TRIGONOMETRY

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Measurement of Angles

There are three systems of measuring angles:

(i) English System

It is called **Sexagesimal** system. In this system

1 Right Angle = 90^0 (degrees),

$1^0 = 60'$ (minutes)

$1' = 60''$ (seconds)

(ii) French System

It is called **Centesimal** System.

1 Right Angle = 100^g (grades),

$1^g = 100'$ (minutes),

$1' = 100''$ (seconds)

(iii) Circular System

1 Right Angle = $\left(\frac{\pi}{2}\right)^c$ (**radians**),

Relation between measuring systems

1 Right Angle = 90^0 = 100^g = $\left(\frac{\pi}{2}\right)^c$

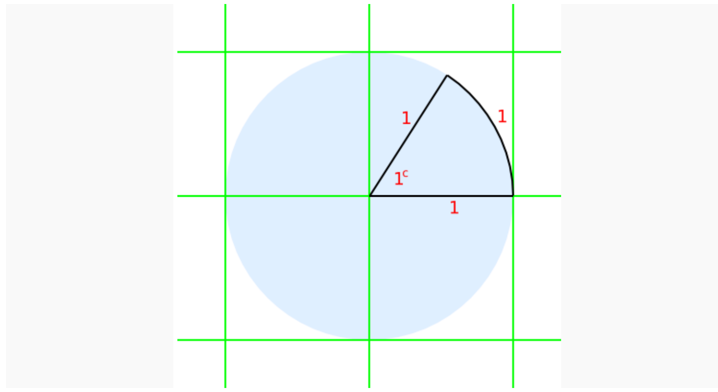
Relation between arc, radius and angle at the centre of a circle

Angle subtended at center of a circle by an arc :

$\theta(in..radians) = \frac{arc}{radius}$; radian is always constant i.e, it has no unit.

Definition of One radian Angle

A single radian is defined as the angle formed in the minor sector of a circle, where the minor arc length is the same as the radius of the circle.



$1rad \approx 57.296°$

Notable facts

(i) **Hiparchus**, a Greek Astronomer is considered the Father of Trigonometry.

(ii) Angle moved by hour hand in one hour=30 degrees.

(iii) Angle moved by hour hand in one minute= (1/2) degrees.

(iv) Angle moved by minute hand in one minute= 6 degrees.

Important Facts

1. sin (-x) = -sin x
2. cos (-x) = cos x
3. tan (-x) = - tan x
4. Trigonometric identity: $\sin^2\theta + \cos^2\theta = 1$
5. Two angles are said to be ***allied*** when their sum or difference is either 0 or multiple of 90 degrees.
6. Algebraic sum of two or more angles are called ***compound*** angles and the angles are called the constituent angles.
7. The ***maximum*** and ***minimum*** values of a cos x + b sin x + c are $c+\sqrt{a^2+b^2}, c-\sqrt{a^2+b^2}$

Range of Trigonometric Functions

8. |sin x| ≤ 1 ⇒ -1 ≤ sin x ≤ 1

9. $|\cos x| \le 1 \Rightarrow -1 \le \cos x \le 1$

10. $|\sec x| \ge 1 \Rightarrow \sec^2 x \ge 1$, $\forall x \in R$ where tan x is defined

11. $|\operatorname{cosec} x| \ge 1 \Rightarrow \operatorname{cosec}^2 x \ge 1$, $\forall x \in R$ where cot x is defined

$-\infty < \tan x < \infty$;

$-\infty < \cot x < \infty$

12. tan x, sec x, and cosec x are unbounded, positive or negative.

13. tan x and cot x can take any value,

14. sec x and cosec x can never lie between -1 and 1.

15. $\sin^2 \alpha + \operatorname{cosec}^2 \alpha \ge 2$

16. $\cos^2 \alpha + \sec^2 \alpha \ge 2$

17. $\tan^2 \alpha + \cot^2 \alpha \ge 2$

18. $\sec^2 \alpha + \operatorname{cosec}^2 \alpha \ge 4$

In a triangle ABC, where A+B+C=180°

1. $\cos A + \cos B + \cos C \le 3/2$
2. $\sin A + \sin B + \sin C \le (3\sqrt{3}/8)$
3. $\tan A + \tan B + \tan C \ge 3\sqrt{3}$
4. $\tan A \tan B \tan C \ge 3\sqrt{3}$
5. $\tan^2 A/2 + \tan^2 B/2 + \tan^2 C/2 \ge 1$

If in a triangle ***ABC***, ***a*** is the length of side opposite to angle ***A***, ***b*** is length of side opposite to ***B***, and ***c*** is the length of side opposite to angle ***C***, s=(a+b+c)/2, then

1. $(s-a)(s-b)(s-c) \le abc/8$
2. $\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$
3. $\Delta \le (s^2/3\sqrt{3})$ where, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Sign of equality holds in case of equilateral triangles.

