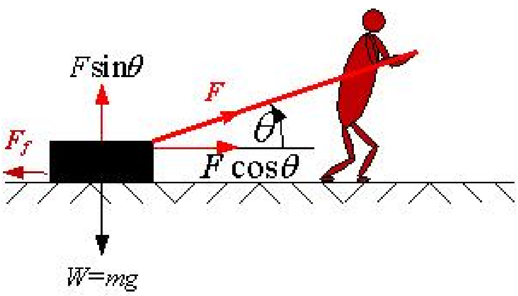
MECHANICS-III:-Dynamics of Rigid Bodies

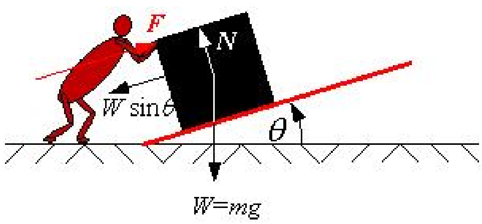
Advent of mechanics has been from predator age of human civilization. Then, people used mechanics to manage their effort to suit their capacity and body posture, out of their experience. Accordingly, they evolved their tools of hunting, pulling of their prey to their destination, cutting, chopping etc. to meet their daily food needs. Galileo and Newton formalized these experiences into a set of laws and equations for a better understanding and appreciation of happening in the surrounding, and to improve effectiveness at work. There are three applications of applications of mechanics which everyone must have used since immemorial times are wedge, liver and pulley. It is time to see how basic laws of kinematics and dynamics apply in these rudimentary examples and set up a platform for wider understanding of physics and its analysis for enhancing effectiveness at work. Extrapolation of particle dynamics, elaborated in here until, has emerged as dynamics of rigid bodies. This part shall deal with the principle of moment and its use in determination of Centre of Mass, Centre of Gravity, Friction, Rotation of Rigid Bodies, Principle of Gravitation, Orbital Motions, and physical properties of solids. Intermittently application of the concepts viz. principle of lever, pulleys, together with D' Alembert's Principle to explain concurrent translational and rotation motion under application of force and Kepler's Laws to explain shall also be elaborated.

The study of kinematics reducing an object to a particle, of negligible size or a point, is unrealistic since each object that we encounter has a measurable size. There is a mass distribution across the object which influences its behaviour from slipping or skidding to rolling. In case of objects considered like a particle, the kind of motion is translational, i.e. trajectory of each particle of the object is parallel to the other, while in real object the motion of particles is translational during slipping and rotational in most of the cases viz. motion of a ball in, motion of a top extending to celestial bodies. These concepts are being elaborated here.

Pulling a heavy object on a horizontal rough surface with an inclined string is a basic example of managing a load within capacity of human body attainable with a working posture. It may be seen that force is applied at an angle θ such that $0 \le \theta \le \frac{\pi}{2}$, when $1 \ge \cos\theta \ge 0$. Thus force F applied to pull the object is greater than the frictional force and its magnitude depends upon the angle θ. Successful pulling essentially requires that- (a) vertical component of force , else the object will get lifted up, and (b) horizontal component of force $F\cos\theta > f$; here f is the frictional force.

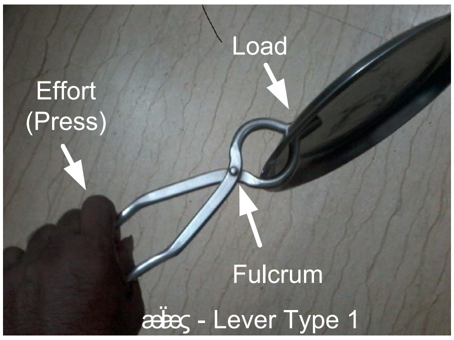


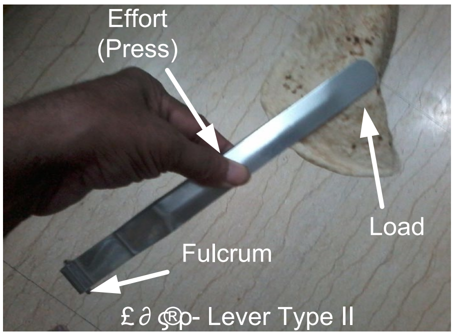
In another case below with a similar posture, body capacity is utilized to push-up a weight along a smooth inclined plane which reduces requirement of force $F = W\sin\theta < W$ since, for $0 < \theta < \frac{\pi}{2} \rightarrow 0 < \sin\theta < 1$. But, in this case person pushing the weight negotiates his body capacity with the weight and angle of inclined plane such that $F << W$ i.e. ***with a lesser force heavier weight is pushed*** In either case posture is important to maximize body capacity so as to perform desired work.

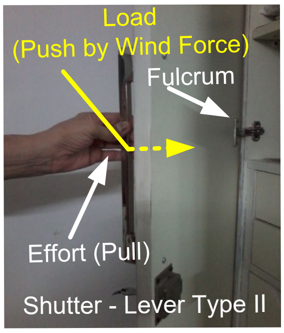


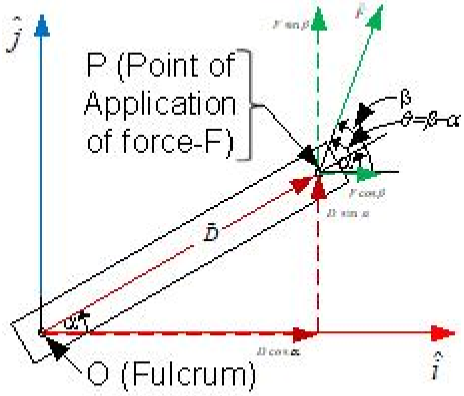
It is may be seen that manipulation of posture is possible either by experience or understanding of mechanics behind it. Learning with experience is a slow process and involves high cost in terms of time and opportunities lost, during the process. Whereas, understanding of basics is a systematic process but it accelerates forward learning process as much its application to enhance experience in an intuitive manner; an ability to perform, the ultimate objective of life. Gradually, such an understanding makes exploring physics a matter of visualization of surrounding in a intuitive manner without carrying burden of learning remembering physics.

Uses of kitchen Garlic Press (in Hindi called Sansi), kitchen Tong (in Hindi called Chamita) and pulling a shutter against wind force are intuitive, but the underlying concepts of physics i.e. ***Torque of force*** $\vec{\tau}$ *involves identifying fulcrum.* ***Fulcrum*** *is point of rotation of a body about which a force is applied, displacement of point of application of force from the fulcrum ($\vec{r}$ a vector), the force ($\vec{F}$ another vector) and* ***Torque*** *is* ***cross product of displacement and force vectors i.e.*** $\vec{\tau} = \vec{r} \times \vec{F}$ needs elaboration. The torque which is capable causing clockwise rotation about the fulcrum is called ***Clockwise Moment*** **(CWM)** and in case of tendency to rotate in anti-clockwise direction it is called ***Anti-clockwise Moment (AWM)***. Analysis of the simple kitchen tools is enough to understand physics behind the concept of Levers, with introduction of three new terms.



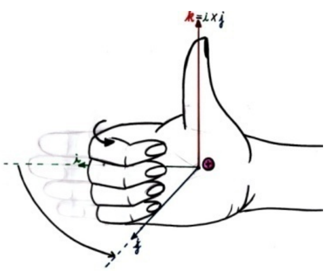






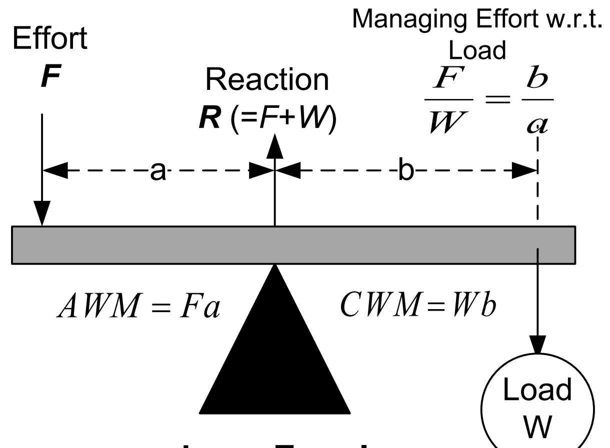
Principle of moment, in a state of rotational equilibrium between CWM and AWM is a subject matter of statics and is analysed by equating the Two i.e. $\vec{\tau}_C = \vec{\tau}_A$. But, in a state of in-equilibrium net torque is $\vec{\tau} = \vec{\tau}_C - \vec{\tau}_A \neq 0$ and it causes clockwise rotational motion if $\vec{\tau}_C > \vec{\tau}_A$ and anticlockwise rotation if $\vec{\tau}_C < \vec{\tau}_A$ which makes it a subject matter of rotational dynamics. This is explained with a generic example where an arm of length ***r*** , at an angle α with reference $\hat{i}$, is hinged at fulcrum **O,** and **force** $\vec{F}$ is applied at point P at an angle $\boldsymbol{\beta}$ with reference $\hat{i}$. The arm (line joining points **O** and **P**) is $\hat{r}$.

Vectorially, the moments CWM $= \vec{\tau}_C = (F\cos\beta)(r\sin\alpha)(-\hat{k})$, and

