# INTEGRAL CALCULUS

**Prof. SB DHAR**

## Definition

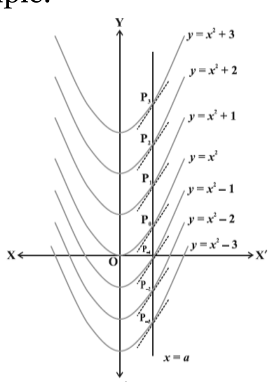
If $\frac{d}{dx}g(x) = f(x)$, then *f(x)* is called the **derivative** of g(x); g(x) is called **anti-derivative** or **primitive** or **integral** of f(x), and is written as $\int f(x)dx = g(x)$.

It is read as **"Integral of f(x) is equal to g(x)".** The symbol ∫ denotes "Integral" and the process of evaluating g(x) is called **"Integration"**.

For example: $\frac{d}{dx}\sin x = \cos x \Rightarrow \int \cos x dx = \sin x$

### Important Facts

1. If two primitives $f_1$ and $f_2$ exist for a function then they differ by a constant.
2. Geometrically, indefinite integral refers to family of curves parallel to a curve upward or downward.
   Example:




3. Derivative of a function is unique but anti-derivative is not unique.
4. A function is differentiable at a point but integrable on an interval.

### Some Important Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1$
2. $\int dx = x$
3. $\int \frac{dx}{x} = \log_e x$
4. $\int a^x dx = a^x \log_a e$
5. $\int e^x dx = e^x$
6. $\int \sin x dx = -\cos x$
7. $\int \cos x dx = \sin x$
8. $\int \tan x dx = \log_e \sec x$
9. $\int \cot x dx = -\log_e \operatorname{cosec} x$
10. $\int \sec x \tan x dx = \sec x$
11. $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x$
12. $\int \sec^2 x dx = \tan x$
13. $\int \operatorname{cosec}^2 x dx = -\cot x$
14. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x = -\cos^{-1} x$
15. $\int \frac{dx}{1+x^2} = \tan^{-1} x = -\cot^{-1} x$
16. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x = -\operatorname{cosec}^{-1} x$
17. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a}$
18. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right|$
19. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right|$
20. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} = -\frac{1}{a}\cot^{-1}\frac{x}{a}$
21. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} = -\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}$
22. $\int \sec x dx = \log(\sec x + \tan x) = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$
23. $\int \operatorname{cosec} x dx = \log(\operatorname{cosec} x - \cot x) = \log\tan\frac{x}{2}$

24. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$

25. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$

26. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

27. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)$

28. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$

29. $\int |x|dx = \frac{1}{2}x|x|$

**Some Important Substitutions**

(a) For integral of type $\int \sqrt{(x-\alpha)(\beta-x)}\,dx$, assume $x=\alpha\cos^2\theta+\beta\sin^2\theta$ .

(b) For integral of type $\int \sqrt{\frac{x-a}{x-b}}\,dx ... or ... \int \sqrt{(x-a)(x-b)}\,dx$ , assume $x = a\sec^2\theta - b\tan^2\theta$ .

(c) For integral of type $\int \frac{1}{\sqrt{(x-a)(x-b)}}dx$ assume $x-a=t^2$.

(d) For integral of type $\int \sqrt{2ax-x^2}\,dx$ , assume $x=a(1-\cos\theta)$ .

**Methods To Find Integrals**

(a) ILATE Rule: $\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$.

u is named the first function and v the second function. The First function is selected through ILATE (order should be Inverse, Logarithmic, Algebraic, Trigonometric, and Exponential).

(b) General formulae for Integration by parts

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - ... + (-1)^{n-1}u^{n-1}v_n - (-1)^{n-1}\int u^n v_n\,dx$

(c) In the Integrals of Type $\int e^{ax}\cos bx.dx$ any one of the two can be taken as the First Function.

(d) In the Integral of Type $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$, $\int \sqrt{ax^2+bx+c}\,dx$ the Denominators/Numerator should be made perfect square and then (x+ b/2a) should be put equal to t and proceed further.

(e) In the Integral of Type $\int \frac{Q(x)dx}{ax^2+bx+c}$, First the Numerator should be made polynomial of lesser than the Denominator by dividing the Denominator and then proper substitution should be made to start.

(f) In the Integral of Type $\int \frac{dx}{a\cos^2 x+b\sin^2 x}$, $\int \frac{dx}{a+b\sin^2 x}$, $\int \frac{dx}{a+b\cos^2 x}$, $\int \frac{dx}{(a\sin x+b\cos x)^2}$ , $\int \frac{dx}{a+b\cos^2 x+c\sin^2 x}$, the numerator and the denominator should be multiplied by $\sec^2$ x and then in the denominator *tan x* should be put = t and start doing integrations after proper substitutions.

(g) In the integrals of Type $\int \frac{dx}{a+b\sin x}$, $\int \frac{dx}{a+b\cos x}$, $\int \frac{dx}{a\sin x+b\cos x}$, $\int \frac{dx}{a+b\sin x+c\cos x}$, the proper way is to use

$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and after putting tan x/2 = t the sum can be done.

(h) In the integral of the Type $\int \frac{(p\sin x+q\cos x+r)dx}{(a\sin x+b\cos x+c)}$, the proper substitution is to put Numerator= λ+ μ( differential coefficient of denominator)+ν. And then after finding the numeric values for the assumed arbitrary constants, the Integrals can be done.

(i) In the integral of Type $\int \frac{Q(x)dx}{P(x)}$, the Numerator should be first made of lesser degree than the Denominator and then if the Denominator is decomposable to factors, the method of partial fractions should be used to split into different fractions and then one of the proper methods can be used to start for Integration.