Trigonometric Equations

1. $\sin x = \sin \alpha$

$\Rightarrow$ general value of $x = n\pi + (-1)^n \alpha$

2, $\cos x = \cos \alpha$

$\Rightarrow$ general value of $x = 2n\pi \pm \alpha$

3 .$\tan x = \tan \alpha$

$\Rightarrow$ general value of $x = n\pi + \alpha$

4. $\sin^2 x = \sin^2 \alpha$

$\Rightarrow$ general value of $x = n\pi \pm \alpha$

5. $\cos^2 x = \cos^2 \alpha$

$\Rightarrow$ general value of $x = n\pi \pm \alpha$

6. $\tan^2 x = \tan^2 \alpha$

$\Rightarrow$ general value of $x = n\pi \pm \alpha$

Where $n \in I$

Note For Solving Trigonometric Equations

1. Squaring should be avoided as far as possible. If at all squaring is done check for extraneous roots.
2. Never cancel the terms which are in product on both sides. (if you cancel your solution contains loss of roots).
3. The solution should never contains such values of the variable which makes any of the given terms undefined.
4. When $\tan\theta$ or $\sec\theta$ are involved in the equation then θ is not equal to odd multiples of π.
5. When $\operatorname{cosec}\theta$ or $\cot\theta$ are involved then θ is not equal to integral multiples of π.
6. Domains of the given equation should not change. If at all change necessary corrections must be made.
7. If the given equation contains $\sin\theta \pm \cos\theta$ and $\sin\theta \cdot \cos\theta$ then put $\sin\theta \pm \cos\theta = t$ and proceed
8. Whenever the terms are in sin, cos, in powers 1, all terms connected with plus sign and number in R.H.S [with (+) or (–) sign] then each term must have its extremum value.
9. In such problems each term will be (+1) when the number in R.H.S is (+)ve and each term will be (–1) when the number in R.H.S is (–1)ve.
10. Whenever the equation is of the form $a\cos\theta + b\sin\theta = c$, the first of all check that whether

real solution exits or not. The condition for this is $|c| \le \sqrt{a^2+b^2}$. If this condition satisfies then proceed the problem by dividing with $\sqrt{a^2+b^2}$ on both sides.

11. The solutions of a trigonometric equation for which $0 \le x < 2\pi$ are called principal solutions.
12. $\sqrt{f(x)}$ is always positive for example : $\sqrt{\cos^2 x} = |\cos x|$ and not $\pm$ cosx.

Properties of Triangles

1. A triangle is an equilateral if
   (a) cotA + cotB + cotC = √3
   (b) $\sin^2 A + \sin^2 B + \sin^2 C = 2$
   (c) cosA + cosB + cosC = 3/2
   (d) $\frac{a}{\cos A} = \frac{b}{\cos B} = \frac{c}{\cos C}$
   (e) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$
2. A triangle is a right angled triangle if
   (a) $8R^2 = a^2 + b^2 + c^2$, where R is the radius of circum-circle
   (b) If cosines of two angles are inversely proportional to the sides opposite to the angles i.e. a cosA=bcosB
   (c) $\frac{a^2-b^2}{a^2+b^2} = \frac{\sin(A-B)}{\sin(A+B)}$
   (d) $2\Delta^2 = \frac{a^2b^2c^2}{a^2+b^2+c^2}$
3. A triangle is an isosceles if
   $a^2\sin(B-C)+b^2\sin(C-A)+c^2\sin(A-B)=0$
4. $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R}$
5. $\cos A = \frac{b^2+c^2-a}{2bc}$
6. $a = b\cos C + c\cos B$
7. $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$
8. $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$
9. $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$
10. $\sin A = \frac{2}{bc}\Delta$
11. $r = \frac{\Delta}{s}$

    $= (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

    $= 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
12. $r_1 = \frac{\Delta}{(s-a)} = s\tan\frac{A}{2}$

    $= 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
13. $r_2 = \frac{\Delta}{(s-b)} = s\tan\frac{B}{2}$

    $= 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$
14. $r_3 = \frac{\Delta}{(s-c)} = s\tan\frac{C}{2}$

    $= 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
15. $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$
16. $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\Delta^2}$
17. $2r \le R$
18. $r_1 + r_2 + r_3 = 4R + r$

19. $r_1r_2 + r_2r_3 + r_3r_1 = s^2 = \frac{r_1r_2r_3}{r}$

20. $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2rR}$

21. $\Delta = 2R^2 \sin A.\sin B.\sin C$

22. The lengths of medians of ΔABC whose sides are *AB=c, BC=a, CA=b,* and D,E and F are the mid-points of BC, CA, AB respectively, are given by

AD= $\frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$ or $\frac{1}{2}\sqrt{b^2 + c^2 + 2bc.\cos A}$

BE= $\frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$ or $\frac{1}{2}\sqrt{c^2 + a^2 + 2ca.\cos B}$

CF= $\frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$ or $\frac{1}{2}\sqrt{a^2 + b^2 + 2ab.\cos C}$

23. **In centre** is represented by I ; **Orthocentre** by O' or H ; **Circumcentre** by O ; **Centroid** by G ; **Ex-centres** by I_1, I_2, I_3 ; **In-radius** by r ; **Ex-radii** by r_1, r_2, r_3 and **Circum-radius** by R.
24. **Pedal triangle** is the triangle formed by joining the feet of the altitudes from the vertices to the sides.
25. The Radius of the circle circumscribing the Pedal triangle is R/2 where R= Radius of the Circum-circle circumscribing the original circle. The circle circumscribing the Pedal triangle bisects the line joining the orthocentre to the circum-centre of the original circle.
26. Distance between the Orthocentre and side BC=2R cosB cosC , between CA=2R cosC cosA , and between AB=2R cosA cosB .
27. **Nine – point Circle** is a circle that passes through (three middle points of the sides, three feet of altitudes from the vertices to the opposite sides and three mid-points of the line joining orthocentre to the vertices).
28. R= $\frac{abc}{4\Delta}$
29. The distance of the ortho-center from the vertices of the triangle are: 2R cos A , 2R

cosB, 2R cos C.