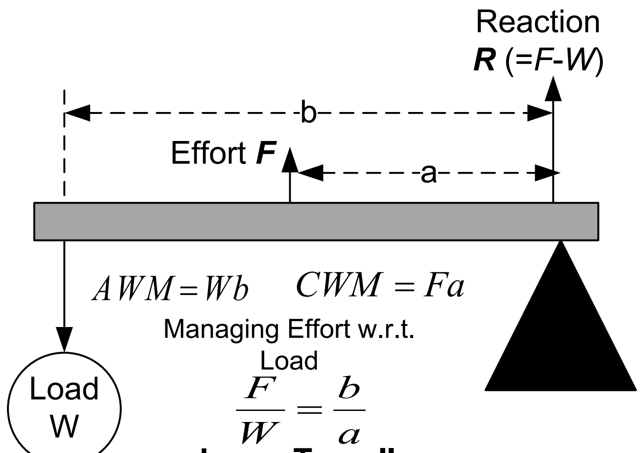


AWM $= \vec{\tau}_A = (F\sin\beta)(r\cos\alpha)\hat{k}$. In this case, for equilibrium to exist $\hat{\tau}_A + \hat{\tau}_C = 0$. It leads to $Fr\sin\beta\cos\alpha\, k - \alpha Fr\widehat{\cos\beta}\sin\alpha\,\hat{k} = 0$; and in turn to $\tan\beta = \tan\alpha \to \alpha = \beta$. Thus, $\theta = \alpha - \beta = 0$ or $\alpha = \pi\beta \to \alpha - \beta = n\pi$, here $n \in I$. Thus, in a state of rotational equilibrium both force vector ($\bar{F}$) and displacement vector ($\vec{r}$) are collinear, irrespective of their direction be it unidirectional or counter-directional. Simplest way of determination of direction of rotation, the effect of torque, is **Right Hand Thumb Rule (RHTR)**. It stipulates that coplanar vector $\hat{r}$ and $\hat{F}$ say on $\left(\hat{i} - \hat{j}\right)$ plane, then a vector appears to be coming out of plane in perpendicular direction is assigned direction $\hat{k}$. Accordingly, a vector which appears to be entering plane in perpendicular direction is assigned a direction $\left(-\hat{k}\right)$. Taking this convention for directions, place palm perpendicular to the plane such that fingers are pointing along vector $\hat{r}$ first multiplicand of $\hat{\tau}$ and then bend the fingers in the direction of vector $\hat{F}$. The direction of thumb is the direction of $\hat{\tau}$ it could be $\hat{k}$ or $\left(-\hat{k}\right)$. It is important to notice that fingers can be folded only towards the palm and not in reverse direction. Thus for torque in $\hat{k}$ directions, the direction of folding of fingers is clockwise, whereas for torque in direction $\left(-\hat{k}\right)$, shall have finger folding anticlockwise. *The* *mathematical formulation of torque states that* $\hat{\tau} = \hat{r} \times \hat{F} = rF \sin\theta\, \hat{k}$. *The angle* θ *takes care of direction of vector* $\hat{\tau}$ *i.e.* $\hat{k}$ *or.* $\left(-\hat{k}\right)$ *; if* $0 < \theta < \pi$ *then* $\sin\theta > 0$ *and of vector* $\hat{\tau}$ *is in anticlockwise direction* $\left(+\hat{k}\right)$*, and as corollary if* $\pi < \theta < 2\pi$ *then* $\sin\theta < 0$ *and of vector* $\hat{\tau}$ *is in clockwise direction* $\left(-\hat{k}\right)$. Greater details of this, is a part of vector analysis in general and cross-product of vectors in particular and are elaborated in chapter **Foundation Mathematics, Common Section**.

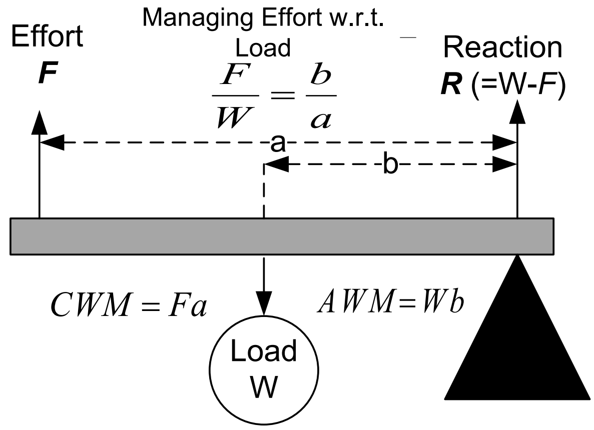
Considering the aforesaid principle of torque vis-a-vis moment, the three types of lever are conceptualized here. In this Fulcrum is the fixed point with respect to load and effort. Accordingly, distances of point of application of effort and load are calculated from it. As regards angle between Effort or Load and line joining the point of its application and fulcrum are taken to be 90^0, i.e. mutually perpendicular. Since, $\sin 90^0 = 1$, hence from the perspective of torque CWM and AWM are reduced to straight product of Load/Effort and their distances from fulcrum. These types of lever are generic in nature and it is just a matter of observation and identifying Effort, Load and Fulcrum in the surrounding and establishes its relevance to one of the three types of lever. Another, important inference is management of effort w.r.t. load by manipulating ratio of arm lengths a and b so as to accomplish required task ogf managing a load within one's capacity to apply effort.



Lever Type-I

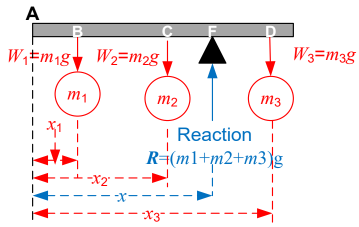
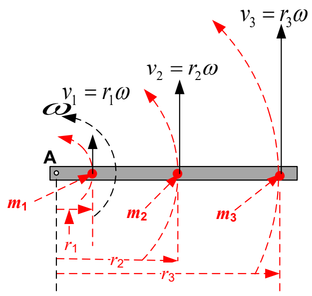


Lever Type-II



Lever Type-IIi

This is the point where understanding of Centre of Mass (COM) and Moment of Inertia (MOI) can be evolved using principle of moment. These two concepts, COM and MOI, are essential to make a headway into the journey of concepts dynamics of rigid bodies. Every rigid body is a conglomeration of particles such that relative distance between particles constituting the body remains un-affected on application of external force. This makes the shape of rigid body to remains fixed. In reality, on application of an external force there is always some deformation of shape, but in respect of rigid bodies the deformation is small enough to be considered as negligible during analysis of dynamics of rigid bodies. Accordingly, for simple illustrations a system of discrete masses m1, m2 and m3, placed laterally along a light bar at distance x1, x2 and x3, respectively, on a line, from end A is considered. A wedge at a point F is placed such that bar remains balanced at it, and is useful for arriving at COM. Whereas, pivot A is considered to be centre of rotation of the bar with three masses fixed to it for arriving at MOI which is based on concept of angular momentum and introduces effect of torque in angular rotation.

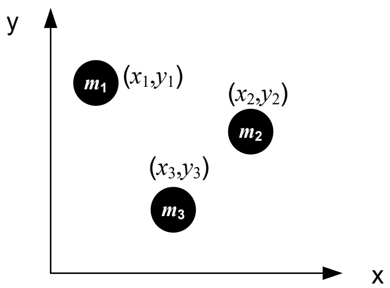
Centre of Mass		Moment of Inertia	
	Let bar with three masses is in equilibrium when wedge is placed at point F, at a distance x from end A.As per NTLM in IFOR, Reaction $R = (m_1 + m_2 + m_3)g$.		Here, system of particles is moving with a constant angular velocity ω, about the pivot A. Thus linear momentum of three masses shall be $P = m1v1{+}m2v2{+}m3v3{=} m1r1\omega{+}m2r2\omega{+}m3r3\omega$.
Further, bar to stay balanced, i.e. in a state of equilibrium, CWM =AWM. Accordingly, $((m_1g)x_1 + (m_2g)x_2 + (m_3g)x_3) = ((m_1 + m_2 + m_3)g)\overline{x}$, or $\overline{x} = \dfrac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$		But, angular momentum of the system about pivot A is: $\vec{L} = \sum \vec{r}_i \times \vec{p}_i = \sum \vec{r}_i \times (m\vec{v}_i) = \sum m_i((r_i\hat{r}_i) \times r i\omega v i$. It leads to $\vec{L} = \sum m_i r_i{}^2\omega(\hat{r}_i \times \hat{v}_i) = \sum m_i r_i{}^2\omega\hat{\omega} = \sum m_i r_i{}^2\vec{\omega}$. It is written as $\vec{L} = I\vec{\omega}$ where, $I = \sum m_i r_i^2 = Mk^2$. Here, $M = \sum m_i$ total of distributed mass and k is called ***radius of gyration;*** *it is like effective radius of distributed masses M, during rotation.*	

It is interesting to note that COM of sun-earth system, taken in isolation in solar system has its COM within the sun and can be verified using known data of sun and earth. Mass of sun is $M_e \approx 2\times10^{30}$ kg and its radius is $r_s \approx 6.95\times10^{8}$ kg and while mass of earth is $M_e \approx 6\times10^{24}$ kg . Mean distance of earth from sun is $d \approx 1.5\times10^{11}$ m. Therefore, COM of the system w.r.t. COM of sun is

$$\overline{x} = \frac{(2\times10^{30})\times 0 + (6\times10^{24})\times(1.5\times10^{11})}{(2\times10^{30}) + (6\times10^{24})} \approx \frac{9\times10^{35}}{2\times10^{30}}\Bigg|_{(2\times10^{30})>>(6\times10^{24})} \approx 4.5\times10^{5}$$

m and is less than radius of the sun.

Likewise, for ***a system of discrete particles distributed on a plane, the concept of COM and MOI*** goes as under:

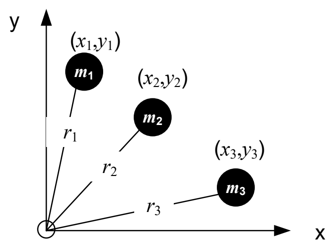


Extending the concept of COM of discrete masses distributed in shall have to be defined with coordinates :

$\overline{x} = \frac{\sum m_i x_i}{\sum m_i}$, and

$$\overline{y} = \frac{\sum m_i y_i}{\sum m_i}$$

This is known as ***Perpendicular Axis Theorem of COM***

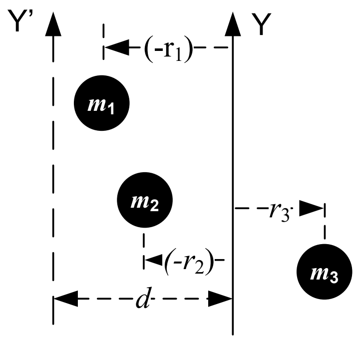


Likewise,

$$I = \sum m_i r_i^2 = \sum m_i(x_i^2 + y_i^2) = \sum m_i x_i^2 + \sum m_i y_i^2 = I_x + I_y$$

Here, I_x is MOI of the distributed masses about Y-axis and I_y is MOI of the distributed masses about X-axis This derivation takes a form $I_z = I_x + I_y$, where I_z is MOI of the mass distributed on a plane about an axis perpendicular to the plane through intersection of X-axis and Y-axis. This is known as ***Perpendicular Axis Theorem of MOI***

And, if the mass distribution is in the space the concept COM and MOI is an extrapolation of the above in three dimension, such that: $\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$, $\bar{y} = \frac{\sum m_i y_i}{\sum m_i}$ and $\bar{z} = \frac{\sum m_i z_i}{\sum m_i}$ while, $I_x = \sum m_i \left(y_i^2 + z_i^2\right)$, $I_y = \sum m_i \left(x_i^2 + z_i^2\right)$, and $I_z = \sum m_i \left(x_i^2 + y_i^2\right)$, but MOI about the origin is $I = \sum m_i \left(x_i^2 + y_i^2 + z_i^2\right)$.



Taking another case of a system of distributed masses rotated about Y'-axis which is not passing through COM, but parallel to the Y-axis passing through COM. This leads to ***Parallel Axis theorem of MOI,*** which is about determining MOI along a *new axis **(Y')***.

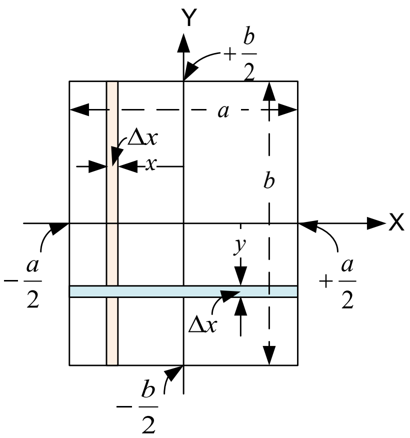
$$I_y' = \sum m_i(d + r_i)^2 = \sum m_i\left(r_i^2 + d^2 + 2r_i d\right) = I + \sum m_i d^2 + 2\sum m_i r_i$$

$= I + \sum m_i d^2$; Since Y is passing through COM and hence, $\sum m_i r_i$,=0 .

