable from the left at b i.e. $\lim_{x \to b^-} \frac{f(x)-f(b)}{x-b}$ exists and is finite.

(d) A function is said to be differentiable function if it is differentiable at every point of its domain.

(e) If A function is differentiable in the open interval (a, b) and also at the end points a and b then it said to be differentiable in the closed interval [a, b].

(f) If a function is differentiable at a point, then it is necessarily continuous at that point but the converse is not true i.e. if it is continuous then it may or may not be differentiable at that point.

(g) If f(x) and g(x) are differentiable, then f(x) $\pm$ g(x) or f(x).g(x) are also differentiable.

(h) If f(x) is differentiable and g(x) is not differentiable then f(x) g(x) may be differentiable.

(i) If f(x) is not differentiable and g(x) is also not differentiable then f(x) g(x) may be differentiable.

(j) A function is not differentiable at ***kink (corner)*** as a unique tangent cannot be drawn at that point i.e. a function is derivable iff its graph is always smooth i.e. there exists no break or corner.

(k) The derivative of a Periodic Function is also a periodic function having the same fundamental period.

(l) The derivative of an even function is an odd function and the derivative of an odd function is an even function.

(m) Differentiability of a function at a point implies the continuity at that point only.

Note: The function may be continuous but not differentiable.

For example:

f(x) = x sin(1/x), $x \neq 0$

=0, otherwise

Is continuous at x=0 but not differentiable at x=0 as the limit does not exist.

Some Facts

(a) $\frac{dy}{dx}$ represents the derivative of ***y w.r.t. x*** and is also the rate of change of y with respect to x. This also represents the slope of the tangent to the curve at (x, y).

(b) If tangent is parallel to x-axis then $\frac{dy}{dx}$ =0 and if it is perpendicular to the x-axis then $\frac{dy}{dx}=\frac{1}{0}$. The value 1/0 should not be written as ∞, as it is not a number but an assumption.

(c) A function is said to be increasing if f'(x) > 0 for all x in its domain.

(d) A function is said to be decreasing if f'(x) < 0 for all x in its domain.

(e) If f'(x) is possibly negative or positive then it is many-one.

(f) If f'(x) >0, for all real x then it in one-one onto.

(g) For comparison of two functions f(x) and g(x), we should check whether ***h(x)= f(x)-g(x)*** is increasing or decreasing.

(h) If a function is strictly increasing in closed interval [a, b] then f(a) is local minimum and f(b) is local maximum.

(i) If a function is strictly decreasing in closed interval [a, b] then f(a) is local maximum and f(b) is local minimum.

(j) In second derivative test for maximum and minimum values, one must note that this method cannot be applied at the points where f'(x) is undefined.

(k) For global maximum and minimum values in the closed interval [a,b] all values including at a and b of f(x) should be evaluated and then noted for maximum and minimum.

Leibnitz formula for successive differentiation of explicit functions:

$$(uv)^{(n)} = u^{(n)}v + {}^nC_1 u^{(n-1)}v' + {}^nC_2 u^{(n-2)}v'' + ... + {}^nC_n uv^{(n)}$$

Use of Newton-Leibnitz formula for definite integral: if

$$I = \int_{\psi(x)}^{\phi(x)} f(x)dx \Rightarrow \frac{dI}{dx} = f\{\phi(x)\}\frac{d\phi(x)}{dx} - f\{\psi(x)\}\frac{d\psi(x)}{dx}$$

Some derivatives

$$\frac{d}{dx}(constt) = 0 \qquad \frac{d}{dx}(|x|) = \frac{x}{|x|}, x \neq 0$$

$$\frac{d}{dx}([x]) = 0 \qquad \frac{d}{dx}(x^n) = n.x^{n-1}$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\cos ec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x.\tan x$$

$$\frac{d}{dx}(\cos ecx) = -\cos ecx.\cot x$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$$

$$\frac{d}{dx}(a^x) = a^x \log_e a$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos ec^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(vers^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{2ax-x^2}}$$

where ***vers x = 1-cosx*** and ***1-sinx = covers x***

$$\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}$$

$$\frac{d}{dx}\{f(x).g(x)\} = g(x).\frac{d}{dx}\{f(x)\} + f(x)\frac{d}{dx}\{g(x)\}$$

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\left(\frac{d}{dx}Nr\right)Dr - \left(\frac{d}{dx}Dr\right)Nr}{Dr^2}$$

Note:

(a) In differentiation of inverse Trigonometric functions if no branch is mentioned then, then the Principal branch should be taken in consideration.