(j) In the integral of Type $\int \frac{dx}{(a+bx)\sqrt{cx+d}}$, put $cx+d=t^2$ and proceed using the method as required.

(k) In the integral of Type $\int \frac{dx}{(a+bx+cx^2)\sqrt{px+q}}$, put $px+q=t^2$ and proceed using the method as required.

(l) In the integral of Type $\int \frac{dx}{(a+bx)\sqrt{px^2+qx+r}}$, put $a+bx=\frac{1}{t}$ and proceed using the method as required.

(m) In the integral of Type $\int \frac{dx}{(a+bx^2)\sqrt{cx^2+d}}$, put $x=\frac{1}{t}$ and proceed using the method as required.

(n) For the integral of type $\int \frac{\cos 2\alpha x - \cos 2\beta x}{1+2\cos 2\gamma x}dx$ where α, β, γ are such that 3γ= α+β multiply the Numerator and the Denominator by sin(γ/2) and then use 2sinAcosB or 2sinAsinB formulae as required.

Example:

$$I = \int \frac{\cos 13x - \cos 14x}{1+2\cos 9x}dx$$

$$= \int \frac{(\cos 13x - \cos 14x)\sin\frac{9x}{2}}{(1+2\cos 9x)\sin\frac{9x}{2}}dx$$

$$= \int \frac{2\sin\frac{27x}{2}.\sin\frac{x}{2}.\sin\frac{9x}{2}}{\sin\frac{27x}{2}}dx$$

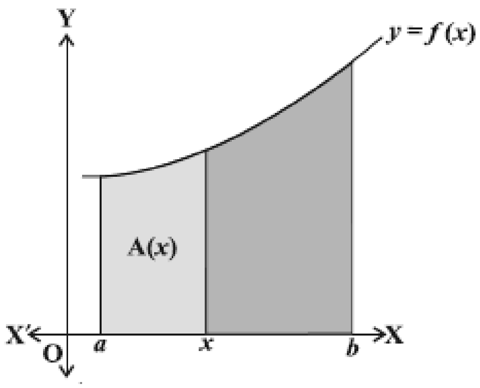
$$= \int 2\sin\frac{x}{2}.\sin\frac{9x}{2}dx = \int (\cos 4x - \cos 5x)dx$$

**Definite integrals**

**Definition**

Definite integral is represented as $\int_{x=a}^{x=b} f(x)dx$. It is read as integration from *x=a to x=b*. ***a*** is called the *lower limit* and ***b*** is called the *upper limit*. This integral represents the bounded region i.e., the area between *x=a* and *x=b*.

The figure represents the integral from *x=a* to *x=b* of function *y=f(x)*. The value of this integral is nothing but the area from *x=a* to *x=b* of the function *y=f(x)* bounded between the regions made by ordinates *x=a, x=b* and above *x-axis*. It is Area function as *ydx* represents a rectangle of sides *dx* and *y*.




**Properties of Definite Integrals**

(a) $\int_a^b f(x)dx = \int_a^b f(t)dt$

(b) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

(c) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$

(d) $\int_0^a f(x)dx = \int_0^{a/2} f(x)dx + \int_0^{a/2} f(a-x)dx$

(e) $\int_a^b f(x)dx = 0, if .. f(a+x) = -f(b-x)$

(f) $\int_a^b f(x)dx = 2\int_a^{\frac{a+b}{2}} f(x)dx, if .. f(a+x) = f(b-x)$

(g) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

(h) $\int_{-a}^a f(x)dx = 0, if .. f(-x) = -f(x)$

(i) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx, if .. f(-x) = f(x)$

(j) $\int_0^{2a} f(x)dx = 0, if .. f(2a-x) = -f(x)$

(k) $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, if .. f(2a-x) = f(x)$

(l) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

(m) $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$ if f(x+T) = f(x)

(n) $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx$ if n is an Integer

(o) $\int_0^{a+T} f(x)dx = \int_0^T f(x)dx$ if n=1.

(p) $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx$

(q) $\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx$ where n is an Integer.

(r) $\int_a^{a+T} f(x)dx$ is independent of a.

**Mean value theorem of Integral Calculus**

If a function *f* is continuous on *[a,b]*, it assumes its mean value in *[a,b]*, that is $\frac{1}{b-a}\int_a^b f(x)dx = f(c)$ for some *c* such that $a \le c \le b$.

**Some Inequlaities**

**(a) Schwarz-BunyanKowsky Inequality**

If f(x) and g(x) are integral on (a,b) then $\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\left(\int_a^b f^2(x)dx\right)\left(\int_a^b f^2(x)dx\right)}$ where $f^2(x)=\{f(x)\}^2$.

(b) If f(x) ≥ g(x) on [a, b] then $\int_a^b f(x)dx \ge \int_a^b g(x)dx$

(c) If f(x) is increasing and has a concave graph in [a,b] then

$$(b-a)f(a) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}$$

(c) If f(x) is increasing and has a convex graph in [a,b] then

$$(b-a)\frac{f(a)+f(b)}{2} < \int_a^b f(x)dx < (b-a)f(b)$$

(d) If m and M be global minimum and global maximum of f(x) respectively in [a,b] then m(b-a) ≤ $\int_a^b f(x)dx \le$ M(b-a)

(e) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$.

(f) If the integral of any function $\int_a^b f(x)dx$ that is continuous on [a,b] and it is not possible to evaluate this Integrand then we use sandwich formula after finding two continuous functions $f_1(x)$ and $f_2(x)$ on [a,b] such that $f_1(x) \le f(x) \le f_2(x)$; $\forall$ x $\in$ [a,b] then $\int_a^b f_1(x)dx \le \int_a^b f(x)dx \le \int_a^b f_2(x)dx$

(g) If f(t) is an odd function then $\phi(x) = \int_a^x f(t)dt$ is an even function.