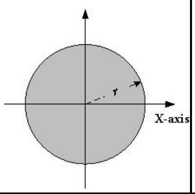
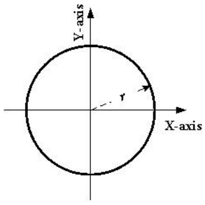
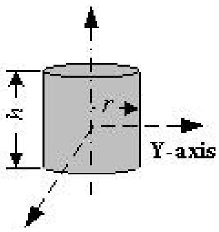
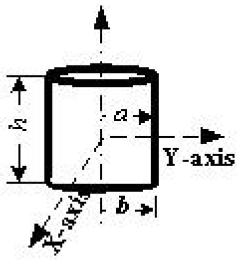
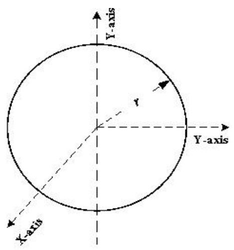
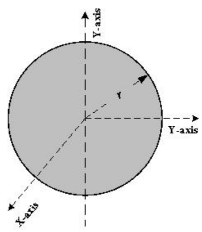
Considering the mathematical formulation of COM, it is a point at which summated moment of mass distribution about a fixed axis is equal to moment of total mass. Likewise, looking at the ***mathematical formulation of MOI it is double moment, i.e. moment of moment, of mass distribution about a fixed axis***.

But, mostly one encounters rigid bodies made of same material in which distribution mass is continuous and uniform. This illustration is initiated through COM and MOI for a rectangular sheet having uniform mass density ρ.

It may be seen that determination of COM and MOI is a case of application of integeral calculus with concept of physics.

Rectangular Sheet	**Centre of Mass**	**Moment of Inertia**
 	Let, X and Y axes are parallel to the length and width of the rectangular sheet and passing through mid of the length and width of the sheet. Let x be the distance of a thin vertical strip of width Δx parallel to Y-axis, and likewise y and Δy are distance and width of the horizontal strip parallel to X axis. Then, $\bar{x} = \int_{-\frac{a}{2}}^{\frac{a}{2}} (\rho b \Delta x)x = \rho b \int_{-\frac{a}{2}}^{\frac{a}{2}} x\Delta x$ $= \rho b \left[\frac{x^2}{2}\right]_{-\frac{a}{2}}^{\frac{a}{2}} = 0$ $\bar{y} = \int_{-\frac{b}{2}}^{\frac{b}{2}} (\rho a \Delta y)y = \rho a \int_{-\frac{b}{2}}^{\frac{b}{2}} y\Delta y$ $= \rho a \left[\frac{y^2}{2}\right]_{-\frac{b}{2}}^{\frac{b}{2}} = 0$ ***Since, $\bar{x} = 0$, and $\bar{y} = 0$ COM of the uniform rectangular sheet is at origin.***	Taking same arrangement of axes, as in case of determination of COM, MOI of the sheet about Y-axis is - $I_y = \int_{-\frac{a}{2}}^{\frac{a}{2}} (\rho b \Delta x)x^2 = \rho b \int_{-\frac{a}{2}}^{\frac{a}{2}} x^2 \Delta x$ $= \rho b \left[\frac{x^3}{3}\right]_{-\frac{a}{2}}^{\frac{a}{2}} = \frac{\rho b}{3}\left[\frac{a^3}{8} - \left(-\frac{a^3}{8}\right)\right]$ $= \frac{\rho a^3 b}{12}$ $I_x = \int_{-\frac{b}{2}}^{\frac{b}{2}} (\rho a y)y^2 = \rho b \int_{-\frac{b}{2}}^{\frac{b}{2}} y^2 \Delta y$ $= \rho a \left[\frac{y^3}{3}\right]_{-\frac{b}{2}}^{\frac{b}{2}} = \frac{\rho a}{3}\left[\frac{b^3}{8} - \left(-\frac{b^3}{8}\right)\right]$ $= \frac{\rho a b^3}{12}$ This, $I_x = \frac{Mb^2}{12}$ and $I_y = \frac{Ma^2}{12}$. Here, $M = \rho ab$

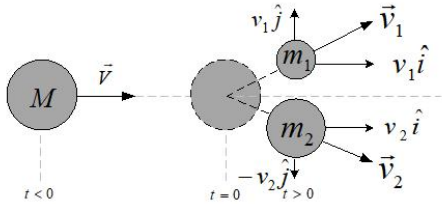
Likewise, some other typical symmetrical shapes of rigid bodies that are generally encountered are summarized below. Derivation of the formulations listed below is left for readers to work upon. If necessary assistance of reference books may be obtained, or referred to us, for necessary assistance.

Shape	Centre of Mass	Moment of Inertia
Circular plate on XY Plane 	COM at the Centre of the Circle	$I_{xx} = \frac{Mr^2}{4};\ I_{yy} = \frac{Mr^2}{4};\ I_{zz} = \frac{Mr^2}{2}$
Thin circular rim 	COM at the Centre of the Circle	$I_{xx} = \frac{Mr^2}{2};\ I_{yy} = \frac{Mr^2}{2};\ I_{zz} = Mr^2$
Sold Cylinder along Z axis 	COM on the axis of Cylinder	$I_{xx} = \frac{Mr^2}{4};\ I_{yy} = \frac{Mr^2}{4};\ I_{zz} = \frac{Mr^2}{2}$
Hollow Cylinder along Z axis (Inner and Outer radii ***a*** and ***b***) 	COM on the axis of Cylinder	$I_{zz} = M\left(\frac{b^2}{2} - \frac{a^2}{2}\right)$
Thin hollow Sphere 	COM at Centre of Hollow Sphere	$I_{zz} = \frac{2}{3}MR^2$
Solid Sphere 	COM at Centre of Sphere	$I_{zz} = \frac{2}{5}MR^2$

The logic of COM and MOI can be extrapolated using integral calculus of the geometrical mass distribution for unsymmetrical objects. The MOI for different shapes has been ***normalized into Radius of Gyration (*ROG *'k')*** such that $I = Mk^2$. It may be observed from the MOI of a few cross-sections brought out above, as mass is distributed along perimeter, created a void within. Thus increase in MOI is achievable, with hollow cross-sections, for a lesser mass. It offers two advantages- a) conserving the material and b) reducing static loading caused by weight $(= Mg)$ of the structure; these are application aspects and shall be encountered during studies in engineering. Therefore, *coming up with different cross-sections conveys and their related MOI and ROG is just not a quest of mathematician or a physicist but it is to the load bearing capacity of the cross-section.* This the reason why use of different cross-sections viz. cylinder pipe, tee , angle, I–section are observed in various structure that have been created around.

It is important to note that, MOI of an object plays a role in rotational motion similar to that mass in translational motion. It will become more explicit when combined effect of rotational and translational motion is elaborated, a little later.

Reduced Mass: This is a very important concept and it is used to analyze splitting of an object splits in absence of an external force, and as demonstrated in the figure**.** This condition must satisfy that $\vec{F}_{ext} = 0 = \frac{d}{dt}\vec{P} \rightarrow \Delta\vec{P} = 0 \rightarrow M\vec{V} - \left(m_1\vec{v}_1 + m_2\vec{v}_2\right) = 0$. It leads to an equation $M\vec{V} = m_1\vec{v}_1 + m_2\vec{v}_2 \rightarrow \frac{d}{dt}\left(M\,\vec{x}\right) = \frac{d}{dt}\left(m_1\vec{x}_1\right) + \frac{d}{dt}\left(m_2\vec{x}_2\right)$. This is in accordance with basic definition of conservation of momentum, seen from the perspective of centre of mass. Incidentally both are abbreviated as COM. Accordingly, velocity of COM $\vec{V}_{cm} = \left.\frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}\right|_{M = m_1 + m_2}$.



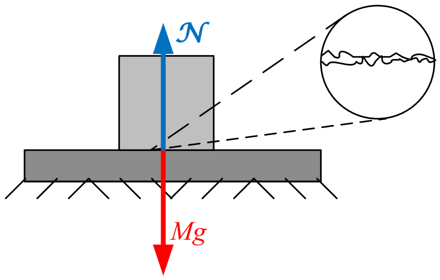
Further, velocity of split mass m_1 relative to COM of pre-split mass is $\vec{v}_{1,cm} = \vec{v}_1 - \vec{V}_{cm}$. It leads to $\vec{v}_{1,cm} = \vec{v}_1 - \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2}\left(\vec{v}_1 - \vec{v}_2\right) = \frac{m_2}{m_1 + m_2}\vec{v}_{1,2}$. Likewise, $\vec{v}_{2,cm} = \vec{v}_2 - \vec{V}_{cm} = \frac{m_2}{m_1 + m_2}\vec{v}_{2,1}$, thus $\vec{v}_{1,cm} = -\vec{v}_{2,cm}$. Thus kinetic energy of system post-splitting is $KE = KE_{split\_masses} + KE_{cm}$, here KE_{split_masses} pertains to total kinetic energy of constituent split masses w.r.t. to their COM, and KE_{cm} pertains kinetic energy of the COM of the split masses. Thus, $KE_{split\_masses} = \frac{1}{2}m_1 v_{1,cm}{}^2 + \frac{1}{2}m_2 v_{2,cm}{}^2 = \frac{1}{2}m_1\left(\frac{m_2}{m_1 + m_2}v_{1,2}\right)^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_1 + m_2}v_{1,2}\right)^2 = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v_{1,2}{}^2 = \frac{1}{2}\mu v_{1,2}{}^2$. Here, $v_{1,2} = \left|\vec{v}_{1,2}\right| = \left|\vec{v}_{2,1}\right| = v_1^2 + v_2^2 - 2\vec{v}_1 \cdot \vec{v}_2$, and μ is called reduced mass which is mathematically $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$, such that $\mu < m_1$ and also $\mu < m_2$. While kinetic energy of COM is $KE_{cm} = \frac{1}{2}\left(m_1 + m_2\right)V_{cm}{}^2$. Thus, $KE_{cm} = \frac{1}{2}\left(m_1 + m_2\right)\left(\frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}\right)^2 = \frac{m_1^2 v_1^2 + m_2^2 v_2^2 + 2\left(m_1 m_2\right)v_1 \cdot v_2}{2\left(m_1 + m_2\right)}$. Accordingly, total kinetic energy is $KE = \left[\frac{m_1 m_2}{2\left(m_1 + m_2\right)}\left(v_1^2 + v_2^2 - 2\vec{v}_1 \cdot \vec{v}_2\right)\right] + \left[\frac{m_1^2 v_1^2 + m_2^2 v_2^2 + 2\left(m_1 m_2\right)v_1 \cdot v_2}{2\left(m_1 + m_2\right)}\right] \rightarrow \frac{1}{2}\left[\frac{m_1\left(m_1 + m_2\right)v_1^2 + m_2\left(m_1 + m_2\right)v_2^2}{\left(m_1 + m_2\right)}\right]$. It leads to $KE = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$. Difference of kinetic energy pre and post splitting of mass $\Delta KE = \frac{1}{2}\left(MV^2 - \left(m_1 v_1^2 + m_2 v_2^2\right)\right)$ is utilized in splitting of mass M into m_1 and m_2.