30. If sin A , sin B, sin C are in AP then the altitudes are in HP.
31. The ratio of the distances of the ortho-center of an acute angled triangle ABC from the sides BC, CA, and AB are sec A: sec B: sec C.
32. If I is the incentre, then ratio of IA:IB:IC= cosec A/2 : cosec B/2 : cosec C/2
33. The distance between the circumcentre(O) and the othocentre(H) of a triangle ABC is=OH= $R\sqrt{1-8\cos A.\cos B.\cos C}$
34. Distance between the circumcentre(O) and the centroid(G) is given by= OG= (1/3)OH
35. Distance between orthocentre and centroid=HG=(2/3)OH.
36. Distance between circumcentre and the incentre =OI= $R\sqrt{1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$.
37. Distance between circumcentre and the escribed centre I_1 touching externally side a=OI_1= $R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$.
38. Distance between circum-centre and the escribed centre I_2 touching externally side b=OI_2= $R\sqrt{1+8\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}}$.
39. Distance between circumcentre and the escribed centre I_3 touching externally side c=OI_3= $R\sqrt{1+8\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}}$.
40. Radius of the circumcentre of triangle $I_1I_2I_3$= 2R.
41. The distance between the circumcentre and the incentre of the triangle ABC is $\sqrt{R^2 - 2Rr}$
42. The line joining the in centre to the circumcentre of a trigangle ABC is inclined to the side BC at $\tan\theta = \left(\frac{\cos B + \cos C - 1}{\sin C - \sin B}\right)$
43. Sum of the opposite angles of a cyclic quadrilateral is always 180 degrees.
44. Sum of the products of the opposite sides is equal to the product of the diagonals.(***Ptolemy's law***)

45. If sum of opposite sides of a quadrilateral is equal, then and only then a circle can be inscribed in the quadrilateral.

46. Area of a cyclic quadrilateral whose sides are AB=a,BC=b,CD=c,DA=d, is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where 2s= a+b+c=d.

47. In cyclic quadrilateral,

cos B= $\frac{a^2+b^2+c^2+d^2}{2(ab+cd)}$

48. Circum-radius (R) of Cyclic quadrilateral ABCD is given by R= $\frac{1}{4}\sqrt{\frac{(ab+cd)(ad+bc)(ac+bd)}{(s-a)(s-b)(s-c)(s-d)}}$

49. A regular polygon is a polygon that has all its **sides** as well as its **interior** and **exterior** angles equal. If the polygon has n sides then the sum of its internal angles is 2(n-2) right angles and each angle is $\frac{(n-2)\pi}{n}$

50. Sum of the exterior angles of a polygon taken in one direction remains constant and is equal to 360 degrees.

51. In the regular polygon the centroid, the circumcenter and the incentre are same.

52. The radii of the inscribed and the circumscribed circles fo a regular polygon if n sides with each side a is $\frac{a}{2}\cot\frac{\pi}{n}, \frac{a}{2}\cos ec\frac{\pi}{n}$

53. The area of regular polygon = $\frac{1}{4}na^2\cot\left(\frac{\pi}{n}\right) = nr^2\tan\left(\frac{\pi}{n}\right) = \frac{1}{2}nR^2\sin\left(\frac{2\pi}{n}\right)$

54. (cosA+cosB)(cos2A+cos2B)($\cos 2^2A+\cos 2^2B$).....($\cos 2^nA+\cos 2^nB$)

$= \frac{\cos 2^{n+1}A-\cos 2^{n+1}B}{2^n(\cos A-\cos B)}$

55. tanA+2tan2A+$2^2\tan 2^2A+2^3\tan 2^3A+\dots+2^n\tan 2^nA+2^{n+1}\cot 2^{n+1}A$=cot A for all n when it is natural number.

56. (2cosA-1)(2cos2A-1)($2\cos 2^2A-1$)...($2\cos 2^nA-1$)

$= \frac{2\cos 2^{n+1}A+1}{2\cos A+1}$ for all n when it is natural number.

57. tanA+cotA= 2 cosec 2A

58. cotA-tanA=2cot2A

59. sin(A+B)=sinAcosB+cosAsinB

60. sin(A-B)=sinAcosB-cosAsinB

61. cos(A+B)=cosAcosB-sinAsinB

62. cos(A-B)=cosAcosB+sinAsinB

63. 2sinAcosB=sin(A+B)+sin(A-B)

64. 2cosAcosB=cos(A-B)+cos(A+B)

65. 2sinAsinB=cos(A-B)-cos(A+B)

66. sinC+sinD=2 sin (C+D)/2.cos (C-D)/2

67. sinC-sinD=2 cos(C+D)/2.sin (C-D)/2

68. cosC+cosD= 2 cos (C+D)/2. Cos(C-D)/2

69. cosC-cosD=2 sin (C+D)/2. Sin (D-C)/2

70. sin2A=2 sinA cos A=2tanA/($1+\tan^2A$)

71. cos2A= $\cos^2A-\sin^2A=2\cos^2A-1=1-2\sin^2A=(1-\tan^2A)/(1+\tan^2A)$

72. tan2A=(2tanA)/($1-\tan^2A$)

73. sin3A=3sinA-$4\sin^3A$

74. cos3A=$4\cos^3A$-3cosA

75. tan3A=($3\tan A-\tan^3A$)/($1-3\tan^2A$)

76. $\tan nA = \frac{{}^nC_1\tan A - {}^nC_3\tan^3 A + {}^nC_5\tan^5 A - \dots}{1-{}^nC_2\tan^2 A + {}^nC_4\tan^4 A - {}^nC_6\tan^6 A + \dots}$

77. sin α+ sin (α+β)+sin(α+2β)+...+ sin {α+(n-1)β}

$= \frac{\sin(\alpha+\frac{n-1}{2}\beta)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$

78. {cos α+ cos (α+β)+cos(α+2β)+.......+ cos {α+(n-1)β}=

$\frac{\cos(\alpha+\frac{n-1}{2}\beta)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$

79. cos α. cos 2α.cos 2^2α.....cos2^{n-1} α = $\frac{\sin 2^n \alpha}{2^n \sin \alpha}$