(h) If f(t) is an even function then $\phi(x) = \int_a^x f(t)dt$ is an odd function.

**(i) Definite integral as the limit of a sum:**

$$\int_a^b f(x)dx = \lim_{h\to 0, n\to\infty} h[f(a)+f(a+h)+f(a+2h)+\ldots$$
$$+ f(a+(n-1)h)] \text{ where b-a=nh}$$

(j) Find the r[th] term and write it as $\lim_{n\to\infty}\sum \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$

**Some useful Functions**

**Gamma function**:

$\int_0^\infty e^{-x}x^{n-1}dx = \Gamma n$ where x ∈ $Q^+$ and n is a positive number. This is also called the Eulerian Integral of 2[nd] Kind.

**Properties**

(a) $\Gamma n$ is pronunciated as Gamma n, and, it is denoted as $\Gamma n = (n-1)\Gamma(n-1)$.

(b) For example: $\Gamma 4 = 3.2.1.$ ; $\Gamma 1 = 1$; $\Gamma 0 = \infty$; $\Gamma{-n} = \infty$ if n is positive integer ; $\Gamma 1/2 = \sqrt{\pi}$

(c) If n is a natural number then $\Gamma(n+1) = n!$ and $\Gamma(1/2) = \sqrt{\pi}$.

(d) $\int_0^{\frac{\pi}{2}} \sin^m x.\cos^n x dx = \dfrac{\Gamma\frac{m+1}{2}\Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$ for all m> -1 and n > -1.

***Beta Function***:

$\int_0^1 x^{m-1}.(1-x)^{n-1}dx$, where m,n>0 is called the Beta Function and is denoted by B(m,n). This is also called ***Eulerian integral of Ist kind***.

Properties

(a) B(m,n)=B(n,m)

(b) B(m,1)= 1/m

(c) $\dfrac{B(m+1,n)}{B(m,n)} = \dfrac{m}{m+n}$.

(d) B(m,n)= $\dfrac{n-1}{m}B(m+1,n-1)$, n > 1

(e) B(m,n)= $\frac{(m-1)!}{n(n+1)(n+2)...(n+m-1)}$ if m is a positive integer.

(f) If m,n are positive integer then B(m,n)= $\frac{(m-1)!(n-1)!}{(m+n-1)!}$.

(g) Another form of Beta Function is given by B(m,n)= $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$ where m,n>0. This form is obtained by replacing x=1/(1+y) in the original format.

**Relation between Gamma and Beta Function:**

(a) B(m,n)= $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, m,n>0

(b) $\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}$, where 0<n<1

(c) $\int_0^\infty \frac{x^{n-1}}{(1+x)}dx=\frac{\pi}{\sin n\pi}$.

(d) $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$.

**Newton-Leibnitz Formula**

$\int_a^b f(x)dx=\left[F(x)\right]_a^b=F(b)-F(a)$ where F(x) is one of the anti-derivative of f(x). This is called as ***Newton-Leibnitz formula***.

**Note:**

This formula is true to compute the definite integral of a function that is continuous on [a,b].

**Integrals with Infinite Limits**

If a function f(x) is continuous for a≤ x<∞ then by definition $\int_a^\infty f(x)dx=\lim_{b\to\infty}\int_a^b f(x)dx$. If there exists a finite limit on the right hand side, then the improper integral is said to be convergent otherwise it is divergent.

**Area Function**

A(x)= $\int_a^x f(x)dx$, if x is a point in [a,b].

**Facts to remember**

(a) The area of the region bounded by $y^2$=4ax, x=c, c>0,a>0 = $\frac{8c\sqrt{ac}}{3}$.

(b) The area of the region bounded by $x^2$=4ay, y=c, c>0,a>0 = $\frac{8c\sqrt{ac}}{3}$.

(c) The area of the region bounded by $y^2$=4ax and its latus rectum(x=a)= $\frac{8a^2}{3}$.

(d) The area of the region bounded by $x^2$=4ay and its latus rectum(y=a)= $\frac{8a^2}{3}$.

(e) The area of the region bounded by $y^2$=4ax and y=mx, = $\frac{8a^2}{3m^3}$.

(f) The area of the region bounded by $x^2$=4ay and x=my, = $\frac{8a^2}{3m^3}$.

(g) The area of the region bounded by $y^2$=4ax and $x^2$=4ay, = $\frac{16a^2}{3}$.

(h) The area of the region bounded by $y^2$=4ax and $x^2$=4by, = $\frac{16ab}{3}$.

(i) The area of the region bounded by $y^2$=4ax and $x^2$=4ay, and x=a, = $\frac{5a^2}{4}$.

**Fundamental Theorems of Integral Calculus**

(a) First fundamental theorem of integral calculus:

If Area function, A(x)= $\int_a^x f(x)dx$ for all x $\geq$ a, & f is continuous on [a,b].Then $A'(x) = f(x)$ for all x $\in$ [a, b].