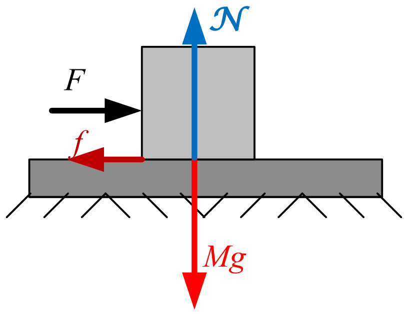
Comparison of Translational and Rotational Mechanics: Translational kinetics and dynamics, like rotational mechanics have identical set of equations except interchange of variables of *Mass* with *Moment of Inertia, which is also called Rotational Inertia (I),* as under –

Displacement:$\vec{\boldsymbol{x}} \leftrightarrow \vec{\boldsymbol{\theta}}$;Velocity:$\vec{\boldsymbol{v}} \leftrightarrow \vec{\boldsymbol{\omega}} = \frac{d\vec{\boldsymbol{\theta}}}{dt}$; Momentum: $\vec{\boldsymbol{P}} = \boldsymbol{m}\vec{\boldsymbol{v}} \leftrightarrow \vec{\boldsymbol{L}} = \boldsymbol{I}\vec{\boldsymbol{\omega}}$; Acceleration: $\vec{\boldsymbol{a}} \leftrightarrow \vec{\boldsymbol{\alpha}} = \frac{d\vec{\boldsymbol{\omega}}}{dt}$; Impulse: $\vec{\boldsymbol{I}} = \int_{t_1}^{t_2}\vec{F}\,dt \leftrightarrow \vec{\boldsymbol{J}} = \int_{t_1}^{t_2}\vec{T}dt$; **Effort:** $\vec{\boldsymbol{F}} = \boldsymbol{m}\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{P}}}{dt} \leftrightarrow \vec{T} = \mathrm{I}\vec{\boldsymbol{\alpha}} = \frac{d\vec{\boldsymbol{\omega}}}{dt}$; **Energy:** $\mathbf{PE} = \boldsymbol{mgh}$ **and** $\mathbf{KE} = \frac{1}{2}\boldsymbol{m}\boldsymbol{v}^2 \leftrightarrow \frac{1}{2}\boldsymbol{I}\boldsymbol{\omega}^2$**.** Accordingly, ***First, Second and Third Equations for Rotational Motion*** (**ERM**) are –

$\omega = \omega_0 + \alpha t$: **FERM**; $\quad \theta = \omega_0 t + \frac{1}{2}\alpha t^2$: **SERM**; $\quad \omega^2 = \omega_0^2 + 2\alpha\theta$: **TERM**.

Cause of conversion of translational motion into rotational motion is caused by friction. Therefore, before dwelling into combined translational and rotational motion, concept of ***friction*** is very much relevant and is being elaborated.

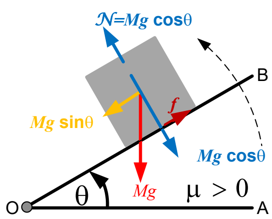




Friction: Resistance to motion is quite frequently talked about and this is called ***friction***. This a resistance is offered by irregularities of surfaces of two object coming in contact during displacement. Howsoever, a polished surface may be, when it is viewed with a magnifier, surface irregularities can be observed, higher the magnifying power more prominently the irregularities become visible. These surface irregularities could be in the form of surface dips or spikes, as shown in the inset, and these my get interlocked while in contact with each other. Unless, an external force is applied to either crush away such interlocks or ride over the spikes relative displacement of the objects cannot take place. In a *frictionless surface, a hypothetical case*, slightest horizontal force on mass ***M*** would be sufficient to cause its acceleration as per NSLM, component ***g*** along the surface, and thus the force $F_{net} = F - \left(Mg\cos 90^0\right)\Big|_{f\to 0} = F - 0$. Thus, on application of an external force the object would start sliding on the surface. But, friction is a reality and a minimum external force (F) required to cause relative displacement and this called Frictional force (f). Nature of this static friction force is non-conservative (f), i.e. it is non-recoverable, unlike the conservative force of spring and gravitational pull. Here, a new term ***Coefficient of Friction*** (***μ***) is introduced. It has two forms ***Static Coefficient of Friction***(μ_s) *and* ***Kinetic Coefficient of Friction*** (μ_k).

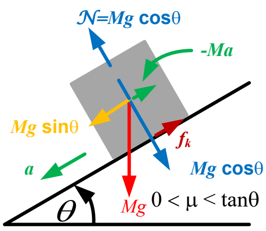
As per laws of conservation of energy (LCE), energy spent in overcoming frictional force gets transformed into heat; this process continues until relative displacement continues. This heat being localized is enough to softening result is a kind of momentary lubrications. As soon as motion stops, softened surface dissipates heat into surrounding matter and surface irregularities reappear. It is to be noted that always $\mu_k < \mu$; this is attributed ***to*** lubrication at the contact surface caused by softening/melting of spikes due ***heat generated by non-conservative frictional force.*** Implications of ***Coefficient of Friction*** on inclined plane are as under.

a. Static Friction On an Inclined Plane: Two planks OA and PB are hinged at O are placed horizontally. A block of mass M is placed on the plank OB. Gradually, the plank OB is lifted to form an angle $\boldsymbol{\theta}$ with OA**.** It is seen that, with ***g*** acting vertically downward components of the weight along the plane is $mg\sin\theta$ and tends to slide the block downward as per NSLM, but it does not start sliding immediately. This continues until the angle $\boldsymbol{\theta}$ reaches a particular value for a specific combination of plank and the block. It the friction that is preventing the relative displacement of the block. The friction is created by the component of weight perpendicular $mg\cos\theta$ to the surface of the plank, causing a reaction N as per NTLM. This causes interlocking of surface irregularities of the plank and the block. In the process, as $\boldsymbol{\theta}$ increases, $mg\sin\theta$ also increases, while force responsible for friction $mg\cos\theta$ decreases until $mg\sin\theta = mg\cos\theta$. At this angle slightest perturbation is enough start sliding down of the block on plane OB. It leads to a critical angle $\boldsymbol{\theta_c}$ which goes into defining ***coefficient of friction*** ($\boldsymbol{\mu}$) such that $\mu = \tan\theta_c = \frac{Force\ of\ Friction}{Normal\ \ Reacton}$. There are other method of determining μ, while this is considered to be the simplest method.

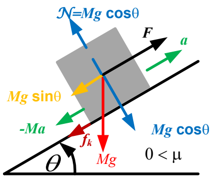


It needs to be reasoned out what happen to friction when $\theta < \tan^{-1}\mu$? does friction cease to exist? if that be the so as per NSLM, the acceleration of the block should start as soon as $\theta > 0$, which does not happen, the reason is that *friction neither an external force nor a self-generating force, it a reaction of the force tending to cause relative displacement, and thus it would never exceed the force causing displacement.*

b. Kinetic Friction Under Gravity: In this setup, angle $\boldsymbol{\theta}$ is so chosen that $\tan\theta > \mu$, and this would lead to $mg\sin\theta > f$, and causes a relative acceleration ($\boldsymbol{a}$) of the block as per NSLM. Accordingly, a pseudo force $-Ma$ appears on the body such that $a = \frac{\boldsymbol{mg\sin\theta - f_k}}{\boldsymbol{M}}$. It is to be noted in this equation of acceleration, frictional force $\boldsymbol{f_k}$ has been used instead of $\boldsymbol{f_s}$. Since body is accelerating, i.e. in a state of motion and hence work done in overcoming friction is causing softening of surface and thus momentary lubrication and hence ***frictional force under dynamic state is less than static friction and so also*** $\boldsymbol{\mu_k < \mu}$ **.**



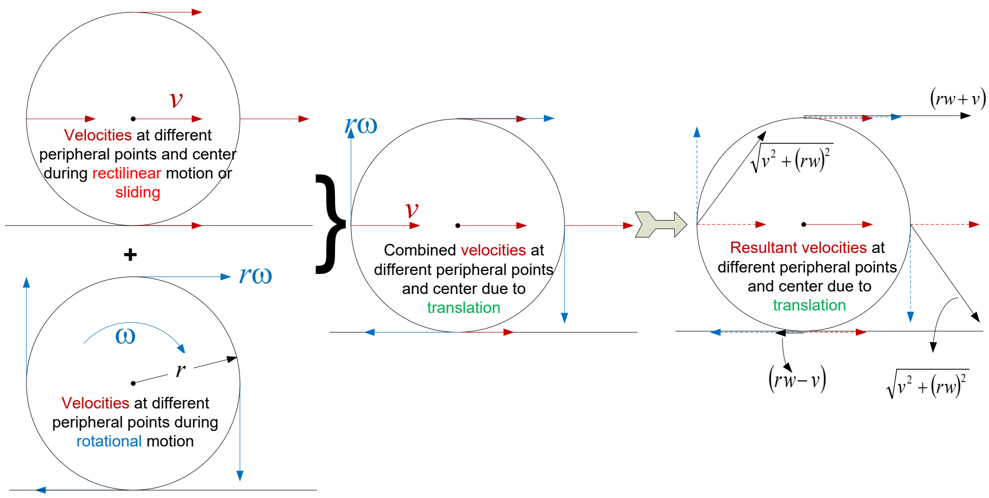
c. Dynamic Friction Under Upward Pull on an Inclined Plane: On a plane inclined where $\theta > \tan^{-1}\mu$, there is a natural tendency of the block to slide, but it can be made to slide upwards only on application of an external force $\boldsymbol{F}$ such that $F > (mg\sin\theta + f)$, and once it starts moving then $F > (mg\sin\theta + f) > (mg\sin\theta + f_k)$. It is to be noted that f_k, in this case is against the cause of action $\boldsymbol{F}$ and motion of the block, therefore it is downwards the slope**.** The acceleration $\boldsymbol{a}$ of the block remains as per NSLM and is $a = F - (mg\sin\theta + f_k)$.



Basic concept of friction is that f_k is always against the cause of action. It leads to some interesting conclusions as analysis of rolling motion starts it is interesting to reveal that f_k is against the cause of action i.e. the force causing motion, but not always against the direction of motion.

It is ironical about friction that at some occasions lubricant are used to reduce its effect viz. cycle chain running hard, while it is useful transfer power, a cause of rotatory motion. Better appreciation and understanding of friction would grow with the in-depth application through problems solving. At this point it is relevant to correlate basic concepts correlating translational and rotational motion and therein force(s) acting on a body.