(b) The differentiation of a determinant is the sum of the differentiations of the determinants of their rows in order, one at a time, i.e.,

$$\frac{d}{dx}\begin{vmatrix} f(x) & g(x) & h(x) \\ g(x) & h(x) & f(x) \\ h(x) & f(x) & g(x) \end{vmatrix} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ g(x) & h(x) & f(x) \\ h(x) & f(x) & g(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ g'(x) & h'(x) & f'(x) \\ h(x) & f(x) & g(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ g(x) & h(x) & f(x) \\ h'(x) & f'(x) & g'(x) \end{vmatrix}$$

(c) $$\frac{d}{dx}\begin{pmatrix} f(x) & g(x) & h(x) \\ g(x) & h(x) & f(x) \\ h(x) & f(x) & g(x) \end{pmatrix} = \begin{pmatrix} f'(x) & g'(x) & h'(x) \\ g'(x) & h'(x) & f'(x) \\ h'(x) & f'(x) & g'(x) \end{pmatrix}$$

(c) The chain rule is expressed as: $\frac{dy}{dx}=\frac{dy}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$

n^{th} Derivatives Of Functions:

(a) $\frac{d^n}{dx^n}x^m = m(m-1)(m-2)..(m-n+1)x^{m-n}$

(b) $\frac{d^n}{dx^n}e^{ax} = a^n e^{ax}$

(c) $\frac{d^n}{dx^n}a^{bx} = b^n a^{bx}(\log_e a)^n$.

(d) $\frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

(e) $\frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$.

(f) $\frac{d^n}{dx^n}e^{ax}\cos(bx+c) = \left(a^2+b^2\right)^{\frac{n}{2}}e^{ax}\cos(bx+c+n\phi)$

where $\tan\phi=\frac{b}{a}$

(g) $\frac{d^n}{dx^n}e^{ax}\sin(bx+c)$

$= \left(a^2+b^2\right)^{\frac{n}{2}}e^{ax}\sin(bx+c+n\phi)$

where $\tan\phi=\frac{b}{a}$

Important Formulae

(a) The ***slope*** of the tangent for the function y=f(x) at point (x_1,y_1) is given by $\tan\psi = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$ = tangent of the angle between the positive direction of x-axis and the tangent.

(b) The slope of the normal is given by $-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}$

(c) ***Tangent*** at (x_1, y_1) is written as
$y-y_1= (dy/dx)(x-x_1)$

(d) ***Normal*** at (x_1, y_1) is written as
$y-y_1= -(dx/dy)(x-x_1)$

(e) If the line is parallel to x-axis
dy/dx=0

(f) If the line is perpendicular to x-axis
dy/dx= 1/0

(g) ***Length of tangent*** $=\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}$

(h) ***Length of normal*** $=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

(i) ***Length of sub-tangent*** $=\frac{y}{\left(\frac{dy}{dx}\right)}$

(j) Length of ***sub-normal =*** $y.\frac{dy}{dx}$

(k) Intercept of tangent on x-axis= $\left|x-y.\left(\frac{dy}{dx}\right)\right|$

(l) Intercept of tangent on y-axis= $\left|y-x.\left(\frac{dy}{dx}\right)\right|$

(m) Two curves touch each other if at the point of contact $m_1 = m_2$

(n) Two curves cut each other orthogonally if $m_1 m_2 = -1$

(o) If function f(x) is continuous on [a,b] such that f'(c) ≥0, or f'(c)>0 for each c ∈(a,b) then f(x) is said to be monotonically (strictly)increasing function on [a,b]

(p) If function f(x) is continuous on [a,b] such that f'(c) ≤0, or f'(c)<0 for each c ∈(a,b) then f(x) is said to be monotonically (strictly) decreasing function on [a,b]

(q) If f(x) and g(x) are monotonically (or strictly) increasing (or decreasing) functions on [a,b] then gof (x) is a monotonically (strictly) increasing function on [a,b]

(r) If one of the function f(x) and g(x) is montonically (or strictly) increasing and other amonotically (or strictly) decreasing , then gof(x) is monotonically (or strictly) decreasing on (a,b)

(s) If f(x) is an ***increasing function***[1] on (a,b) then tangent makes an acute angle with +ive direction of x-axis ie dy/dx > 0.

(t) If f(x) is ***decreasing function***[2] on (a,b) then tangent makes an obtuse angle with the +ive direction of x-axis ie dy/dx<0.

(u) The sign of the derivative gives a **sufficient condition** for the function to be increasing or decreasing but this condition is by means **necessary** . The function $f(x)=x^3$ produces a counter example as this is differentiable and increasing on (-1,1) and everywhere else except at x=0 where it is 0.

Important Theorems

Fermat Theorem : Let a function ***y=f(x)*** be defined on a certain interval and have a maximum or a minimum value at an interior point x_0 of the interval.

If there exists a derivative $f'(x_0)$ at the point x_0 then $f'(x_0) = 0$.