(b) Second fundamental theorem of integral calculus:

Let $f$ be a continuous function of x in the closed interval [$a$, $b$] and let F be another function such that $\frac{d}{dx}F(x) = f(x)$ for all x in domain of f, then $\int_a^b f(x)dx = [F(x)+c]_a^b = F(b) - F(a)$

**Some Typical Results:**

(a) $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} = \pi, if, \beta > \alpha$

Ex. $\int_2^3 \frac{dx}{\sqrt{(x-2)(3-x)}} = \pi$

(b) $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}dx = \frac{\pi}{8}(\beta-\alpha)^2$

(c) $\int_\alpha^\beta \sqrt{\frac{x-\alpha}{\beta-x}}dx = \frac{\pi}{2}(\beta-\alpha)$

(d) $\lim_{x\to 0}\left|\frac{\int_0^x f(x)dx}{x}\right| = f(0)$

Ex. $\lim_{x\to 0}\left|\frac{\int_0^x e^x dx}{x}\right| = e^0 = 1$

(e) $\int_a^b f(x)dx = \frac{1}{n}\int_{na}^{nb} f(x)dx$

(f) $\int_{a-c}^{b-c} f(x+c)dx = \int_a^b f(x)dx$

(g) $\int_a^b f(x)dx = (b-a)\int_0^1 f[(b-a)t+a]dt$

(h) $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx = \frac{1}{2}(b-a)$

(i) $\int_0^{\pi/2} \frac{a.\sin x + b.\cos x}{\sin x + \cos x}dx = \int_0^{\pi/2} \frac{a.\sec x + b.\operatorname{cosec} x}{\sec x + \operatorname{cosec} x}dx$

$= \int_0^{\pi/2} \frac{a.\tan x + b.\cot x}{\tan x + \cot x}dx$

(j) $\int_a^b (|x-a|+|x-b|)dx = (b-a)^2$ Ll

(k) $\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \frac{(n-1)(n-3)....4.2}{n(n-2)....3.1}$, when n is odd

$= \frac{(n-1)(n-3)....3.1}{n(n-2)....4.2}$ when n is even

(l) $\int_0^a \frac{dx}{1+e^{f(x)}} = \frac{a}{2}$, if f(a-x)=f(x)

Ex. $\int_0^\pi \frac{dx}{1+e^{\sin x}} = \frac{\pi}{2}$

(m) $\int_0^{\pi/4} (\tan^n x + \tan^{n-2} x)dx = \frac{1}{n-1}$

(n) $\int_0^{\pi/4} (\cot^n x + \cot^{n-2} x)dx = \frac{1}{n-1}$

(o) Let a function f(x,$\alpha$) be a continuous for a$\leq$ x$\leq$ b and c $\leq\alpha\leq$ d . Then for any $\alpha \in$ [c,d], if **I**($\alpha$)= $\int_a^b f(x,\alpha)dx$ then **I**'($\alpha$)= $\int_a^b f'(x,\alpha)dx$ where I'($\alpha$)is the derivative of **I**($\alpha$) wrt $\alpha$ and f'(x,$\alpha$) is the derivative of f(x,$\alpha$ ) wrt $\alpha$ keeping x constant.

(p) $\int_a^b [x]dx = \frac{(b-a)(b+a-1)}{2}$ where a,b are integers and [x] is a greatest integer function

(q) $\int_0^n [x]dx = \frac{n(n-1)}{2}$ where n is a positive integer.

(r) $\int_0^n \{x\}dx = \frac{n}{2}$ where n is a positive integer.

(s) $\int_0^n [x]dx = [n].\left(\frac{n+\{n\}-1}{2}\right)$ where n is a real number.

(t) $\int_0^n \{x\}dx = \frac{(b-a)}{2}$ where a ,b are integers.

(u) $\int_0^n [kx]dx = \frac{nk(nk-1)}{2k}$ where n,k are positive integers.

(v) $\int_0^n [x^2]dx = -\left(1+\sqrt{2}+\sqrt{3}+....+\sqrt{n^2-1}+\left(n^2-1\right)n\right)$ where n is an integer.

(w) $\int_0^n [x^k]dx = -\left(1+2^{\frac{1}{k}}+3^{\frac{1}{k}}+....+(n^k-1)^{\frac{1}{k}}+(n^k-1)n\right)$ where n,k are positive integers.

(x) $\int_0^{\frac{2\pi}{k}} [\cos kx]dx = -\frac{\pi}{k}$ where k is a positive integer.

(y) $\int_0^{\frac{2\pi}{k}} [\sin kx]dx = -\frac{\pi}{k}$ where k is a positive integer.

(z) $\int_a^b [x]dx = \int_a^{[a]+1} [x]dx + \int_{[a]+1}^{[b]} [x]dx + \int_{[b]}^b [x]dx$ where a, b are any real numbers.

(aa) $\int_a^b \{x\}dx = \int_a^{[a]+1} \{x\}dx + \int_{[a]+1}^{[b]} \{x\}dx + \int_{[b]}^b \{x\}dx$ where a,b are any real numbers.

(bb) $\int_0^n \left[\frac{x}{k}\right]dx = pn - \frac{p(p+1)k}{2}$ where p=[n/k] and n,k are positive integers.

(cc) $\int_0^n \{kx\}dx = \frac{n}{2}$ where n is a positive integer.

(dd) $\int_0^n \left\{\frac{x}{k}\right\}dx = \frac{n^2}{2k} - pn + \frac{p(p+1)k}{2}$ where p=[n/k] and n,k are positive integers.

**Partial fraction methods are as under:**

(a) $\frac{1}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$ ,

(b) $\frac{1}{(x-a)(x-b)^2} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-b)^2}$

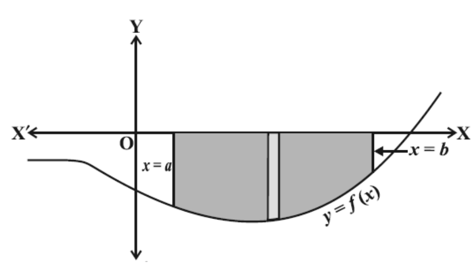
(c) $\frac{1}{(x-a)(x^2+b)} = \frac{A}{(x-a)} + \frac{Bx+C}{(x^2+b)}$

(d) $\frac{1}{(x-a)(x^2+b)^2} = \frac{A}{(x-a)} + \frac{Bx+C}{(x^2+b)} + \frac{Dx+E}{(x^2+b)^2}$

## Application of Integration

Integration is very useful in finding the area of a curve bounded by either ordinates or abscissas. The area is always a positive quantity, hence we denote the Area Function as $\left|\int_{x=a}^{x=b} f(x)dx\right|$ or $\left|\int_{y=a}^{y=b} f(y)dy\right|$ .