Translational and Rotational motion: In the kinematics covered earlier ***translational or rectilinear motion*** *(even called linear motion) were considered discretely. It is the time to extend both the types of motions in conjunction in a rigid body. It is essential to define rigid body as the one which traverses displacement without changing its either shape or relative position of particles w.r.t. each other.* This is true in case of sliding, but in case of rolling while a body performs rotational motion, it changes its position also and such a kind of motion is called translational motion. Simplest example is rolling of a wheel. All particles in cylinder performing rectilinear motion travel same distance at any time. In a cylinder performing rotational motion, if its axis remains stationary, all particles traverse same angular displacement at any point of time. Here, both the motions are emulated in same cylinder. Analysis of motion is made by super imposing tangential velocity caused by rotational motion on rectilinear velocities of particles on four



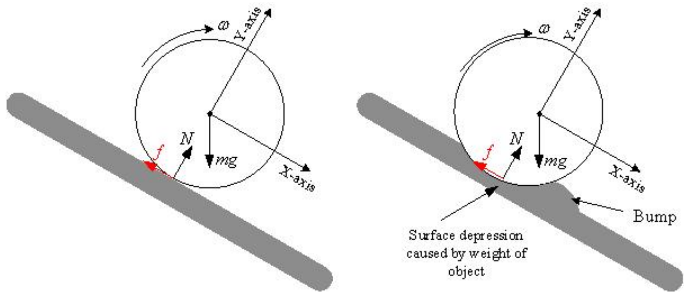
diametric opposite points. These are shown in different modes – a) separately, b) combined and c) resultant velocities. Depending upon magnitude of tangential velocity and rectilinear velocity three cases emerge and elaborated below.

Case 1 $(|\bar{\boldsymbol{r}} \times \bar{\boldsymbol{\omega}}| = |\bar{\boldsymbol{v}}_t| = \boldsymbol{v}_t)$: This is a case of matching tangential velocity and translational velocity at the line of contact point of the cylinder with horizontal surface, i.e. perfect rolling. In this velocity of the line of contact shall be Zero Velocity. And with slightest increase in $\boldsymbol{\omega}$ the rotational motion would lead to case 2 below. In this case frictional force would cause rotational torque about COM of the cylinder or when. This will set in rotational motion, Thus, *direction of frictional force would against the translational motion of the object*,

Case 2$(\boldsymbol{r\omega} > \boldsymbol{v}_t)$**:** In this case rotational motion of the cylinder would be ***slipping*** in the direction of angular velocity i.e. $\boldsymbol{\omega}$, which is against translational velocity. This would set in a relative motion of point of contact of cylinder with horizontal surface in a direction opposite to the direction of translational motion. Therefore, *it would set in frictional force on the point of contact of the cylinder in direction of translational motion*. In turn it would cause a torque which will cause retardation of $\boldsymbol{\omega}$. This can be observed from slipping of tyre of a vehicle on mud accelerator being raised.

Case 3$(\boldsymbol{r\omega} < \boldsymbol{v}_t)$: In this case rotational motion of the cylinder would start ***skidding*** in the direction of translational motion i.e. and hence friction at this point would be backward. Thus, *it would set in acceleration of angular velocity i.e.* $\boldsymbol{\omega}$ *and retardation of translational velocity* $\boldsymbol{v}_t$. Example of s*kidding is a vehicle in high speed fails to stop translational motion despite applying brakes.*

At this point it is relevant to discriminate between kinetic friction, caused by translational motion and rolling friction caused by rotational motion. This requires a close examination of contact surface area in case of rolling which tends to be a point in case of sphere and a line in case of cylinder. At this point or line of contact a high pressure is exerted by weight of the rolling object; this causes widening of the contact surfaces, which is nothing but a surface-depression along with a bump in the direction of rolling as shown in the figure. In reality for metals it is very small, but a significant to affect coefficient of friction. This phenomenon can be experienced by rolling a bar on a compressible surface or a plastic material dough of wheat flour while making chapati .Thus obstruction to the motion in case of rolling is not the spikes, as discussed during elaboration of friction. Thus the object is required to ride over depression-cum-bump to keep rolling.



These concepts, are sufficient to start analysis of kinematics of translational-cum-rotational motion which are, otherwise, mutually independent. It requires use of ***D' Alembert's Principle to*** *analyse independence of dynamics of translational motion and rotational motion*. Though explicit mention of this principle is encountered in advanced mechanics and in engineering studies, but its understanding is based on basic principle of Newton's Laws of Motion, which a student in class 11^{th} is able apply proficiently in problem solving.

Any number forces acting on an object can be resolved into an equivalent force $\left(\vec{F}\right)$, *which is Zero in case of equilibrium. But, in case on in-equilibrium and when non-zero the object is experiencing translational motion as per NSLM*. Each of the force may or may not exert a torque, which depends on point of application of force. Accordingly, for convenience of expressions, resultant force shall be taken along $\hat{x}$ and rotational torques on $\hat{y} - \hat{z}$ plane.

Translational Motion of a Rigid Body: Let there be a particle of mass *m* at a point (x, y, z)) and is experiencing a force $\vec{F}_x = m\frac{d^2\vec{x}}{dt^2} = F_x\hat{x}$. Taking this particle to be a part of a rigid body. Therefore, under influence of an *external force* $\vec{F}_{e-x}$ along $\hat{x}$, which create an *internal force* $\vec{F}_{i-x}$ as a reaction of the external force. Accordingly, *effective force* on the particle would be $\vec{F}_x = \vec{F}_{e-x} + \vec{F}_{i-x}$ This is true along all the three orthogonal axes ***x, y,*** and ***z***. Accordingly, $\vec{F}_y = m\frac{d^2\vec{y}}{dt^2}$ and also $\vec{F}_z = m\frac{d^2\vec{z}}{dt^2}$.

Applying NSLM to the system of particles, constituting a rigid body, $\sum\vec{F}_x = \sum m\frac{d^2\vec{x}}{dt^2}$; $\sum\vec{F}_y = \sum m\frac{d^2\vec{y}}{dt^2}$; and $\sum\vec{F}_z = \sum m\frac{d^2\vec{z}}{dt^2}$. Here, $\sum m = M$, i.e. mass of the rigid body. These set of equations are being used to analyse translational motion of a rigid body. Let, (x_0, y_0, z_0) be the coordinates of the Centre of Mass (COM), and (x', y', z') be coordinates of a particle in the rigid body w.r.t. to COM, then $x = x_0 + x'$; $y = y_0 + y'$; and $z = z_0 + z'$. This leads to an equations: $\sum\vec{F}_x = \sum m\frac{d^2\vec{x}}{dt^2} = \sum m\frac{d^2}{dt^2}(x_0 + x')\hat{x} = \left(\sum m\frac{d^2}{dt^2}x_0 + \sum m\frac{d^2}{dt^2}x'\right)\hat{x}$. During translational motion of a rigid body about COM $\sum m\frac{d}{dt}x' = \sum m\frac{d^2}{dt^2}x' = 0$, hence $\sum\vec{F}_x = \left(M\frac{d^2}{dt^2}x_0\right)\hat{x}$. This is valid along Y-axis and Z-axis. Thus, ***it is observed that external forces on a rigid body when resolved along any of the axis, it causes a translational acceleration of the body along the axis as if whole mass is concentrated at its COM.***

Rotational Motion of a Rigid Body: Extending the above concept of translational motion of a rigid body to rotational motion about ***x*-axis** the equation would be $\vec{\tau}_x = \tau_x\hat{x} \rightarrow \sum(\vec{y}\times\vec{F}_z + \vec{z}\times\vec{F}_y) = \left(\sum(yF_z - zF_y)\right)\hat{x}$. It leads to $\vec{\tau}_x = \left(\sum m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right)\right)x = \left(\sum m\left((y_0 + y')\frac{d^2}{dt^2}\widehat{(z_0 + z')} - (z_0 + z')\frac{d^2}{dt^2}(y_0 + y')\right)\right)\hat{x}$. Magnitudes of this equation can be organized into:

$$\tau_x = M\left(y_0\frac{d^2}{dt^2}z_0 - z_0\frac{d^2}{dt^2}y_0\right) + \sum m\left(y'\frac{d^2}{dt^2}z_0 - z'\frac{d^2}{dt^2}y_0\right) + \sum m\left(y_0\frac{d^2}{dt^2}z' - z_0\frac{d^2}{dt^2}y'\right) + \sum m\left(y'\frac{d^2}{dt^2}z' - z'\frac{d^2}{dt^2}y'\right)$$

$$= M\left(y_0\frac{d^2}{dt^2}z_0 - z_0\frac{d^2}{dt^2}y_0\right) + \left(\frac{d^2z_0}{dt^2}\sum my' - \frac{d^2y_0}{dt^2}\sum mz'\right) + \left(y_0\sum m\frac{d^2}{dt^2}z' - z_0\sum m\frac{d^2}{dt^2}y'\right) + \sum m\left(y'\frac{d^2}{dt^2}z' - z'\frac{d^2}{dt^2}y'\right)$$

$$= \sum m\left(y'\frac{d^2}{dt^2}z' - z'\frac{d^2}{dt^2}y'\right)$$

This simplification is based on the fact that (a) $\sum my' = \sum mz' = 0$ as per COM, (b) $\sum m\frac{d^2}{dt^2}y' = \sum m\frac{d^2}{dt^2}z' = 0$, an extension of COM as elaborated in Translational motion of a rigid body. on the lines similar to that, while determining acceleration of rigid body. Moreover, while, object is performing translational motion along $\hat{x}$, the y_0 and z_0 remain constant therefore, $\frac{d^2}{dt^2}y_0 = 0$ and likewise $\frac{d^2}{dt^2}z_0 = 0$.

Transforming, Cartesian coordinates (***y',z'***) of each point w.r.t. COM into polar form $r'\angle\theta'$ such that $y' = r'\cos\theta'$, and $z' = r'\sin\theta'$. In the expresseion of Two double derivative of a function involving sine and cosine of an angle of rotation w.r.t. time (t). Accordingly, first derivative is $\frac{d}{dt}\sin\theta = \frac{d}{d\theta}\sin\theta\cdot\frac{d}{dt}\theta = \cos\theta\cdot\omega$. In presence of torque ω is also dependent time (t), therefore $\frac{d^2}{dt^2}\sin\theta = \frac{d}{dt}\left(\frac{d}{dt}\sin\theta\right) = \frac{d}{dt}\omega.\cos\theta$. It is a case of product of two functions, therefore, $\frac{d^2}{dt^2}\sin\theta = \frac{d}{dt}\omega.\cos\theta = \omega\cdot\frac{d}{dt}\cos\theta + \cos\theta\cdot\frac{d}{dt}\omega = -\omega^2\sin\theta + \alpha\cdot\cos\theta$. Likewise, $\frac{d^2}{dt^2}\cos\theta = \frac{d}{dt}(-\omega.\sin\theta) = -\omega\cdot\frac{d}{dt}\sin\theta - \sin\theta\cdot\frac{d}{dt}\omega = -\omega^2\cos\theta - \alpha\cdot\sin\theta$. Applying, these general mathematical deductions into the case being analyzed leads to-

$$-\tau_x = \sum m\left(y'\frac{d^2}{dt^2}z' - z'\frac{d^2}{dt^2}y'\right) = \sum m\left(r'^2\left(\alpha\cos^2\theta' - \omega^2\sin\theta'\cos\theta'\right) - \left(-r'^2\left(\alpha\sin^2\theta' + \omega^2\sin\theta'\cos\theta'\right)\right)\right)$$

$$= \alpha\sum mr'^2 = I\alpha$$

Thus, it is evident that angular acceleration (AA) of a body <u>is about its COM</u> is the same as if it were fixed and the same set of forces act upon the body.