80. Interior angle of a regular polygon of n sides = $\frac{(2n-4)}{n} \cdot 90^0$

Perimeter, Area, and Volume

1. Area of an equilateral triangle = $\frac{\sqrt{3}}{4} a^2$ where a is the side of the triangle.
2. Area of a right angled triangle = (1/2) (base)×(height).
3. Area of a parallelogram = base × height
4. Area of Rhombus = (1/2) (product of diagonals)
5. Area of trapezium =(1/2) (sum of parallel sides)×Height.
6. Area of an ellipse = π ab where 2a= length of the major axis and 2b= length of the minor axis
7. Perimeter or circumference of an ellipse $= \pi \sqrt{2(a^2 + b^2)}$ approx.
8. Volume of cube= a^3 where a = length of an edge
9. Total surface area of the cube = $6a^2$
10. Diagonal of a cube = $\sqrt{3}\, a$.
11. Volume of a cuboid= $l \times b \times h$
12. Area of curved surface of a cylinder= 2 π rh
13. Volume of a cylinder= π r^2h
14. Volume of a cone= (1/3) π r^2h
15. Curved surface area of a cone = πrl where l= slant height
16. Volume of a sphere= (4/3) π r^3
17. Surface area of a sphere= 4 π r^2.
18. Volume of a Prism= $\frac{\sqrt{3}}{4} a^2.h$ where a= side of the base and h= height of the prism.
19. Total surface area of the prism= lateral surface area + sum of the areas of two ends= 3ah+$\frac{\sqrt{3}}{4} a^2$.
20. Volume of a Pyramid = (1/3) height × Area of base.
21. Lateral surface area of pyramid = (1/2) (perimeter)×(slant height)

Complementary and Supplementary angles

1. sin (90°-A) = cos A
2. cos(90°-A) = sin A
3. sin (90°+A) = cos A
4. cos (90°+A) = - sin A
5. sin (180°-A) = sin A
6. cos(180°-A) = - cos A
7. sin (180°+A) = - sin A
8. cos (180°+A) = - cos A
9. sin (360°± A) = ± sin A
10. cos (360°± A) = cos A
11. $\sin 18^0 = \frac{\sqrt{5}-1}{4}$
12. $\cos 36^0 = \frac{\sqrt{5}+1}{4}$

Inverse Trigonometric Functions Map Real Number To Angles

1. The branch of $\sin^{-1}$ function with range is the principal branch. So $\sin^{-1} : [-1,1] \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
2. Inverse of sine function denoted by $\sin^{-1}$x or arc (sin x) is defined on [-1,1] and range may be any of the intervals: $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right], \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$,...
3. The value of an inverse trigonometric function which lies in its ***principal value branch*** is called the ***principal value*** of that inverse trigonometric function.
4. The graph of $\sin^{-1}$ is obtained from the graph of sin x by interchanging the x and y axes.
5. Graph of the inverse function is the mirror image (i.e. reflection) of the original function along the line y = x.
6. $\cos^{-1} x + \cos^{-1} y =$

   $\cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}\, x, y \in [-1,1] : LHS \in [0,\pi]$
7. $\sin^{-1} x + \sin^{-1} y =$

   $\sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}\, x, y \in [-1,1] :$
   $LHS \in [-\pi/2, \pi/2]$

8. $\sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x, |x|\le 1$

9. $\sin^{-1}\frac{2x}{1+x^2}=2\cot^{-1}x, |x|\ge 1$

10. $\cos^{-1}\frac{1-x^2}{1+x^2}=2\tan^{-1}x, |x|\ge 0$

11. $\tan^{-1}\frac{2x}{1-x^2}=2\tan^{-1}x, |x|<1$

12. $\tan^{-1}\frac{1+x}{1-x}=\frac{\pi}{4}+\tan^{-1}x, x<1$

13. $\tan^{-1}\frac{1-x}{1+x}=\frac{\pi}{4}-\tan^{-1}x, x>-1$

14. $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}, |x|\le 1$

15. $\cos ec^{-1}x+\sec^{-1}x=\frac{\pi}{2}, |x|\ge 1$

16. $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}, x\in R$

17. $\sin^{-1}x : |x|\le 1, [-\pi/2, \pi/2]$

18. $\cos^{-1}x : |x|\le 1, [0,\pi]$

19. $\tan^{-1}x : x\in R, (-\pi/2, \pi/2)$

20. $\cos ec^{-1}x : |x|\ge 1, [-\pi/2, 0)\cup(0,\pi/2]$

21. $\sec^{-1}x : |x|\ge 1, [0,\pi/2)\cup(\pi/2,\pi]$

22. $\cot^{-1}x : x\in R, (0,\pi)$

23. $\sin(\cos^{-1}x)=\cos(\sin^{-1}x)=\sqrt{1-x^2}, |x|\le 1$

24. $\sec(\cos ec^{-1}x)=\cos ec(\sec^{-1}x)=\frac{|x|}{\sqrt{x^2-1}}, |x|>1$

Notes:

$\sin^{-1}(\sin x)=x....domain=R;..\&...range=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$; this is periodic with 2 π .

$\sin(\sin^{-1}x)=x...domain=\left[-\frac{\pi}{2},\frac{\pi}{2}\right],...range=[-1,1]$ and this is not periodic.

$\sin^{-1}x+\sin^{-1}y=\pi$ iff x=y=1

$\sin^{-1}x+\sin^{-1}y=-\pi$ iff x=y=-1

$\cos^{-1}x+\cos^{-1}y=0$ iff x=y=1

$\cos^{-1}x+\cos^{-1}y=2\pi$ if x=y=-1

25. While calculating the period of the periodic functions, one must remember that trigonometric functions may have period but algebraic functions do not have except the fractional part function {x} that has period 1.

26. While calculating period of the mixed functions like trigonometric and others , one must remember that LCM or HCF is calculated only for all the rationales or all the irrationals. The LCM or HCF of 2 and π is not possible because one is rational and the other is irrational number.

27. Trigonometric functions are not one-one and onto over their natural domains and ranges and hence their Inverse do not exist.

28. Inverse exists for the functions that are one-one and onto.

29. If we restrict sine to have domain only $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ instead of R then sine becomes one-one and onto with range [-1,1].