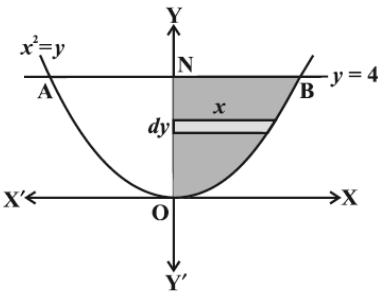
The value of integral is negative when the curve lies below x-axis.




Example:

Find the area of the region bounded by the curve $y=x^2$ and the line $y=4$.

Solution:




The figure shows the required area as

$$A=\left|\int_{y=0}^{y=4} f(y)dy\right| = \left|\int_{y=0}^{y=4} \sqrt{y}dy\right| = \left(\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right)_{y=0}^{y=4} = \frac{16}{3}$$

**Some Illustrations**

**1.** Find the integral of $\frac{1-\sin x}{\cos^2 x}$.

***Solution:***

$$\int \frac{1-\sin x}{\cos^2 x}dx = \int \frac{1}{\cos^2 x}dx - \int \frac{\sin x}{\cos^2 x}dx$$

$$= \int \sec^2 x dx - \int \sec x \tan x dx$$

=tanx-secx+C,

where C is a constant of Integration.

**2.** Evaluate: $\int \frac{\sin(\tan^{-1} x)}{1+x^2}dx$

***Solution:***

Assume $\tan^{-1}x=t$.

On differentiating both sides, we get

$$\frac{1}{1+x^2}dx = dt$$

Hence

$$\int \frac{\sin(\tan^{-1} x)}{1+x^2}dx = \int \sin t dt = -\cos t + C$$

**3.** Evaluate: $\int \frac{dx}{1+\tan x}$

***Solution:***

$$\int \frac{dx}{1+\tan x} = \int \frac{dx}{1+\frac{\sin x}{\cos x}} = \int \frac{\cos x}{\sin x + \cos x}dx$$

$$= \frac{1}{2}\int \frac{\cos x + \sin x + \cos x - \sin x}{\sin x + \cos x}dx$$

$$= \frac{1}{2}\int \left(\frac{\cos x + \sin x}{\sin x + \cos x} + \frac{\cos x - \sin x}{\sin x + \cos x}\right)dx$$

$$= \frac{1}{2}\int \left(1 + \frac{\cos x - \sin x}{\sin x + \cos x}\right)dx$$

$$= \frac{1}{2}(x + \log(\sin x + \cos x) + C$$

**4.** Evaluate: $\int \sin 2x \cos 3x dx$

***Solution:***

Let us use the Identity

2sinA cosB=sin(A+B)+sin(A-B)

Hence

$$\int \sin 2x \cos 3x dx =$$

$$\frac{1}{2}\int [\sin(2x+3x) + \sin(2x-3x)]dx$$

$$= \frac{1}{2}\int [\sin 5x - \sin x]dx$$

$$= \frac{1}{2}\left(-\frac{\cos 5x}{5} + \cos x\right) + C$$

**5.** Evaluate: $\int \frac{dx}{\sqrt{5x^2 - 2x}}$

***Solution:***

We know that

$$5x^2-2x=5\left(x^2-\frac{2}{5}x\right)=5\left(x^2-\frac{2}{5}x+\frac{1}{25}-\frac{1}{25}\right)$$

$$=5\left(\left(x-\frac{1}{5}\right)^2-\left(\frac{1}{5}\right)^2\right)$$

Therefore,

$$\int\frac{dx}{\sqrt{5x^2-2x}}=\int\frac{dx}{\sqrt{5\left[\left(x-\frac{1}{5}\right)^2-\left(\frac{1}{5}\right)^2\right]}}=$$

$$\frac{1}{\sqrt{5}}\int\frac{dt}{\sqrt{t^2-a^2}} \text{ where } t=x-\frac{1}{5}, a=\frac{1}{5}$$

$$=\frac{1}{2a\sqrt{5}}\log\frac{t-a}{t+a}+C$$

Replace t and a by its values.

**6.** Find $\int\frac{dx}{(x+1)(x+2)}$ .

***Solution:***

Write $\frac{1}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

On solving the equation, A=1, B=-1

Hence $\frac{1}{(x+1)(x+2)}=\frac{1}{(x+1)}-\frac{1}{(x+2)}$

Now $\int\frac{dx}{(x+1)(x+2)}=\int\frac{dx}{(x+1)}-\int\frac{dx}{(x+2)}$

$$=\log_e(x+1)-\log_e(x+2)+C$$

**7.** Find $\int x\cos x dx$

***Solution:***

Use ILATE rule

$$\int x\cos xdx=x\int\cos xdx-\int\left(\frac{d}{dx}x\right)\left(\int\cos xdx\right)dx$$

=x(sinx)- $\int 1.(\sin x)dx$ =x(sinx)+cosx+C

**8.** Find $\int_0^2 x^2dx$ as the limit of a sum.

***Solution:***

We know that

$$\int_{x=a}^{x=b} f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\{f(a)+f(a+h)+f(a+2h)+...+f(a+$$