Thus important conclusions of D' Alembert's principle, propounded in 1743, are that when a forces acts on a rigid body it produces –

- ***Translational Motion as it the force is acting on its COM,***
- ***Rotational Motion about its COM, as if were a point of rotation.***

A typical problem of rotation-cum translation of a ring on frictional surfaces is elaborated here under-

Problem: A boy is pushing a ring of mass 2 kg and radius 0.5 m with a stick as shon in the figure. The stick applies a force of 2 N on the ring and rolls it without slipping with an enough acceleration of 0.3 m/s^2. The coefficient of friction between the ground and the ring is large that rolling always occurs and the coefficient of friction between the stick and the ring is $\left(\frac{P}{10}\right)$. Find the value of P.

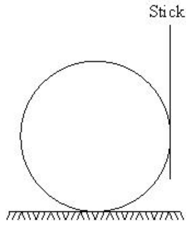
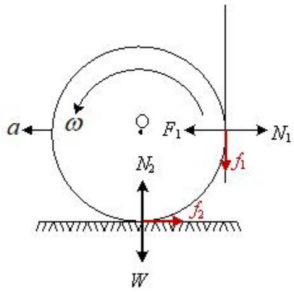


Illustration: In the instant case rolling of ring without slipping is specified on ground but the probem is silent in respect of in repect of pure rolling between the stck and the ring. Therefore, frictional force between ring and the ground $f_2 < \mu_2 N_2$, while $f_1 = \mu_1 N_1$ to aloow for slipping between stick and the ring.



Moment of inertia of the ring is $I = MR^2$ Let F_1 is the push provided by the stick to the ring and, $F_1 - f_2 = Ma = 2\times 0.3 = 0.6$ N .. **(a)** . Angular acceleration of the ring is caused by f_2 frictional reaction on the ring in backward direction due to friction on the road in direction of rolling. Whereas, at stick pushing the ring frictional force on the ring f_1 is downward. Thus net trque experienced by the ring is $\tau_2 = (f_2 - f_1)R = I\alpha = (MR^2)\left(\frac{a}{R}\right) \rightarrow (f_2 - f_1) = Ma$.

It leads to $Ma = F_1 - f_2 = f_2 - f_1 = 0.6$ .. **(b)**. Combining (a) and (b) we get $F_1 - f_1 = 1.2\ldots$ **(c)**

Further, with the given data $F_1 = N_1$ and thus $f_1 = \mu_1 N_1 = \left(\frac{P}{10}\right)N_1 = (0.1P)F_1$ **…(d)** .Combining (c) and (d) $F_1 - (0.1P)F_1 = 1.2 \rightarrow (1-0.1P)F_1 = 1.2 \rightarrow F_1 = \frac{1.2}{1-0.1P}\ldots$**(e).**

As regards the force applied by the stick is 2 N it has two components F_1 and f_1 such that $f_1^2 + F_1^2 = 2^2$. It leads to $((0.1P)F_1)^2 + F_1^2 = 4 \rightarrow F_1^2 = \frac{4}{1+0.01P^2}\ldots$**.(f)**

Combining (e) and (f) we get $\frac{4}{1+0.01P^2}=\left(\frac{1.2}{1-0.1P}\right)^2 \rightarrow (1-0.1P)^2=(1+0.01P^2)(36\times10^{-2})$. It leads to a quadratic expression $(0.06P)^2+0.2P-0.64=0 \rightarrow 36\times10^{-4}\times P^2+20\times10^{-2}\times P-64\times10^{-2}=0$. This can be simplified into $0.36\times P^2+20P-64=0$, it leads to $P=\frac{-20\pm\sqrt{20^2-4(0.36)(-64)}}{2\times0.36}$. It further solves into $P=\frac{-20\pm\sqrt{400+92.16}}{0.72}=\frac{-20\pm\sqrt{492.16}}{0.72}=\frac{-20\pm\sqrt{492.16}}{0.72}=\frac{-20\pm22.185}{0.72}=\frac{2.185}{0.72}=3.03\approx3$. Thus value of $P=3$, is the answer.

N.B.: **(a)** The force of 2 N is stated to be exerted by stick on the ring. It is resulting into push for rolling which is radially horizontal at the point of contact and frictional force which is tangential and vertically downward. Both these components of the 2 N force are variables shich satisfy the remaining conditions of the problem. Takimg this force to be simplistically horizontal is incorrect.

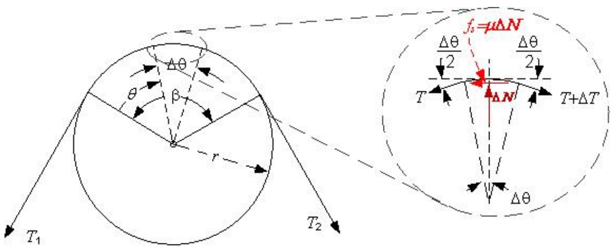
(b) Frictional force during rolling of this ring combines frictional forces on Two wheels of bicycle. Ring with the earth experiences frictional force in a direction backward (in direction of linear velocity of the at point of contact, similar to that of front (driven) wheel of the bicycle. While the ring with stick experiences force vertically downward (in direction opposite to that of the linear velocity of ring at the point of contact, similar to that of rear (driving) wheel of the bicycle. Thus this example beautiffuly combines frictional effect of two separate wheels of bicycle into a single ring.

(c) This involves many variables and hence equation, retaining them in algebric form for substituition of values at theend is liable for errors, and simplification in numeric form at each stage will make the solution simple.

(d) In this problem numerical values over a wide range were involved and, therefore, it requires cautious handling and conversion of numerical values for ease of calculations.

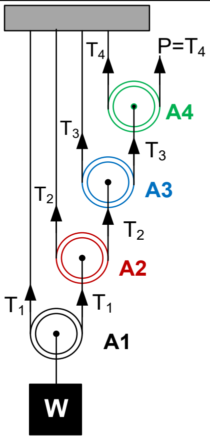
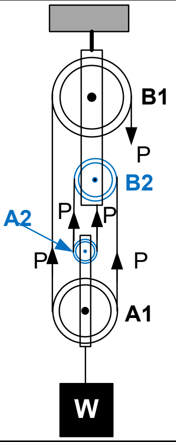
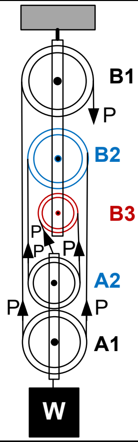
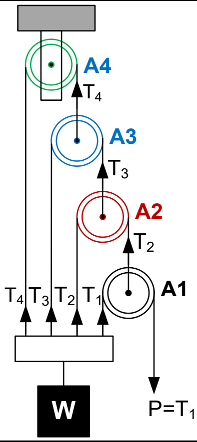
Dynamics of Rotational-cum-Translational Motion on an Inclined Plane: *It is interesting to analyse dynamics of rotational motion, which in presence of friction becomes translational and vice-versa and how does the friction becomes a conservative force.* Consideration of rotational motion requires object to be either spherical or cylindrical, and wheel was one of the most important discovery which changed course of development of human civilization. Analysis on an inclined plane is forms a generic case; it can be easily transformed to motion on a plane by substituting making angle of inclination $\theta = 0$. In this three types of forces and associated accelerations come into play. These forces are - **a)** Force causing motion, **b)** Gravitational Force and **c)** Frictional Force. Various cases of rolling-cum-translational motion of an object viz. circular ring, hollow/solid cylinder, circular disc solid/punched, hollow/solid sphere, having mass M and radius r, on an inclined plane at an angle θ can be classified as: a) Rolling in the direction of slope, b) Rolling against direction of slope. Each of the case has three possibilities- **(i)** slipping, **(ii)** pure rolling and **(iii)** skidding. These possibilities have been brought out earlier.

In pure rolling, from consideration of friction, at point of contact relative velocity between two surfaces is zero and this makes it a case of static friction, ***this makes friction force in pure rolling conservative***. While, in case of slipping or skidding the relative velocity is non-zero and hence it becomes a case of ***kinetic frictional force which is non-conservative***. Here, while driving shaft torques is ignored, initial angular velocity of the rolling body $(\omega_0)$is assumed.

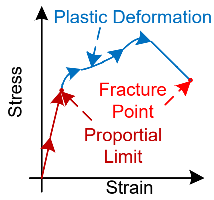


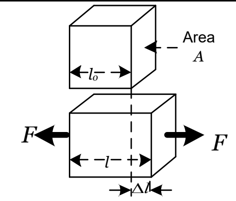
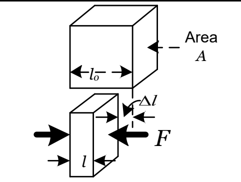
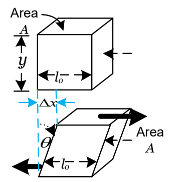
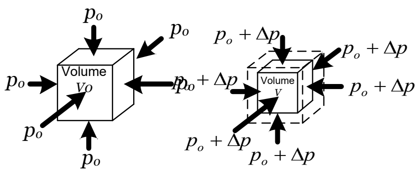
Friction of a Pulley on a Circular Surface: This is another interesting case of friction when a rope or a belt passes over a circular surface, it could be a static pulley or a drum, as shown in the figure. In this case unlike ideal pulley tension along the rope or belt is not uniform. Different tensions T_1 and T_2 on two ends of the rope having angle of contact β is analysed within inset for an element of rope with contact angle $\Delta\theta$. Applying conditions of equilibrium along tangential line and radial lines independently $\sum F_t = 0;\ -T\cos\frac{\Delta\theta}{2} + (T+\Delta T)\cos\frac{\Delta\theta}{2} - \mu\Delta N = 0$ and likewise, $\sum F_e = 0;\ -T\sin\frac{\Delta\theta}{2} - (T+\Delta T)\sin\frac{\Delta\theta}{2} + \Delta N = 0$. Under limiting condition $\Delta\theta \to 0$, these equation lead to $dT = \mu dN$, and $Td\theta = dN = \frac{dT}{\mu}$, or $Td\theta = \frac{dT}{\mu}$. Alternatively, $\frac{dT}{T} \approx \mu d\theta$. Integrating this equality over contact angle, $\int_{T_1}^{T_2} \frac{1}{T} dT = \int_0^{\beta} d\theta$. It leads to $\ln\frac{T_2}{T_1} = \beta$. Alternatively, $T_2 = T_1 e^{\beta}$.

Pulley Systems: Despite friction being a retardant in rigid pulleys, rolling pulleys are in extensive use, where it is assumed that pulley moves with the rope, but has a frictionless rotation about its axel. This leads to $T_1 = T_2$ and development of pulley systems, involving more than one pulley, which are classified into three main types as under.