30. This restriction is also for intervals $\left[-\frac{3\pi}{2},-\frac{\pi}{2}\right]$, $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ etc. with the same range [-1,1] but $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is called the principal value and $\sin^{-1}x:[-1,1]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and the graph that lies in this range is called the Principal value branch.

31. It is obvious: $\sin(\sin^{-1}x)=x$ if $-1 \le x \le 1$ and $\sin^{-1}(\sin x)=x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

32. $\sin^{-1}(-x) = -\sin^{-1}x$

33. $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$

34. $\tan^{-1}(-x) = -\tan^{-1}x$

35. $\cos^{-1}(-x) = \pi - \cos^{-1}x$

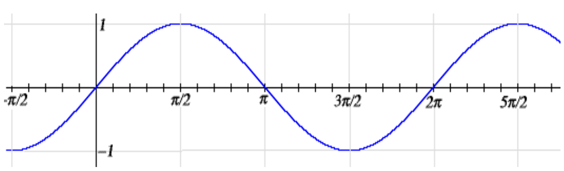
36. $\cot^{-1}(-x) = \pi - \cot^{-1}x$

37. $\sec^{-1}(-x) = \pi - \sec^{-1}x$

38. While converting radians into degrees and degrees into radians, the value of π should be taken 22/7 (approx) and not 180 etc.

Sine function

A good way to understand a function is to look at its graph. Let's start with the graph of sin *t*. Take the horizontal axis to be the *t*-axis (rather than the *x*-axis as usual), take the vertical axis to be the *y*-axis, and graph the equation *y* = sin *t*. It looks like this.

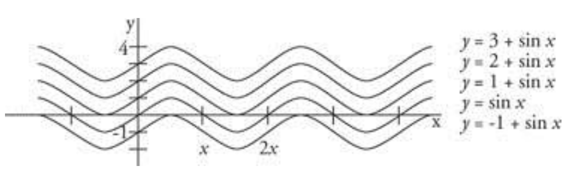


Note:

a. It is periodic of period 2π. Geometrically, it means that if one takes the curve and slides it 2 π either left or right, then the curve falls back on itself.

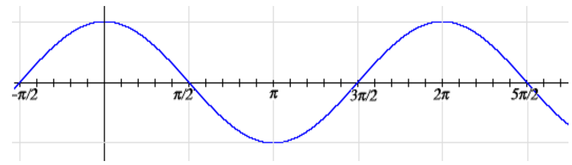
b. The graph is within one unit of the x-axis. Not much else is obvious, except where it increases and decreases. For instance, sin x grows from 0 to π /2 since the *y*-coordinate increases as the angle increases from 0 to π/2.

c. The graph of *y* = *sin x* changes if the additional term A is introduced to *y* = *A+sin x*. The graph shifts vertically upward or downward depending on the value of A (positive or negative) as shown below:



Cosine function

Look at the graph of cosine. Again, take the horizontal axis to be the *t*-axis, but now take the vertical axis to be the *x*-axis, and graph the equation *x* = cos *t*.

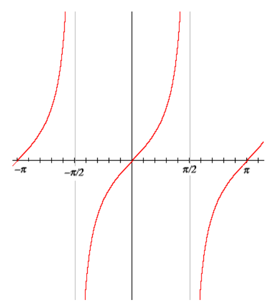


Note

It looks just like the graph of sin *t* except it's translated to the left by π /2. That's because of the identity cos *t* = sin (π /2 + *t*).

Tangent function

The graph of the tangent function has a vertical asymptote at x = π/2. This is because the tangent approaches infinity as *t* approaches π/2. (Actually, it approaches minus infinity as *t* approaches π/2 from the right as it is obvious on the graph.



Note:

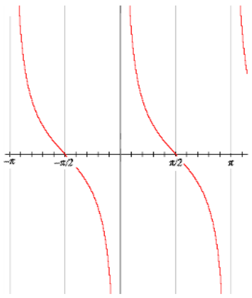
(a) Tangent has period π

(b) There are vertical asymptotes every πunits to the left and right.

(c) Algebraically, this periodicity is expressed by tan (*t* + π) = tan *t*.

Cotangent function

The graph for cotangent is very similar.

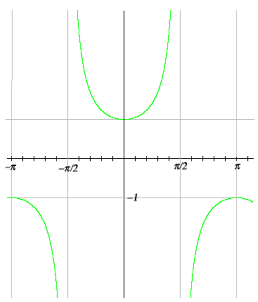


This similarity is simply because the cotangent of t is the tangent of the complementary angle $\pi - t$.

Secant function

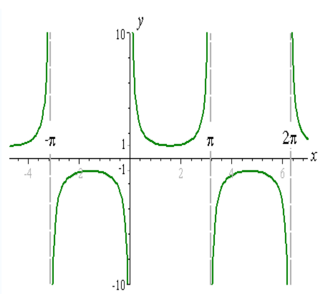
(a) The secant is the reciprocal of the cosine, and as the cosine only takes values between -1 and 1, therefore the secant only takes values above 1 or below -1, as shown in the graph.

(b) Secant has a period of 2π .



Cosecant function

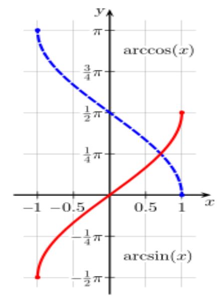
The graph of the cosecant looks much like the graph of the secant as below:



Graphs of Inverse functions:

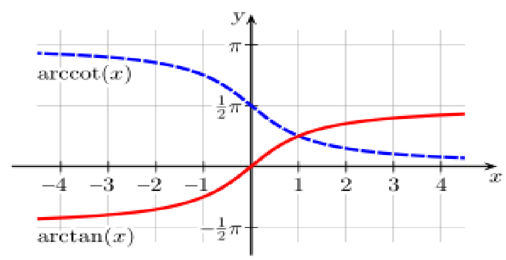
The notations $\sin^{-1}$, $\cos^{-1}$, etc. are often used for arcsin, arccos, etc., but this convention logically conflicts with the common semantics for expressions like $\sin^2(x)$, which do not refer to function composition but rather multiplication, and therefore may result in confusion between multiplicative inverse and compositional inverse.