Where, $h=\frac{b-a}{n}$

a=0, b=2, therefore nh=2

$$\int_{x=0}^{x=2} x^2dx$$

$$=2\lim_{n\to\infty}\frac{1}{n}\{f(0)+f(0+h)+f(0+2h)+...+f(0+(n-1)h\}$$

$$=2\lim_{n\to\infty}\frac{1}{n}\{0+h^2+4h^2+...+(n-1)^2h^2\}$$

$$=2\lim_{n\to\infty}\frac{1}{n}\sum(n-1)^2h^2$$

g

$$=2\lim_{n\to\infty}\frac{1}{n}\sum(n^2-2n+1)h^2$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left(\sum n^2-2\sum n+\sum 1\right)h^2$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left(\frac{n(n+1)(2n+1)}{6}-2\frac{n(n+1)}{2}+n\right)h^2$$

$$=2\lim_{n\to\infty}\left(\frac{(n+1)(2n+1)}{6}-2\frac{(n+1)}{2}+1\right)h^2$$

$$=\frac{2}{6}\lim_{n\to\infty}\left((n+1)(2n+1)-6(n+1)+6\right)h^2$$

$$=\frac{2}{6}\lim_{n\to\infty}\left(2n^2-3n+1\right)h^2$$

$$=\frac{1}{3}\lim_{n\to\infty}\left(2n^2h^2-3nh^2+h^2\right)$$

$$=\frac{1}{3}(2\times4)=\frac{8}{3}$$

**9.** Evaluate: $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

***Solution:***

Let , $I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$ ....(i)

Applying property $I = \int_0^{a} f(x)dx = \int_0^{a} f(a-x)dx$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx \quad ....(ii)$$

By (i)+(ii)

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}dx \Rightarrow I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow I = -\frac{\pi}{2}\int_1^{-1} \frac{dt}{1+t^2}dx = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}dx$$

$$\Rightarrow I = \frac{\pi}{2}\left(\tan^{-1} t\right)_{-1}^{1} = \frac{\pi}{2}\left(\tan^{-1} 1 - \tan^{-1}(-1)\right)$$

$$\Rightarrow I = \frac{\pi}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi^2}{4}$$

**10.** Evaluate: $\int_0^{\frac{\pi}{2}} \log \sin x dx$

***Solution:***

Let $I = \int_0^{\frac{\pi}{2}} \log \sin x dx$ .....(i)

Using Property $I = \int_0^{a} f(x)dx = \int_0^{a} f(a-x)dx$

$$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right)dx = \int_0^{\frac{\pi}{2}} \log \cos x dx \quad ...(ii)$$

From (i)+(ii),

$$2I = \int_0^{\frac{\pi}{2}} \left(\log \sin x + \log \cos x\right)dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log(\sin x \cos x)dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log \frac{2\sin x \cos x}{2}dx$$

$$2I = \int_0^{\frac{\pi}{2}} (\log \sin 2x - \log_e 2)dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log \sin 2x dx - \int_0^{\frac{\pi}{2}} \log_e 2 dx$$

$$2I = \frac{1}{2}\int_0^{\pi} \log \sin t dt - (\log_e 2)\frac{\pi}{2}$$

$$2I = \int_0^{\pi/2} \log \sin t dt - (\log_e 2)\frac{\pi}{2}$$

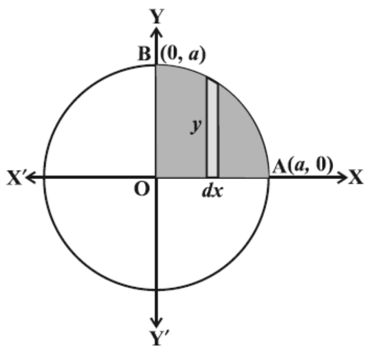
$$2I = \int_0^{\pi/2} \log \sin x dx - (\log_e 2)\frac{\pi}{2}$$

$$2I = I - (\log_e 2)\frac{\pi}{2}$$

$$\therefore I = -(\log_e 2)\frac{\pi}{2} = \frac{\pi}{2}\log_e \frac{1}{2}$$

**11.** Find the area enclosed by the circle $x^2+y^2=a^2$.

***Solution:***




The required area is $A = 4\int_0^a y\,dx$

$$A = 4\int_0^a \sqrt{a^2 - x^2}\,dx$$

Using standard Integral

$$\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$A = 4\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)_0^a = \pi a^2$$

**12.** If {x} represents the fractional part of x, then evaluate $\int_0^{100} \{\sqrt{x}\}dx$.

***Hint:*** Write {x}=x-[x]

And hence the integrand can be written as $\int_0^{100} \left(\sqrt{x} - [\sqrt{x}]\right)dx = \int_0^{100} \sqrt{x}dx - \int_0^{100} [\sqrt{x}]dx$

Rewrite the second integrand as $\int_0^1 [\sqrt{x}]dx + \int_1^4 [\sqrt{x}]dx + \int_4^9 [\sqrt{x}]dx + .... + \int_{81}^{100} [\sqrt{x}]dx$

Or $\int_0^1 0.dx + \int_1^4 1.dx + \int_4^9 2.dx + .... + \int_{81}^{100} 9.dx$

Solve and find the required value.

**13.** Evaluate $\lim_{n\to\infty} \frac{1}{n^2}\sum_{r=1}^{n} r.e^{\frac{r}{n}}$

***Hint:*** Rewrite the expression as

$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{r}{n}.e^{\frac{r}{n}}$. Replace (r/n) by x and (1/n) by dx and put limit x=0 as lower limit and x=1 as the upper limit and integrate.