Pulley System : Type-I	**Pulley System : Type-II**		**Pulley System : Type-III**
			
Let, n be the number of pulleys system	a. It uses a continuous rope and hence each pulley experiences equal pull. b. Distance between Two blocks of Pulleys is large enough to consider, each section of continuous rope is nearly vertical. c. Let , ***n*** *be the number of pulleys in lower block*, Each block of pulleys is rigid		Let , n be the number of pulleys system
$T_1 = \frac{W}{2}; T_2 = \frac{T_1}{2} = \frac{W}{2^2};$ $T_3 = \frac{T_2}{2} = \frac{W}{2^3}; T_4 = \frac{T_3}{2} = \frac{W}{2^4}$ Hence, in a system of ***n*** Pulleys, $P = \frac{W}{2^n}$	$P = \frac{W}{2n}$	$P = \frac{W}{2n+1}$	P=T1, T2=2T1, T3=2T2, T4=2T3, W=T1+T2+T3+T4=P+2P+4P+8P, It is geometric progression. Hence, in a system of n pullets N Pulleys $W = \left(\frac{2^n - 1}{2-1}\right)P = \left(2^n - 1\right)P$, or $P = \frac{W}{\left(2^n - 1\right)}$
Upward displacement of load is : x Disp. of A2=2^1x, Disp. of A3=2^2x Disp. of A4=2^3x, Disp. of P=2^4x Thus displacement of effort for a n pulley system is $2^n x$	Disp. of load is : x Disp. effort =$2nx$	Disp. of load is : x Disp. effort =$(2n+1)x$	Upward displacement of load is : x Disp. of A3=2^1x, Disp. of A2=2^2x Disp. of A1=2^3x, Disp. of Effort=$(2^4-1)x$
a. In this analysis, pulleys and rope are assumed to be weightless and frictionless **b. Work done on Load is equal to work done by effort**			

Physical Deformation of Solids: *Action of force or torque on a solid body if causes relative displacement of molecules, it leads to physical deformation, which causes internal stresses till an equilibrium is reached between external force and* *internal stress. **Robert Hook** formally stated this phenomenon in 1676, later **Thomas Young** established proportionality of stress to the deformation of the body. This proportionality constant is known as **Young's Modulus**. A typical stress-strain curve shows loss of proportionality beyond a stress level where plastic deformation starts. It causes decrease in cross-section of the material subjected to tensile force, which further increases stress and ultimate fracture. Study of this deformation and its restoration after removal of external stress is broadly called **Elasticity**. Intrinsically there are three types of deformations and physical property of solids are classified as under –*



Linear *Deformation*		**Surface Deformation**	**Bulk Deformation**
Tensile Stress	Compressive Stress	Shear Force	Compressive Force
Elongation	Compression	Slippage	Compression
			
$\text{Stress}[\boldsymbol{\sigma}] = \frac{\text{Force}[F]}{\text{Cross}-\text{sectional area Perpedicular to the area}[A]}$ Dimension of Stress: $[ML^{-1}T^{-2}]$ Unit: N/m^2		Shear Stress$[\boldsymbol{\tau}]$ $= \frac{\text{Tangential Force}[F]}{\text{Cross}-\text{sectional of the Tangential surface}[A]}$ Dimension of Stress: $[ML^{-1}T^{-2}]$ Unit: N/m^2	Pressure on all faces (P), it is uniform all along the surface area of the solid Dimension of Stress: $[ML^{-1}T^{-2}]$ Unit: N/m^2
$\text{Strain}[\boldsymbol{\varepsilon}] = \frac{\text{Change in length under stress}[\Delta l]}{\text{Original length}[l_0]}$ Dimension Less Unit : Per Unit		Shear Strain $[\boldsymbol{\gamma}] = \frac{\text{Slip of opposite faces under shear stress}[\Delta x]}{\text{Perpendistance between opposite faces}[y]}$ $= \tan\theta$ Dimension Less Unit : Per Unit	Volumetric Srain $[\boldsymbol{\theta}]$ $= \frac{\text{Change in volume under compresseion}[\Delta V]}{\text{Perpendistance between opposite faces}[V_0]}$ Dimension Less Unit : Per Unit
Young s Modulus of Elasticity $[\boldsymbol{E\ or\ Y}] = \frac{Stress\ [\sigma]}{Strain\ [\varepsilon]}$ Dimension of $\boldsymbol{E}$ or $\boldsymbol{Y}$: $[ML^{-1}T^{-2}]$ Unit of $\boldsymbol{E}$ or $\boldsymbol{Y}$: N/m^2		*Shear* Modulus of Elasticity$[\boldsymbol{G\ or\ S}] = \frac{Stress\ [\boldsymbol{\tau}]}{Strain\ [\boldsymbol{\gamma}]}$ Dimension of $[\boldsymbol{G\ or\ S}]$: $[ML^{-1}T^{-2}]$ Unit of $[\boldsymbol{G\ or\ S}]$: N/m^2	Bulk Modulus of Elasticity$[\boldsymbol{K\ or\ B}] = \frac{Stress\ [\boldsymbol{P}]}{Strain\ [\boldsymbol{\theta}]}$ Dimension of $[\boldsymbol{K\ or\ B}]$: $[ML^{-1}T^{-2}]$ Unit of $[\boldsymbol{K\ or\ B}]$: N/m^2
Visible effect:		**Visible Effect:** Rectangles become Parallelograms	**Visible Effect:** Volume changes but shape does not change.
Longer and Thinner	Shorter and Fatter		
Effect on exceeding elastic limits:		**Effect on exceeding elastic limits**:	**Effect on exceeding elastic limits**:
Starts with Plastic elongation and ends up in breakage	Starts with plastic compression and ends up in fusion or crushing.	Starts with plastic slippage with chipping i.e. shearing off of material.	Starts with plastic deformation and ends up in fusion or crushing

Here, Force is the cause, while stress is the intensity of the cause. Deformation is the effect, while strain is the relative deformation. And the proportionality constant between intensity of cause and relative deformation is the physical property of the material.

Torque causing rotational motion of a shaft causes torsional stress in it and it is akin to shear stress at cross-sections of the shaft. Torque is at one end of the shaft and driven load revolving at the other end creates a gradual twist of the shaft along its length. A typical representation of failure of shaft of a pump under torsion, in the event of excessive corrosion leading to reduction in area of cross-section of the shaft is shown in the figure. This is more a case of engineering application of the concept. ***In every man-made system stress-strain analysis and limiting deformation within permissible limit of elasticity is a property of material used, is guiding factor in design and engineering of an application.***

Gravitation: Understanding of gravitation has its roots in understanding of cosmos. Sky, rise and fall of sun, changes in phases of moon, and stars in sky in various formations must have been mystery as well as a fascination to observe. This had become been an integral part of culture and spiritual believes prevalent in Egyptian, Indian, Chinese and Greek cultures, and each of them have their own model of universe. It was Nicolas Copernicus who published heliocentric model in 1543, in which Sun is positioned near the centre of universe. After him, Tycho Brahe in later half of 16th century recoded his observations with naked eye on Moon orbiting Sun and planets around the Sun. But, mistakenly considered Sun to be revolving around the Earth. Johannes Kepler analysed observations of Tycho and in 1609 he published Two laws of planetary motion and published third law in 1619. Galileo Galilei, developed telescope to propound the authenticity of heliocentric model of Copernicus, proposed about a century ago. He also analysed motion of free fall of bodies known as Kinematics. By then sequence of discoveries had speeded up and it prompted Isaac Newton to propound Laws of Motion and Gravitational Force so as to authenticate Kepler's Laws of planetary motion. In 1915, almost Three centuries later Albert Einstein in his General Theory of Relativity predicted Gravitational waves, and a century later in Sept'2015 ripples of gravitational waves were detected using highly sophisticated Two Laser Interferometer Gravitational-wave Observatory (LIGO) detectors in USA and Australia. These ripples were caused by collision of Two black holes estimated to have occurred about 1.3 billion years ago.

This makes Kepler's laws quite inquisitive to appreciate beauty of mathematics and physics which aims at discovery of nature. Accordingly, after illustrating Laws of Gravitation, derivation of Kepler's Laws would help to connect the two as a natural consequence of painstaking and honest observations, and analysis made by these forerunner in the field of science.

Newton's Laws of Gravitation:

Inquisitive Newton tried to relate planetary motion to factors that determine acceleration of celestial bodies and guessed that –

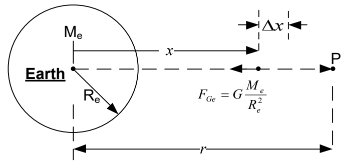
a. Acceleration (α) of a body towards earth is inversely proportional to the square of distance of the body from the centre of earth (r), i.e.$\alpha = \frac{1}{r^2}$.
b. Force (F) of attraction towards centre of earth is proportional to the mass of object (m), i.e. $F\alpha \frac{m}{r^2}$, and is in accordance with NSLM.
c. Force (F) on a body due to earth a body of mass m is proportional to mass of earth (M), in accordance with NTLM and thus $F\alpha \frac{Mm}{r^2}$.

Combining these Three postulates he proposed a ***universal constant of gravitation*** $(\boldsymbol{G})$ and the statement $\vec{F} = -G\frac{Mm}{r^2}\hat{r}$ is known as ***universal law of gravitation (ULG)***. Experimentally value of $G = 6.67 \times 10^{-11}\text{N} - \text{m}^2/\text{kg}^2$ has been determined. Experimental set-up and method of determining G has been brought out in references.

Gravitational Field: *In accordance with the ULG* of a body of mass M on another unit mass displaced by $\vec{r}$ from the centre of the body is $\vec{E} = -G\frac{M}{r^2}\hat{r}$.*On the earth's surface* $\vec{E} = -G\frac{M_e}{R_e^2}\hat{r} = \vec{g}$, *the gravitational field is also called acceleration due to gravity* $\vec{g}$ and its magnitude on the surface of the earth is $= G\frac{M_e}{R_e^2} = 9.8\ m.s^{-2}$.

Gravitational Potential: Work done in moving a unit mass from Earth's surface, against the gravitational pull, upto a point P at a radial distance r is $W = \int_{R_e}^{r}\left(G\frac{M_e}{x^2}(-\hat{x})\right)\cdot d\bar{x} = GM_e\left[\frac{1}{x}\right]_{R_e}^{r} = GM_e\left[\frac{1}{r}-\frac{1}{R_e}\right] = \left.\frac{GM_e}{r}\right|_{taking\ ,\frac{GM_e}{R_e}=0}$

Here, gravitational force, is in direction $-\hat{x}$; while displacement Δx is in direction $\hat{x}$.