Rolle's Theorem : ***f(x)*** is continuous on ***[a,b]***, derivable in ***(a,b)*** and ***f(a) = f(b)*** then there exists atleast one point $c \in (a,b)$ such that $f'(c)=0$.

Lagrange's Mean Value Theorem : If ***f(x)*** is continuous on ***[a, b]***, derivable in ***(a, b)*** then there exists atleast one point $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Cauchy's Theorem : Let ***f(x)*** and ***g(x)*** be two functions continuous in the interval ***[a,b]*** and have finite derivatives at all interior points of the interval. If these derivatives do not vanish simultaneously and $g(a) \neq g(b)$, then there exists $\varepsilon \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\varepsilon)}{g'(\varepsilon)}$

Sandwich Theorem (Squeeze Theorem) : This is sometimes also called Pinching Theorem. It states that if ***g(x)*** is squeezed between ***f(x)*** and ***h(x)*** at ***x = a***,

i.e. if $f(x) \le g(x) \le h(x), \forall x \in (a-\delta, a+\delta)$

and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l_1$$

Then

$$\lim_{x\to a} g(x) = l$$

Memorable facts :

(a) If δx is an error in the variable then $\frac{\delta x}{x}.100$ is called the percentage error in x.

(b) A function f(x) is said to have a local maximum value at x=a if there exists a nbd (a-δ,a+δ) of **a** such that f(x)<f(a) for all x∈ (a-δ, a+δ), [x≠ a] or f(x)-f(a) <0 for all x∈ (a-δ,a+δ), [x ≠a]. f(a) is called the local maximum value of f(x) at x=a.

(c) A function f(x) is said to have a local minimum value at x=a if there exists a nbd (a-δ,a+δ) of **a** such that f(x)>f(a) for all x∈ (a-δ, a+δ), [x≠ a] or f(x)-f(a) >0 for all x∈ (a-δ,a+δ), [x ≠a]. f(a) is called the local minimum value of f(x) at x=a.

(d) The points at which the function has either the local maxima or minima are called extreme values of f(x).

(e) The values of x for which f'(x)=0 are called stationary values or critical values of x and the corresponding values of f(x) are called the stationary or turning values of f(x). The points at which f'(x) does not exist are also called critical points.In nutshell the ***critical*** points are the values of x for which **f(x) is undefined**, **f'(x)=0** and/or **f'(x) does not exist**.

(f) Point of ***Inflexion*** is a point where $d^2y/dx^2=0$ but d^3y/dx^3 is not zero.

(g) First derivative test
   i. If f(x) is differentiable at x=a and f'(a)=0 and f'(x) changes sign from + to – as x passes through then f(x) is said to have the local ***maximum*** value at x=a.

   ii. If f(x) is differentiable at x=a and f'(a)=0 and f'(x) changes sign from - to + as x passes through then f(x) is said to have the local ***minimum*** value at x=a.

(h) If y is maximum or minimum then log y is also maximum or minimum provided y >0.

(i) n^{th} derivative test for Relative Extrema

Find the critical number for $x=x_0$.

Find also $f'(x_0)$.

If $f'(x_0)>0$, f(x) is minimum at $x=x_0$

If $f'(x_0)<0$, f(x) is maximum at $x=x_0$

If $f'(x_0)=0$, neither maxima nor minima but this point is called the point of inflexion if $f''(x_0)\neq 0$.

Repeat this process till we obtain $f^n(x_0)\neq 0$.

If n is odd f(x) has neither maxima nor minima.

If n is even and $f^n(x_0)>0$, f(x) is minimum at $x=x_0$

If n is even and $f^n(x_0)<0$, f(x) is maximum at $x=x_0$.

Examples:

(a) ***f(x) = x^4***

f'(0)=0, f''(0)=0, f'''(0)=0 and ***$f^4(0)>0$***

Hence minimum at x=0

(b) f(x) = - x^4

f'(0)=0,f''(0)=0,f'''(0)=0 and

$f^4(0) < 0$;

Hence maximum at x=0

(c) ***f(x)=x^3***

f'(0)=0,f''(0)=0 and

$f^3(0)=6$;

Hence neither maxima nor minima at x=0

Note For Greatest And Least Values : A ***maximum*** value of ***f(x)*** at ***$x=x_0$*** in an interval ***[a,b]*** does not mean that it is the greatest value of ***f(x)*** in that interval. There may be a value of the function greater than a maxi mum value. As a matter of fact there may exist a minimum value of the function which is greater than or equal to some maximum value of the function in ***[a,b]***.

L'Hospital Rule: If the function f(x) and g(x) are differentiable in the certain neighborhood of the point **a** , except, may be, at the point **a** itself, and $g'(x)\neq 0$ and if $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0..or..\infty$ then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists. The point ***a*** may be either finite or improper i.e. $+\infty$ or $-\infty$.