(a) $\sin^{-1} x$ and **$\cos^{-1} x$**



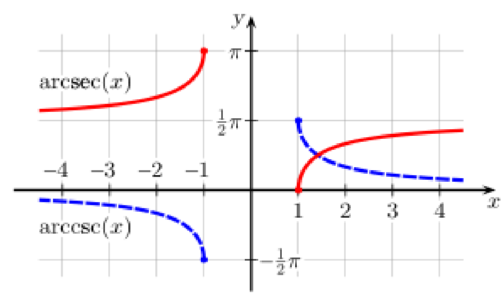
The usual principal values of the ***f*(*x*) = arcsin(*x*)** and ***f*(*x*) = arccos(*x*)** functions graphed on the cartesian plane.

(b) $\tan^{-1} x$ and **$\cot^{-1} x$**



The usual principal values of the ***f*(*x*) = arctan(*x*)** and ***f*(*x*) = arccot(*x*)** functions graphed on the cartesian plane.

(c) $\sec^{-1} x$ and **$\mathrm{cosec}^{-1} x$**



Principal values of the ***f*(*x*) = arcsec(*x*)** and ***f*(*x*) = arccsc(*x*)** functions graphed on the cartesian plane.

Some Illustrations

1. If $\sin x + \sin^2 x = 1$, then evaluate $\cos^8 x + 2\cos^6 x + \cos^4 x$.

Hint:

Write $\sin x = 1 - \sin^2 x = \cos^2 x$

Or $\sin^2 x = \cos^4 x$

$1 - \cos^2 x = \cos^4 x$, Or $\cos^2 x + \cos^4 x = 1$

Square both sides to get the required result.

2. ***If x=y cos 2π/3= z cos 4π/3 then evaluate xy + yz + zx***

Hint:

Note x=-y/2=-z/2

Hence x= m, y=-2m, z= -2m

Put these values in xy + yz+zx to get value =0

3. ***If sinα + sinβ =0 and cosα +cosβ=0 , then find the value of cos 2α + cos 2β.***

Hint:

Square and subtract (i) from (ii) to get the required result= -2 cos(α -β)

4. ***If cot (A/2)= (b+c)/a, then show that the triangle is a right angled triangle.***

Hint:

Use sine formulae:

(sinA)/a = (sinB)/b = (sinC)/c=λ

Replace a,b and c and write

sinB+sinC= 2 {sin(B+C)/2}{ cos(B-C)/2}

and simplify writing

sin(B+C)/2 = sin(180^o-A)/2= cos A/2.

5. ***If tan(π cosθ)= cot(πsinθ), then find the value of cos {θ -(π/4)}.***

Hint:

Write the expression as

tan(π cosθ)= cot(πsinθ)= tan{π/2-(π sinθ)}

ie πcosθ = π/2-(π sinθ)

simplify to find the result.

6. ***If in a triangle ABC, angles A,B,C are in AP, then find the value of*** $\frac{a+c}{\sqrt{a^2-ac+c^2}}$

Hint:

Given 2B=A+C

B=60^o

cosB=cos 60^o= $\frac{c^2+a^2-b^2}{2ca}$

simplify and get the result.

7. ***If A+B+C=270^o, then find the value of cos2A+cos2B+cos2C+4 sinAsinB sinC***

Hint:

Use: cosC+cosD= 2 cos (C+D)/2 cos (C-D)/2.

Write the given expression

=2 cos(A+B) cos(A-B)+1- 2 sin^{2}C+ 4 sinA sinB sinC

=2 cos (270^o-C) cos (A-B)+ 1- 2 sin^{2}C+ 4 sinA sinB sinC

Simplify and find the value =1

8. ***Find the value of cos x. cos2x.cos3x.......cos 999x if x= 2π/1999.***

Hint:

Assume , P= cos x. cos2x. cos3x.......cos 999x

And , Q= sin x. sin2x. sin3x........sin 999x2^{999}

PQ=2sinxcosx)(2sin2xcos2x)......(2sin999x.cos999x)

= sin2x. sin 4x. sin 6x.......sin1998x

= sin 2x. sin4x. sin 6x.....sin999x.{-sin(2π-1000x)}{-sin(2π-1002x)}.....{-sin(2π-1998x)}

=sin2x.sin4x.sin6x.....sin998x.sin999x.sin997x....sin x as 2π=1999x

=Q

Hence

P= (1/2^{999}) is the required result.

9. ***Find the set of solutions of [tan^{-1} x] + [cot^{-1} x] = 2 where [.] denotes greatest integer function.***

Hint:

Assume

Case I

[tan^{-1} x] = 1 and [cot^{-1} x]=1

⇒ 1 ≤ tan^{-1} x < 2 and 1 ≤ cot^{-1} x <2

⇒ x ∈[tan 1, ∞) and x ∈ (cot2, cot 1]

But cot 1 < tan 1

Hence no such x possible that satisfy the both.

Case II

Assume [$\cot^{-1}$ x]=2 and [$\tan^{-1}$ x]=0

$\Rightarrow$x $\in$(cot 3, cot 2] and x$\in$[0, tan 1)

Again no such x exists as cos2 < 0.

Case III

Assume [$\cot^{-1}$ x]=3 and [$\tan^{-1}$x]=-1

$\Rightarrow$ x$\in$(-∞, cot 3] and x$\in$[-tan1,0)

Again no such x exists as cot 3 < - tan1

Hence no x exists for which the equation is 2.