**14.** Evaluate $\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + .... + \frac{1}{2n}\right)$

***Hint:*** Find out the rth term and convert it into (r/n) and (1/n) form to be replaced by x and dx respectively and integrate between limits x=0 to x=1.

**15.** If F(x)= $\frac{1}{x^2}\int_4^x \{14t^2 - 2F'(t)\}dt$ then find F(4).

***Hint:*** Obviously if x=4, the integrand becomes from x=4 to x=4 i.e. 0

**16.** Find the value of $\int_0^1 \frac{d}{dx}\left(\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right)dx$.

***Hint:*** Use Newton-Leibnitz formula for definite integral treating upper limit x and then putting x=1 to have the value π/2, as below:

$$= \sin^{-1}\left(\frac{2x}{1+x^2}\right) - 0 = \sin^{-1}\left(\frac{2.1}{1+1}\right)$$

Newton-Leibnitz formula for differentiation of a definite integral:

$$\frac{d}{dx}\int_{f(x)}^{g(x)} F(x)dx = F(g(x))d(g(x)) - F(f(x))d(f(x))$$

**17.** Find the mistake in the following evaluation of the integral

$$\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{\sec^2 x dx}{1+3\tan^2 x} = \frac{1}{\sqrt{3}}\int_0^{\pi} \frac{d(\sqrt{3}\tan x)}{1+(\sqrt{3}\tan x)^2}$$

$= \left[ \frac{1}{\sqrt{3}} \tan^{-1}\left(\sqrt{3}\tan x\right) \right]_0^{\pi} = 0$.

The integral of a function positive everywhere turns out to be zero.

***Hint:*** Note that the function tan⁻¹(√3 tanx) is discontinuous at (x = π/2) in the interval [0,π ].

As LHL ≠ RHL at x=π/2.

Hence the correct result can be evaluated as below:

$$\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{\sec^2 x dx}{1+3\tan^2 x} = 2\int_0^{\pi/2} \frac{\sec^2 x}{1+(\sqrt{3}\tan x)^2} dx$$

using the property of Definite Integral for even function.

$$= 2\int_0^{\pi/2} \frac{\sec^2 x}{1+(\sqrt{3}\tan x)^2} dx = \frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi/2}$$

$$= \pi/\sqrt{3}$$

**18.** Evaluate $\int_0^{\infty} \frac{e^{-ax}\sin bx}{x} dx$ and then deduce the value of $\int_0^{\infty} \frac{\sin bx}{x} dx$ .

***Hint:*** Assume a variable function of b

g(b)= $\int_0^{\infty} \frac{e^{-ax}\sin bx}{x} dx$

differentiate both sides w.r.t b

$$\frac{dg(b)}{db} = \int_0^{\infty} \frac{x.e^{-ax}\cos bx}{x} dx = \int_0^{\infty} e^{-ax}\cos bx.dx$$

On integration by Parts, the RHS becomes

$$\left[e^{-ax}.\frac{\sin bx}{b}\right]_0^{\infty} + \frac{a}{b}.\int_0^{\infty} e^{-ax}\sin bx.dx$$

On again integration by parts, the Intrgrand becomes

$$\frac{a}{b^2} - \frac{a^2}{b^2}\frac{dg(b)}{db}$$

i.e. $\frac{dg(b)}{db} = \frac{a}{a^2+b^2}$ .since b is a variable and a is a constant hence

g(b)= tan⁻¹ (b/a)+c

when b=0, g(b)=0 and hence c=0

therefore,

g(b)= tan⁻¹(b/a).

by putting a=0,

the other result can be evaluated as below

$$\int_0^{\infty} \frac{e^{-ax}\sin bx}{x} dx = \tan^{-1}\frac{b}{a}$$

$$\int_0^{\infty} \frac{\sin bx}{x} dx = \lim_{a\to 0} \tan^{-1}\frac{b}{a} = \frac{\pi}{2}.$$

**19.** Evaluate $\int_0^{\infty} \frac{\tan^{-1}(bx)}{x(1+x^2)} dx$ .

***Hint:*** Assume a variable function of b

g(b)= $\int_0^{\infty} \frac{\tan^{-1}(bx)}{x(1+x^2)} dx$ .

differentiate w.r.t b

$$\frac{dg(b)}{db} = \int_0^{\infty} \frac{1}{\left(1+b^2x^2\right)\left(1+x^2\right)} dx.$$

Use partial fraction method to evaluate the Integral as $\frac{dg(b)}{db} = \frac{\pi}{2(1+b)}$ .

Now integrate both sides to get g(b)= $\frac{\pi}{2}\log(1+b)+c$ and c=0 when b=0 so the function is $\frac{\pi}{2}\log(1+b)$

**20.** Show that $\left|\int_{10}^{19} \frac{\sin x}{1+x^8} dx\right| < \frac{1}{10^7}$

***Hint:*** This problem is related to Inequality

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

$$\left|\int_{10}^{19} \frac{\sin x}{1+x^8} dx\right| \le \int_{10}^{19} \left|\frac{\sin x}{1+x^8}\right| dx = \int_{10}^{19} \frac{|\sin x|}{1+x^8} dx \le \int_{10}^{19} \frac{1}{1+x^8} dx ... as .. |\sin x| \le 1$$

$\le \int_{10}^{19} \frac{1}{x^8} dx$ and this can be done now easily.