Note : If a function is defined and continuous in some interval, and if this interval is not a closed one then it can have neither the greatest nor the least value.

Questions on Limits

1. ***If f(x) is an odd function and*** $\lim_{x\to 0} f(x)$ ***exists, then find the limit.***

Hint: Given f(-x)= -f(x) as f(x) is an odd function.

Limit exists, i.e. LHL=RHL

$\Rightarrow f(0+h) = f(0-h)$

$\Rightarrow f(h) = f(-h) = -f(h)$

$\Rightarrow 2f(h) = 0$

$\Rightarrow .f(h)=0$

2. ***Evaluate:*** $\lim_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]}$.

Hint: As $x\to 0$, $\cos x\to 1$

$\Rightarrow [\cos x] = [<1] = 0$

3. ***Evaluate:*** $\lim_{x\to 0}\frac{2^x+2^{3-x}-6}{2^{-x/2}-2^{1-x}}$.

Hint: Factorize the Numerator after simplification as $(2^{x/2}-2)(2^{x/2}+2)(2^x-2)$ and proceed as usual.

4. ***Evaluate:*** $\mathop{Lim}_{x\to\infty}\left(\frac{3x^2+2x+1}{x^2+x+1}\right)^{\frac{6x+1}{3x+2}}$

Hint: Rewrite the given expression as

$$\mathop{Lim}_{x\to\infty}\left(\frac{3+\frac{2}{x}+\frac{1}{x^2}}{1+\frac{1}{x}+\frac{1}{x^2}}\right)^{\lim_{x\to\infty}\frac{6+\frac{1}{x}}{3+\frac{2}{x}}}=\left(\frac{3}{1}\right)^{\frac{6}{3}}$$

.

5. ***Evaluate:*** $\mathop{Lim}_{n\to\infty}\left(\frac{n!}{(n+1)!-n!}\right)$.

Hint: Simplify to $\frac{1}{(n+1)-1}=\frac{1}{n}$ and evaluate the limit.

6. Evaluate: $\lim\limits_{x\to\infty} 3^x \sin\left(\frac{4}{3^x}\right)$.

Hint: Use $\lim\limits_{x\to 0}\frac{\sin mx}{mx}=1$

Rewrite the expression as below:

$$\lim_{x\to\infty}\frac{\sin\left(\frac{4}{3^x}\right)}{\frac{4}{3^x}}.4=4$$

7. Evaluate: $\lim\limits_{x\to\infty}\left(\frac{\log x}{x^m}\right)$ if m>0.

Hint: Use L'Hospital Rule to get the Limit=0

8. Evaluate: $\lim\limits_{x\to\infty}\left(\frac{x^6}{6^x}\right)$.

Hint: Rewrite the given expression as

$$\left(\frac{x^6}{e^{x\log_e 6}}\right)=\frac{x^6}{1+(x\log_e 6)+\frac{(x\log_e 6)^2}{2!}+...}$$

Divide N^r and D^r by x^6 and evaluate.

9. Evaluate: $\lim\limits_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2}$.

Hint: Assume ***y = (1+x)$^{1/x}$***

$\Rightarrow$*$\log_e y$=(1/x) log(1+x)*

or,

y= $e^{(1/x)\log(1+x)}$

$$=e^{1-\frac{x}{2}+\frac{x^2}{3}-...}$$

$$=e.e^{\left(-\frac{x}{2}+\frac{x^2}{3}-\right)}$$

$$=e.\left\{1+\left(-\frac{x}{2}+\frac{x^2}{3}-...\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-..\right)^2+...\right\}$$

$$\Rightarrow y=e.\left\{1-\frac{x}{2}+\frac{x^2}{3}+\frac{x^2}{8}+\frac{x^4}{18}-\frac{x^3}{6}+...\right\}$$

$$=e.\left\{1-\frac{x}{2}+\frac{11x^2}{24}-\frac{5x^3}{12}+...\right\}$$

On putting this value in the original expression,

$$\lim_{x\to 0}\frac{(1+x)^{1/x}-e+\frac{1}{2}ex}{x^2}$$

$$=\lim_{x\to 0}\frac{e-\frac{ex}{2}+\frac{11ex^2}{24}-\frac{5ex^3}{12}-e+\frac{1}{2}ex}{x^2}$$

$$=\lim_{x\to 0}\frac{+\frac{11ex^2}{24}-\frac{5ex^3}{12}}{x^2}$$

$$=\lim_{x\to 0}\left(\frac{11e}{24}-\frac{5ex}{12}+g(x)\right)$$

where g(x) is the expression containing x.