10. ***Find the Minimum value of cos(cosx).***

Hint:

Note, -1 $\le$ cosx $\le$ 1 for all real values of x

Hence if cos x is replaced by cos(cos x) then the inequality becomes

-1 $\le$ cos(cos x) $\le$ 1

Hence the minimum value is -1

11. ***If A = $\cos^2x + \sin^4x$ then for all values of x, find the maximum and minimum values of A.***

Hint:

Rewrite

A = 1 – $\cos^2$ x + $\cos^4$ x

$\Rightarrow$ 1- A = $\cos^2$ x – $\cos^4$ x

= $\cos^2$ x $\sin^2$ x=(1/4)$(\sin2x)^2$

$\Rightarrow$ 0 $\le$ (1-A) $\le$ (1/4) as 0 $\le$ $(\sin2x)^2$ $\le$ 1.

$\Rightarrow$ (3/4) $\le$ A $\le$ 1

12. ***Find the minimum value of sin(cosx)***.

Hint:

Note cos x$\in$ [-1, 1] and sin x is an increasing function on [-π /2 ,π /2]

Hence maximum value of sin(cos x) is

sin (maximum of cos x)= sin 1

13. ***If [x] denotes the greatest integer less than or equal to x and f(x) = sin x + cos x . Then find the most general solution of***

$$f(x) = \left[f\left(\frac{\pi}{10}\right) \right].$$

Hint:

Obviously, $f(\pi/10) = \sin 18^0 + \cos 18^0 = \sqrt{2}\sin 63^0$.

Since, sin 63° > sin 45° = $\frac{1}{\sqrt{2}}$ and sin 63° < 1

$$\Rightarrow 1 < f\left(\frac{\pi}{10}\right) < 2 \Rightarrow \left[f\left(\frac{\pi}{10}\right) \right] = 1 \Rightarrow \sin x + \cos x = 1$$

And hence the required solution can be evaluated.

14. ***Find the number of solutions of the equation $x^3 + x^2 + 4x + 2\sin x = 0$ in $0 \le x \le 2\pi$.***

Hint:

Rewrite the given expression as

$x^3 + (x+2)^2 + 2\sin x = 4$

obviously x=0 is a solution.

Hence if 0 < x $\le$ π then

$x^3 + (x+2)^2 + 2\sin x > 4$

And if π < x $\le$ 2π then

$x^3 + (x+2)^2 + 2\sin x$ 27+25-2

Hence, x=0 is the only solution.

15. ***Find the number of real solutions of $\sin e^x . \cos e^x = 2^{x+2} + 2^{-x-2}$.***

Hint:

Rewrite the given expression

$$\frac{1}{2}\left(\sin 2e^x\right) = 2^{x+2} + 2^{-x-2}.$$

Apply AM – GM inequality

$$\frac{2^{x+2} + 2^{-x-2}}{2} \ge \sqrt[2]{2^0} \Rightarrow 2^{x+2} + 2^{-x-2} > 2 \Rightarrow \sin(2e^x) > 4$$

Hence no solution.

16. ***Find the value(s) of a for which the equation $2\sin^2x-(a+3)\sin x+2a-2=0$ has a real solution.***

Hint:

Find sinx $\frac{(a+3)\pm\sqrt{(a+3)^2-8(2a-2)}}{4}=2,\frac{a-1}{2}$

For real solutions

-1 ≤ sinx ≤ 1 ⇒ -1 ≤ $\frac{a-1}{2}$ ≤ 1⇒ a∈ [-1,3]

17. ***Find the general solution of the equation $\sin^{100}x-\cos^{100}x=1$.***

Hint:

Rewrite the given expression as below:

$\sin^{100}x=1+\cos^{100}x$

LHS ≤ 1 but RHS ≥ 1 except when cos x=0

Hence general solution is

$x=n\pi + (-1)^n (\pi/2)$

18. ***Solve the equation: sinx=[1+sinx][1-cosx] where [.] is a greater integer function.***

Hint:

Case I: x = - π/2 or 3π/2 ⇒ -1 = +1 i.e. an absurd result.

Case II: x=0⇒ 0 =1 i.e. an absurd result.

Case III: x=π/2 ⇒ sin x =3, an absurd result.

Case IV: x=π⇒sin x =3, an absurd result.

Case V: x∈(-π/2, 0) gives absure result

Case VI: x∈(0, π/2) gives absurd result

Case VII: x∈(π/2,π0) gives absurd result

Case VIII: x∈(π,3π/2) gives absurd result. Hence, no solution.

19. ***Find the condition for k if k cosx – 3 sin x = k+1 is solvable.***

Hint:

Simplify as below

$$\cos(x+\theta)=\frac{k+1}{\sqrt{k^2+9}}$$

And for a solution,

$$-1\le\cos(x+\theta)\le1\Rightarrow-1\le\frac{k+1}{\sqrt{k^2+9}}\le1\Rightarrow k\le4\Rightarrow x\in(-\infty,4]$$

20. ***Find the number of solutions of cosx = /1 + sinx/ on the interval [0 , 3π]***

Hint:

Obviously 1+sin x ≥ 0 hence cos x- sin x =1 if the solution holds.

Rewrite as cos (x+π/4)=1/√2

⇒x=0,3π/2,2π⇒ 3 solutions.

21. ***Find the number of solutions of $2^{\cos x}$ = /sinx/ on the interval [-2π, 2π].***

Hint:

The equation is true for only x=π/2 hence sin x = ± 1 and hence 4 solutions.

22. ***Find the number of solutions of sin {x} = cos {x} on [0,2π], where {.} denotes the fractional part.***

Hint:

Use graphs of sin {x} and cos {x} to find the solution

Obviously the intersection points are in number =6 in given interval.