**21.** Evaluate: $\int_0^{\pi/2} \log_e\left(\alpha^2 \cos^2\theta + \beta^2 \sin^2\theta\right) d\theta$

***Hint:*** Assume the Integrand

I= $\int_0^{\pi/2} \log_e\left(\alpha^2 \cos^2\theta + \beta^2 \sin^2\theta\right) d\theta$

Differentiate w.r.t $\alpha$ under integral sign

$$\frac{dI}{d\alpha} = \int_0^{\pi/2} \frac{2\alpha \cos^2\theta}{\left(\alpha^2 \cos^2\theta + \beta^2 \sin^2\theta\right)} d\theta$$

$$= \frac{2\alpha}{\left(\alpha^2 - \beta^2\right)} \int_0^{\pi/2} \left\{1 - \frac{\beta^2}{\left(\alpha^2 - \beta^2\right)\cos^2\theta + \beta^2}\right\} d\theta$$

Integrate and Simplify using traditional method of multiplying the second part of the Integrand's Nr and Dr by $\sec^2\theta$.

$$\frac{dI}{d\alpha} = \frac{\pi}{\alpha + \beta}$$

Now Integrate,

I= $\pi \log_e(\alpha + \beta) + c$

Put $\alpha=\beta$ and evaluate I= $\int_0^{\pi/2} \log_e \alpha^2\left(\cos^2\theta + \sin^2\theta\right) d\theta = \pi \log \alpha$

$\Rightarrow \pi.\log_e \alpha = \pi.\log_e 2\alpha + c \Rightarrow c = \pi \log_e(1/2)$.

$\Rightarrow I = \pi \log_e(\alpha+\beta)/2$.

**22.** Let a, b, c be non-zero real numbers such that,

$$\int_0^1 \left(1+\cos^8 x\right)\left(ax^2+bx+c\right)dx$$

$$= \int_0^2 \left(1+\cos^8 x\right)\left(ax^2+bx+c\right)dx$$

then show that the quadratic equation $ax^2+bx+c=0$ has at least one root in (1,2).

***Hint:*** Let a function

f(x)= $\int_0^x \left(1+\cos^8 x\right)\left(ax^2+bx+c\right)dx$

It is continuous on[1,2] and differentiable on(1,2).

Also note f(1)=f(2)

Then there exists a k such that f'(k)=0 by Rolle's Theorem.

f'(x)= $\left(1+\cos^8 x\right)\left(ax^2+bx+c\right)$ where $(1+\cos^8 x) \neq 0$ hence $ax^2+bx+c=0$

And $k \in (1,2)$.

**23.** Form the differential equation of the family of all parabolas with focus at the origin and the x-axis as the axis.

***Hint:*** Assume the parabola $y^2=4a(x+a)$ and eliminate the arbitrary constant a by twice differentiating both sides.

**24.** Express in Beta function: $\int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx$ .

***Hint:*** Rewrite the given integrand as below:

$$\int_0^1 x^2.(1-x^5)^{-1/2} dx = \int_0^1 x^{-2}.(1-x^5)^{-1/2}.x^4 dx$$ .

$= \frac{1}{5}\int_0^1 y^{-2/5}.(1-y)^{-1/2} dy$ on putting $x^5=y$

= (1/5) B(3/5, ½)

**25.** Prove: $\int_0^a (a-x)^{m-1}.x^{n-1}dx = a^{m+n-1}B(m,n)$.

***Hint:*** Assume x=ay and get the required answer.

**26.** Let f be an odd function defined and integrable everywhere and also periodic with period 2 as below: g(x)= $\int_0^x f(t)dt$

then

(a) Find the value of f(4).

**Hint:** f(x) is odd function and defined everywhere, hence f(0)=0 and so f(0)=f(2)=f(4)=0 because it is periodic also with period 2.

(b) Find the value of g(4).

***Hint:*** g(x)= $\int_0^x f(t)dt \Rightarrow g(4) = \int_0^4 f(t)dt$

assume t=u+2.

So dt=du and $g(4) = \int_{-2}^{2} f(u+2)du = \int_{-2}^{2} f(u)du$

Because f(u+2)=f(u) as it is periodic with period 2 and is also an odd function so by the property of definite integral, its value is zero.

(c) Find the value of g(x+2).

***Hint:*** g(x+2)

$$= \int_0^{x+2} f(t)dt = \int_0^x f(t)dt + \int_x^{x+2} f(t)dt$$

$$= g(x) + \int_0^2 f(t)dt = g(x) + g(2)$$

(d) If f'(-2)= -2 then find the value of f'(2).

***Hint:*** f is an odd function, it will be symmetric about origin specially in the domain (-2,0) and (0,2). Hence the slope at x=2 and x=-2 will be same.

i.e. f'(-2)=f'(2)=-2 as it is given.

(e) If $g(x^2) = x^2(1+x)$ then find the roots of the equation $x^2 - f(x^2) = 0$.

***Hint:*** Differentiate w.r.t x

$2x\, g'(x^2) = 2x(1+x) + x^2$

Or, $g'(x^2) = 1 + (3/2)x$

Also g'(x)=f(x) by using Newton-Leibnitz formula in the given relation.

So, the given equation becomes:

$x^2 - 1 - (3/2)x = 0$ and hence the roots may be evaluated.

Dr S.B. Dhar, is **Editor of this Quartrerly e-Bulletin**. He is an eminent mentor, analyst and connoisseur of Mathematics from IIT for preparing aspirants of Competitive Examinations for Services & Admissions to different streams of study at Undergraduate and Graduate levels using formal methods of teaching shared with technological aids to keep learning at par with escalating standards of scholars and learners. He has authored numerous books – Handbook of Mathematics for IIT JEE, A Textbook on Engineering Mathematics, Reasoning Ability, Lateral Wisdom, Progress in Mathematics (series for Beginner to Class VIII), Target PSA (series for class VI to class XII) and many more.

**e-Mail ID:** maths.iitk@gmail.com