$$=\frac{11e}{24}$$

10. Evaluate: $\lim\limits_{x\to 1}(2-x)^{\tan\frac{\pi x}{2}}$.

Hint: Rewrite the limit as below:

$$\lim_{x\to 1}(2-x)^{\tan\frac{\pi x}{2}}=\lim_{x\to 1}(1+(1-x))^{\tan\frac{\pi x}{2}}$$

$$=\lim_{x\to 1}(1+(1-x))^{\frac{\tan\frac{\pi x}{2}}{1-x}}=e^{2/\pi}$$

11. Evaluate: $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$.

Hint: Rewrite as below:

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$$

$$=\lim_{x\to 0}\left(1+\frac{\sin x-x}{x}\right)^{\frac{x}{\sin x-x}.\frac{\sin x}{x-\sin x}}=e^{-1}$$

12. Evaluate: $\lim_{x\to 0} x^x$

Hint:

$$\lim_{x\to 0} x^x = e^{\lim_{x\to 0} x\log x} = e^{\lim_{x\to 0}\frac{\log x}{1/x}} = e^{\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}} = e^{-x} = e^0 = 1$$

13. Evaluate: $\lim_{x\to\infty} x^{\frac{1}{x}}$

Hint:

$$\lim_{x\to\infty} x^{\frac{1}{x}} = e^{\lim_{x\to\infty}\frac{1}{x}\log x} = e^{\lim_{x\to\infty}\frac{\log x}{x}} = e^{\lim_{x\to\infty}\frac{\frac{1}{x}}{1}} = e^{\frac{1}{\infty}} = e^0 = 1$$

.

*14. **Find the value of a if*** $\lim_{x\to a}\frac{a^x - x^a}{x^x - a^a} = -1$ ***and*** *a>0.*

Hint: Use L'Hospital Rule.

Differentiate Numerator and Denominator separately once.

Put *x=a*

have $\frac{\log a - 1}{1+\log a} = -1$ and hence *a=1*.

15. Evaluate: $\lim_{m\to\infty}\lim_{n\to\infty}\{1+\cos^{2m}(n!\pi x)\}, \forall x\in R$

Hint: Assume 2 cases

Case I : x=p/q , (i.e.a rational number)

$\Rightarrow$ (n!πx) = n! π. (p/q)

= an integral multiple of π as n→∞

$\Rightarrow \cos^2$(n!πx) =1 and hence $\cos^{2m}$(n!πx)=1

$\Rightarrow$ the required limit (1+1)=2

Case II : x an irrational number.

Then (n!πx)= an irrational number, i.e. not an integral multiple of π

$\Rightarrow$ -1 < cos(n! πx) < 1

$\Rightarrow$ 0 < $\cos^2$ (n! πx) < 1

$\Rightarrow$ $\cos^{2m}$ (n! πx)→0 as m→∞

Hence the required limit= 1+0=1

16. Evaluate: $\lim_{x\to[a]}\frac{e^{\{x\}}-\{x\}-1}{\{x\}^2}$ ***where {x} represents the fractional part and [x] represents the integral part of a.***

Hint: RHL at *a*

$$= \lim_{x\to[a]+}\frac{e^{\{[a]+h\}}-\{[a]+h\}-1}{\{[a]+h\}^2}$$

$$= \lim_{h\to 0}\frac{e^h - h - 1}{h^2} = \frac{1}{2}$$

As

{[a]+h}=[a]+h-[[a]+h]=[a]+h-[a]=h

LHL at *a*

$$= \lim_{x\to[a]-}\frac{e^{\{[a]-h\}}-\{[a]-h\}-1}{\{[a]-h\}^2}$$

$$= \lim_{h\to 0}\frac{e^{1-h}-(1-h)-1}{(1-h)^2} = e-2$$

As

{[a]-h}=[a]-h-[[a]-h]=[a]-h-[a]+1=1-h

17. Find h(x) in terms of f(x) and g(x) if h(x) = $\lim_{n\to\infty}\frac{x^{2n}f(x)+g(x)}{1+x^{2n}}$

Hint: Assume 3 cases

Case I: $x^2<1$

The expression $h(x) = \frac{0.f(x)+g(x)}{1+0} = g(x)$

Case II: $x^2>1$

The expression h(x)

$$= \lim_{n\to\infty}\frac{f(x)+\frac{1}{x^{2n}}g(x)}{\frac{1}{x^{2n}}+1} = \frac{f(x)+0}{0+1} = f(x)$$

Case III : $x^2=1$

The expression h(x)

$$= \lim_{n\to\infty} \frac{1.f(x)+g(x)}{1+1} = \frac{f(x)+g(x)}{2}$$

18. Evaluate: $\lim_{x\to 0}\left[\frac{a.\sin x}{x}\right]+\left[\frac{b.\tan x}{x}\right]$ ***where a,b are integers and [.] denotes the greatest integer function.***

Hint: $\frac{\sin x}{x}<1$, and, $\frac{\tan x}{x}>1$

Hence the required limit = *a-1+b*

19. Prove that $e^{2x}+e^{x}+2\sin^{-1}x+x-\pi=0$ ***has atleast one real solution in [0,1].***

Hint: Note that

f(0)= 2-π < 0 and

f(1)= e^2 + e -1 >0

Hence there exists one solution f(c)=0 of f(x)=0 between the given limit as the function f(x) is continuous between 0 and 1.