23. ***Solve: cos x cos y =1.***

Hint:

-1 ≤ cos x ≤ 1 and

-1≤ cos y ≤1

⇒ cos x cos y =1

⇒ cos x =1 and cos y =1

⇒x= 2nπ , n∈ I and y= 2mπ , m∈ I

Also

cos x = -1 and cos y=-1

⇒ x=(2n +1)π and y=(2m+1)π

24. ***Solve : x+y = 2 π/3 and cos x + cos y = 3/2.***

Hint:

Rewrite cos x + cos y = 3/2 as below:

2cos {(x+y)/2}. cos {(x-y)/2}= 3/2

⇒cos{ (x-y)/2}= 3/2 that is absurd.

⇒no solution.

25. *Find the number of solutions of $e^{\sin x} - e^{-\sin x} - 7 = 0$.*

Hint:

Note $-1 \le \sin x \le 1$

$\Rightarrow e^{-1} \le e^{\sin x} \le e^{1}$

But the original equation gives

$e^{\sin x} = 7 + (1/e^{\sin x})$

For no value of x the two sides LHS and RHS can be equal and hence no solution exists.

26. ***Evaluate:*** $\sum_{k=1}^{\infty} 3^{k-1} \sin^3 \frac{a}{3^k}$

Hint:

Use the formula: $\sin 3x = 3\sin x - 4\sin^3 x$.

And rewrite the given expression

$$= \sum_{k=1}^{\infty} \frac{1}{4}\left(3^k \sin\frac{a}{3^k} - 3^{k-1}\sin\frac{a}{3^{k-1}}\right)$$

$$= \lim_{k\to\infty} \frac{1}{4}\left(3^k \sin\frac{a}{3^k} - \sin a\right)$$

as all other terms cancel out if we put k=1,2,3,..., and this can be evaluated by changing to the form (0/0) and using the standard limit of

$$\lim_{k\to\infty} \frac{\sin\frac{a}{3^k}}{\frac{a}{3^k}} = \lim_{k\to\infty} \frac{\sin 0}{0} = 1.$$

27. *If $x \ne (n\pi/2)$ then solve the equation* $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1.$.

Hint:

Obviously, $\cos x \ne 0, 1, -1$ as $x \ne (n\pi/2)$.

$\Rightarrow \sin^2 x - \sin x + 2 = 0$

$\Rightarrow (\sin x - 2)(\sin x - 1) = 0$

$\Rightarrow \sin x = 1$ as sin x cannot be 2.

$\Rightarrow \cos x = 0$ but cos x cannot be zero as $0^{\infty} \ne 1$ hence no solution is possible.

28. *If the arcs of the same lengths in two circles subtend angles 65^o and 110^o at the centre, then find the ration of their radii.*

Hint:

Let r_1, r_2 be the radii of the two circles. Then

$$\theta_1 = 65^0 = \frac{\pi}{180} \times 65 = \frac{13\pi}{36} \text{ radians}$$

$$\theta_2 = 110^0 = \frac{\pi}{180} \times 110 = \frac{22\pi}{36} \text{ radians}$$

We know that if ***l*** be the length of equal arcs then

$l = r_1\theta_1$

$l = r_2\theta_2$

Then the required ratio is $\frac{r_1}{r_2} = \frac{\theta_2}{\theta_1} = \frac{22}{13}$

29. *If* $\cos x = -\frac{3}{5}$*, x lies in the third quadrant, find the values of other five trigonometric functions.*

Hint:

From $\sin^2 x + \cos^2 x = 1$,

$\sin^2 x = \frac{16}{25} \Rightarrow \sin x = \pm\frac{4}{5}$

Since x lies in third quadrant, hence sin x will be negative. Therefore, $\sin x = -\frac{4}{5}$.

Similarly, other ratios keeping their existence with sign may be calculated.

30. *Find the principal solution of the equation* $\sin x = \frac{\sqrt{3}}{2}$.

Hint:

We know that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\left(\pi - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

Hence $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ are the principal solutions because principal solutions are the solutions that lie in the interval $[0,2\pi)$

31. If $\sin x = \frac{3}{5}$ ***and*** $\cos y = -\frac{12}{13}$***, where x and y both lie in the second quadrant, find the value of sin(x+y).***

Hint:

We know that sin(x+y)=sinx cosy+cosx siny

Sinx is given hence from the identity $\sin^2 x+\cos^2 x=1$,

$\cos^2 x = \pm\frac{4}{5}$ but x lies in second quadrant, hence cosx should be negative.

So, $\cos x = -\frac{4}{5}$

Similarly, $\sin y = \pm\frac{5}{13}$. Here siny will be positive as y lies in second quadrant.

So, $\sin y = \frac{5}{13}$

Now, on putting the essential values, sin(x+y) can be evaluated as $-\frac{56}{65}$

32. Find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$**.**

Hint:

Let us assume

$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$=x

$\Rightarrow$ sinx=$\left(\frac{1}{\sqrt{2}}\right)$ or

x=$\left(\frac{\pi}{4}\right)$ or $\left(\pi-\frac{\pi}{4}\right)$.

But the principal value branch of $\sin^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Since $\left(\frac{\pi}{4}\right)$ lies in this interval, hence this is the principal value.

Dr S.B. Dhar, is **Editor of this Quartrerly e-Bulletin**. He is an eminent mentor, analyst and connoisseur of Mathematics from IIT for preparing aspirants of Competitive Examinations for Services & Admissions to different streams of study at Undergraduate and Graduate levels using formal methods of teaching shared with technological aids to keep learning at par with escalating standards of scholars and learners. He has authored numerous books – Handbook of Mathematics for IIT JEE, A Textbook on Engineering Mathematics, Reasoning Ability, Lateral Wisdom, Progress in Mathematics (series for Beginner to Class VIII), Target PSA (series for class VI to class XII) and many more.

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