20. If f(x) = x-[x], k ≤ x < k+0.5
= [x], k +0.5 ≤ x < k+1, k∈ I

and g(x) = $\sin^4 x+\cos^4 x$,

then find the value of f(g(x)).

Hint: Obviously,

g(x)= 1- (1/2) $\sin^2 2x$.

⇒ (1/2) ≤ g(x) ≤ 1

⇒ g(x)=1 if sin2x=0

⇒x = nπ/2

Hence,

f(g(x)) = f(1) = 1-[1] = 0

⇒f(g(x)) = 0 for all x∈ R.

21. Show that the equation x=1+sin x has a root.

Hint: Assume a function

f(x)= x-1-sin x .

Check the continuity of the function.

It is continuous for all x ∈ R.

Also f(π/2) < 0 and f(π) > 0,

Hence there exists certainly a value of x between π/2 and π such that ***f(x=c)=0*** .

Hence, at least one solution exists.

Note: This question can be done by the **Graph Method also.**

Draw the graphs of ***y = x-1*** and ***y = sin x***. If they intersect each other, then there exists a solution otherwise there is no solution.

22. ***Show that f(x) is a constant function if f(x)satisfies*** $f:R\to R; |f(x)-f(y)|\le |x-y|^3$.

Hint: $|f(x)-f(y)|\le |x-y|^3, x\ne y$

$$\Rightarrow \left|\frac{f(x)-f(y)}{x-y}\right|\le |x-y|^2$$

$$\Rightarrow \lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right|\le \lim_{y\to x}|x-y|^2=0$$

$$\Rightarrow |f'(x)|=0\Rightarrow f(x)=c,$$

i.e. a constant function.

23. Find dy/dx when y=f(x) where $f\left(x+\frac{1}{x}\right)=x^4+\frac{1}{x^4}$

Hint: Rewrite the function as below:

$$f\left(x+\frac{1}{x}\right)=x^4+\frac{1}{x^4}$$

$$=\left(x^2+\frac{1}{x^2}\right)^2-2=\left[\left(x+\frac{1}{x}\right)^2-2\right]^2-2$$

Replace the expression $\left(x+\frac{1}{x}\right)$ by z and rewrite as $f(z)=(z^2-2)^2-2$ and then the function becomes

$f(x)=(x^2-2)^2-2 = x^4-4x^2+2$

Now differentiate to find the required derivative.

24. If $f\left(\frac{x+2y}{3}\right)=\frac{f(x)+2f(y)}{3}, \forall x,y\in R$ ***and f'(0) exists and is finite, then show that f(x) is continuous on the whole Number Line.***

Hint: It is given that f(x) is differentiable at x=0

$$\Rightarrow f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}$$

It is given that this is finite and also exists.

$\Rightarrow$ by definition of differentiability $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f\left(\frac{3x+3h}{3}\right)-f\left(\frac{3x+0}{3}\right)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(\frac{3x+2.\frac{3h}{2}}{3}\right)-f\left(\frac{3x+2.0}{3}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{f(3x)+2f\left(\frac{3h}{2}\right)}{3}-\frac{f(3x)+2f(0)}{3}}{h}$$

$$=\lim_{h\to 0}\frac{2f\left(\frac{3h}{2}\right)-2f(0)}{3h}$$

$$=\lim_{h\to 0}\frac{f\left(\frac{3h}{2}\right)-f(0)}{\frac{3h}{2}}=f'(0)=c$$

(say a constant)

On Integration of both sides

f(x)=cx + d where d is also another arbitrary constant. It is linear, hence it is continuous.

25. ***Find the equations of all tangents to the curve y = cos(x+y); (-2π ≤ x ≤ 2π) that are parallel to the line x+2y=0***

Hint: Differentiate the curve and find (dy/dx). Equate it the slope of the given line i.e. (– ½).

$$\Rightarrow \frac{dy}{dx}=-\sin(x+y).\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\sin(x+y)}{1+\sin(x+y)}=-1/2$$

⇒sin(x+y)=1

⇒cos(x+y)=0

⇒y=0 from the equation of the given curve and hence sinx=1

⇒x= (-3π/2), (π/2)

Write the points P(π/2,0) and Q(-3π/2,0) and then the equations of the tangents using one point and slope formula i.e. $y-y_1 = m_T (x-x_1)$

26. ***Assume that a raindrop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of radius of the rain drop.***

Hint: Given $\frac{dV}{dt}=-kS$

where

V=Volume of the spherical raindrop

=(4/3)πR^3,

S =Surface area of the raindrop =4π R^2 and

K= an arbitrary constant.

Differentiate V w.r.t to t and get the required equation: dR/dt=-k.

27. ***Find the intervals in which the function f(x)= sin 3x, x∈ [0, π/2] is increasing or decreasing***.

Hint: f'(x)= 3 cos 3x

for increasing or decreasing f'(x) ≥ 0, or, ≤ 0

f'(x)=0

since x ∈ [0, π/2]

⇒ 3x= π/2, 3π/2 ⇒ x = π/6 , π/2

⇒ x = π/6 divides the interval [0, π/2] into two disjoint intervals [0, π/6) and (π/6, π/2]

Obviously,

f'(x)>0 for [0, π/6) i.e. increasing and f'(x)<0 for (π/6, π/2] i.e. decreasing.

28. ***Find the interval in which the function $f(x)= \sin x + \cos x$, $0 \le x \le 2\pi$,is strictly increasing or strictly decreasing.***

Hint: $f'(x)= \cos x – \sin x$

$\Rightarrow f'(x)=0$

$\Rightarrow \cos x- \sin x=0$

$\Rightarrow x = \pi /4 , 5\pi/4$ as $x \in[0, 2\pi]$

$\Rightarrow$ these values of x divide the given interval into 3 disjoint intervals

i.e.$[0, \pi/4), (\pi/4, 5\pi/4), (5\pi/4, 2\pi]$

obviously,

$f'(x) >0$ for $[0, \pi/4) \cup (5\pi/4, 2\pi]$ i.e increasing

and

$f'(x)<0$ for $(\pi/4, 5\pi/4)$ i.e decreasing.

29. ***Show that $f(x)= [x]$ on $(1,\infty)$ is monotonic increasing but not strictly increasing.***

Hint: Assume $x_1, x_2 \in(1,\infty)$ such that $x_1 < x_2$. Obviously there exists n_1 and n_2 s.t. $n_1 < n_2$ for $x_1 \in[n_1,n_1+1)$ and $x_2 \in[n_2,n_2+1)$.

Certainly f(x) is monotonic increasing but not strictly increasing.

But $f(5/4)=[5/4]=1$ and $f(3/2)=[3/2]=1$

Here $(5/4)< (3/2)$ but $f(5/4)=f(3/2)$.

30. ***Find the Absolute maximum and the absolute minimum values of $f(x)=2x^3-5x^2+4x-1$ on $[-1,2]$.***

Hint: Find the critical points by solving $f'(x)=0$ or at the x where $f'(x)$ does not exist.

Obviously $x=2/3,1$.

It lies in $(-1,2)$.

So find out all the values of f(x) at $x=-1, 2/3, 1, 2$.

The smallest of these will give the absolute minimum and the greatest will be the absolute maximum one.

31. ***Find the interval in which λ should lie so that $f(x)= \sin^3 x + \lambda \sin^2 x$, where $-\pi/2 < x < \pi/2$, has exactly one maxima and exactly one minima.***

Hint: Find, $f'(x) = 3\sin^2 x \cos x + \lambda 2 \sin x \cos x$

$= \sin x \cos x (3\sin x +2\lambda)$

For maxima or minima $f'(x)=0$,

hence

Either $\sin x=0$ or $\cos x=0$ or $3\sin x+2\lambda =0$

Since $- \pi/2 < x < \pi/2$ hence $\cos x \ne 0$

Then $\sin x=0$ or $\sin x =- 2\lambda /3$

$\Rightarrow x=0$ or $x= \sin^{-1}(-2\lambda /3)$

$\Rightarrow x=0$ or $-1 < - 2\lambda /3 < 1$

$\Rightarrow \lambda \in (-3/2, 3/2)$

But if $\lambda=0$ then $x=0$ and then only one solution will be there hence $\lambda\ne 0$

Therefore $\lambda \in (-3/2, 0) \cup (0, 3/2)$

For two solutions.

32. ***Show that $|\sin u – \sin v| \le |u-v|$.***

Hint: Assume $f(x)=\sin x$.

Mean Value Theorem's essentials exist as it is continuous and differentiable in[u,v] for all x in the specified interval.

Hence there exists some $c\in (u,v)$ such that $f'(c)=\{f(u)-f(v)\}/(u-v)$

As $|\cos c| \le 1$ hence $\{f(u)-f(v)\} \le |u-v|$

33. ***Show that $2 \sin x + \tan x \ge 3x$ where $0 \le x < \pi/2$***

Hint: Assume $f(x)= 2 \sin x + \tan x – 3 x$

Prove f(x) , an increasing function in the given interval. If it comes true then it is well otherwise assumption is False.

$$f'(x)= \frac{(\sec x-1)^2(\sec x+2)}{\sec x}$$

for $0 < x < \pi /2$, $f'(x) > 0 \Rightarrow f(x)$ is an increasing function in $(0, \pi/2)$.

For $x=0$, $f(x)=0$ hence on combining the above two results, f(x) results true.

34. ***If $x = a \cos^3\theta$, $y= a \sin^3\theta$ then evaluate $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$***

Hint: Use differentiation method for parametric form.

Find $$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

And simplify to get the answer.

35. ***If $x = t^2$, $y = t^3$, then find the value of $\frac{d^2y}{dx^2}$***

Hint: note $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

36. ***Evaluate*** $\frac{d^{20}}{dx^{20}}(2\cos x\cos 3x)$

Hint: Change the given expression as below

2cosAcosB=cos(A-B)+cos(A+B)

And then use formula for n^{th} derivative of cos x as

$$\frac{d^n}{dx^n}\cos x = \cos\left(\frac{n\pi}{2}+x\right)$$

37. ***If f(x)=*** $\int_{-1}^{x}|t|dt, x \geq -1$ ***, then show that f and f' are continuous for x+1>0.***

Hint: Note $\int_{-1}^{x}|t|dt = \left[\frac{t|t|}{2}\right]_{-1}^{x}$

Hence f(x)= $\frac{x|x|}{2}+\frac{1}{2}$

And f'(x)=|x| using Newton-Leibnitz formula of definite integral.

Obviously f(x) and f'(x) are continuous for x > -1.

38. ***Let $f(x) = x^p \cos(1/x)$, when $x \neq 0$ and f(x)=0 when x =0. Then show that f(x) will be differentiable at x=0 if p>1.***

Hint: Use

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h^p\cos\frac{1}{h}}{h} = \lim_{h\to 0} h^{p-1}\cos\frac{1}{h}$$

It is obvious that if p-1 $\leq$ 0 then the limit will not exist finitely. Hence for existence of the limit , p must be >1.

39. ***If f(x) = 1+ 2 sin x + 3 $\cos^2$ x, 0 ≤ x ≤ 2π/3 then show that it is minimum at x =π/2***

Hint: Differentiate once and find f'(x). find value(s) of x. find d^2y/dx^2 at x and then show that it is positive for this x. (i.e. use second derivative test for maxima and minima)

40. ***Find the value of n^{th} differential coefficient of $\frac{x^3}{x^2-1}$ for x=0, if n is even and also if n is odd and greater than 1.***

Hint: Rewrite the given expression as

$$y= \frac{x^3}{x^2-1} = x+\frac{x}{x^2-1} = x+\frac{A}{x-1}+\frac{B}{x+1}$$

use method of partial fraction to decompose into two fractions and then evaluate A and B.

the n^{th} derivative is

$$y_n = \frac{1}{2}(-1)^n(n!)(x-1)^{-n-1}+\frac{1}{2}(-1)^n(n!)(x+1)^{-n-1}$$

$$\Rightarrow (y_n)_{x=0} = \frac{1}{2}(-1)^n(n!)(-1)^{-n-1}+\frac{1}{2}(-1)^n(n!)$$

Two cases arise

Case I: when n is even i.e. n=2m $\Rightarrow y_n$ at x=0 is 0

Case II: when n is odd i.e. n=2m+1$\Rightarrow$ y_n at x=0 is – (n!)

41. ***Show that there exists a function f(x) satisfying f(0)=1, f'(0)=-1,f'(x)>0 for all x and f''(x)<0 for all x.***

Hint: f(x) is continuous and differentiable under given conditions for all x , then f(x) is decreasing for x>0 and concave down. Therefore f''(x)<0.

42. ***Find the maximum value of $f(x)=2x^3-15x^2+36x-48$ on set A= { x: $x^2+20 \leq 9x$ }.***

Hint: From $x^2+20 \leq 9x \Rightarrow x\in[4,5]$

f'(x) is decreasing as

f''(x)= - sin x –(6/π) is negative i.e. < 0, for all x∈[0,π/2]

f'(x)>0 as x< π/2.

Since f'(x) is decreasing,

hence f'(x)> f'(π/2)⇒f(x) is increasing. i.e.

For x≥0, f(x) ≥f(0) ⇒ sinx + 2x- 3x(x+1)/π≥0.