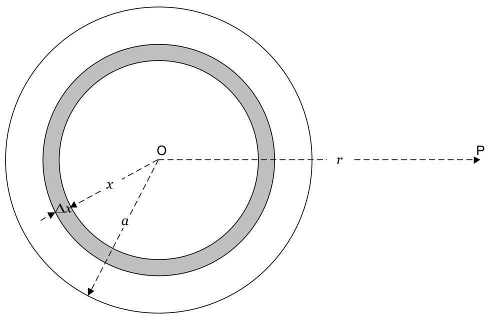
Further, in above derivation, it is assumed that Potential at Earth's surface is Zero and hence the PE at point P, calculated above can be called as relative Potential or Difference in Potential w.r.t. Earth's surface. The moment mass of the object being moved is considered, i.e. other than unity, it becomes **Potential Energy** of the mass at that point.

These illustrations lead to the following conclusions:

(a) It is to be noted that PE is scalar, while Gravitational Field is vector like acceleration due to gravity.

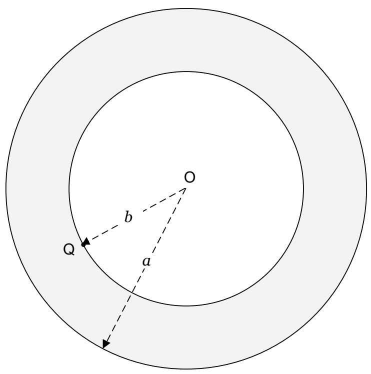
(b)(-ve) sign to $\bar{E}$ or $\bar{g}$ indicates that direction of Field or acceleration is opposite to $\hat{r}$, unit displacement vector.

Concept of gravitational field for typical, but uniform mass distribution, using ULG, is application of definite integral as elaborated earlier for COM and MOI, and left for readers as an exercise. Nevertheless, gravitational field in case of earth at its surface, below the surface and above it, is of specific relevance and is illustrated here under. This analysis, in different form, is applicable in Electrostatics, separate chapter of Mentors' Manual.

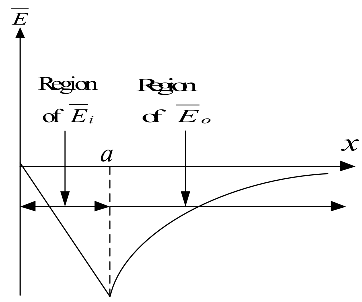


The problem is being solved considering integration of gravitational field, caused at a point P, at a distance r from the centre O of a solid sphere of uniform density ρ, due to an elemental-thin-hollow-spheres of thickness of mass dm such that $dm = (4\pi x^2 dx)\rho$, and $\rho = \frac{M}{\frac{4}{3}\pi a^3}$. Thus $\vec{E} = \int d\vec{E}$. It leads to $\vec{E} = -G\int_0^a \frac{dm}{r^2}\hat{r} = -G\left[\int_0^a \frac{(4\pi x^2 dx)\left(\frac{M}{\frac{4}{3}\pi a^3}\right)}{r^2}\right]\hat{r} = \left[-G\frac{3M}{a^3r^2}\int_0^a x^2dx\right]\hat{r} = \left[-G\frac{3M}{a^3r^2}\left[\frac{x^3}{3}\right]_0^a\right]\hat{r} = \left[-G\frac{3M}{a^3r^2}\frac{a^3}{3}\right]\hat{r} = -G\frac{M}{r^2}\hat{r}$.

This is same as a gravitational at point P field point mass M is placed at O.



Now consideration is made of gravitational field at point Q inside a solid sphere which is at a distance b from O. This problem is decomposed into two parts, First part is $\bar{E}_1$ due to mass contained in that portion of concentric shells which fall outside radial b, and Second Part is $\bar{E}_2$ constituting mass contained in concentric shells starting from O upto radial b. Accordingly, $\bar{E}_1 == -G\int_0^b \frac{dm}{r^2}\hat{r} = -G\frac{M_i}{b^2}\hat{r}$. This is in accordance with analysis of solid sphere, brought out above. Here, $M_i = \left(\frac{M}{\frac{4}{3}\pi a^3}\right)\frac{4}{3}\pi b^3 = M\frac{b^3}{a^3}$.



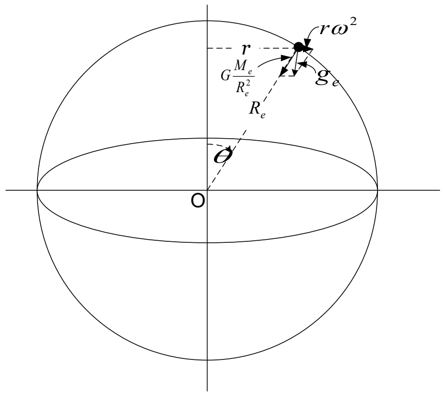
At any point inside a hollow spherical shell gravitational field intensity is $\vec{E}_1 = 0$. And on surface of the a sphere $\vec{E}_2 = -G\left(M\frac{b^3}{a^3}\right)\frac{1}{b^2}\hat{r} = -G\left(\frac{M}{a^3}b\right)\hat{r}$. Thus gravitational field at any point inside a solid sphere is: $\vec{E}_{i-b} = \vec{E}_1 + \vec{E}_2$, it leads to: $\vec{E}_{i-b} = -G\left(\frac{M}{a^3}b\right)\hat{r}$.

This is the equation of central force acting towards a central point and proportional to the distance from the central point. This is the case of Simple Harmonic Motion (SHM), which would be illustrated separately in a chapter Waves and Motion of this manual. Nevertheless, hypothetically if a diametric tunnel is dug through the centre of the earth and an object is placed on the one end of tunnel; it would keep oscillating in the tunnel. The combined effect of the E due to a solid sphere, at a point inside it is represented as $\vec{E}_i$. and outside the sphere as $\vec{E}_o$ e are shown in the adjoining graph.

Variation of acceleration due to gravity $\vec{g}$ above the earth's surface is dependents on many parameters and are illustrated below-

a. Height above Earth's Surface: Let, $h$be the height above Erath's surface and this would lead to acceleration due to gravity $g' = G\frac{M_e}{(R_e+h)^2} = G\frac{M_e}{R_e^2\left(1+\frac{h}{R_e}\right)^2} = G\frac{M_e}{R_e^2}\left(1+\frac{h}{R_e}\right)^{-2} = g\left(1+\frac{h}{R_e}\right)^{-2}$. But, **when $\boldsymbol{h \ll R_e}$, $\boldsymbol{\frac{h}{R_e} \to 0}$** and as per as per *Binomial theorem* $\boldsymbol{g' \to g\left(1-2\frac{h}{R_e}\right) \to g.}$

b. Rotation of Earth: Rotation of earth creates an another centrifugal acceleration vector $r\omega^2$ which is radially outwards and perpendicular to the axis of rotation of the earth. While, acceleration due to gravity $g = G\frac{M_e}{R_e^2}$ is radial and towards the centre of the earth O. **Accordingly, effective acceleration $\boldsymbol{g_e}$ is the vector sum of the earlier Two, but its direction is not towards O, rather drifted away from it towards the object.**



c. Shape of the earth: Mean radius of the earth is 6.37x10^3 km, while at the equator radius is 21 km ($\approx$ 0.3%) larger than that at the pole$\approx$ 0.3%. This shape of earth influences $\boldsymbol{g}$ in three way, -

i) at poles effective radius is lesser that at equator, and it is inversely proportional to the square of the radius,

ii) effective mass of earth which is in cubic proportion to radius at the point of object,

iii) Influence of rotation, discussed above is highest at equator, and Zero at poles, since the radial distance from axis of rotation is also a parameter in determining effective $\boldsymbol{g}$.

Resultant acceleration due to gravity at poles is greater than that at equator.

d. Non-Homogeneity of Earth: All the above derivation are based on a homogeneous mass distribution, but sea, minerals, mountains and valley do affect the mass distribution and locally influence value of **g.**

Escape Velocity: It is the minimum velocity with which a particle projected vertically would reach beyond a point where gravitational field of the earth becomes ineffective and then it would not return back to it. Initial energy of the projectile at the earth's surface is = KE + PE = $\frac{1}{2}mu^2 + G\frac{M_e m}{R_e}$, where u is the vertical velocity of the projectile. Let, at height h, vertical velocity of the projectile is v, then its total energy would be $= \frac{1}{2}mv^2 + G\frac{M_e m}{R_e+h}$. Assuming, that travel of the projectile is resistance free then as per principle of conservation it would imply that $\frac{1}{2}mu^2 + G\frac{M_e m}{R_e} = \frac{1}{2}mv^2 + G\frac{M_e m}{R_e+h}$. For the projectile to escape the earth's gravity $v \geq 0$ and thus: $v^2 = \left(u^2 - \frac{GM_e}{R_e}\right) + \frac{GM_e}{R_e+h}$. One inference can be drawn that if $u^2 - \frac{GM_e}{R_e} = 0$, or minimum value of the initial velocity of projectile: $u_{min} = \sqrt{\frac{GM_e}{R_e}}$, would lead to $v \geq 0$ such that at any value of h, vertical velocity of the projectile be always +ve, and hence it would not return to earth. Accordingly, u_{min} is called escape velocity of a projectile and it is independent of the mass of the projectile.

Kepler's Laws: A beginning into the understanding of Kepler's Laws is made from their statements and are as under -

Kepler's First Law (KFL): Each planet describes an ellipse having sun in one of its foci.

Kepler's Second Law (KSL): Area described by the radii, in an orbit, drawn from the planet to the sun is proportional to the times of describing it.

Kepler's Third Law (KTL): Square of the periodic times of the various planets are proportional to the cube of the major axis of their orbit.

Circular motion of a point object around a central mass is understandable with balancing of **Centripetal Force(CPF)** caused by *gravitational force due to central mass as per Newton's Universal Law of Gravitation*, and **Centrifugal Force (CFF)**, a pseudo force, caused by *acceleration responsible for circular motion.* This goes well to describe KSL and KTL, for motion of Earth around Sun in a circular orbit as under-

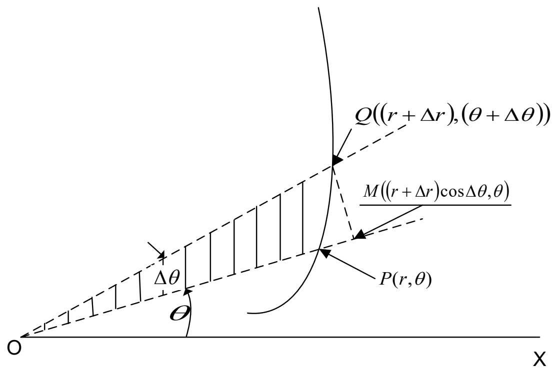
CPF=$\frac{GM_sM_e}{R^2}$; here G – Gravitational constant, M_s – mass of Sun and M_e- Mass of Earth, and R – radial distance between centre of mass of Sun and Earth.

CFF=$M_eR\omega^2 = M_eR(2\pi/T)^2 = \frac{4\pi^2 M_eR}{T^2}$; here T- is the time period of circular motion of earth.

Basic premise of circular motion is that angular velocity (ω) remains constant in circular motion and hence area described by radii (A) over a period is shall be $A = \pi R^2\left(\frac{\omega T}{2\pi}\right) = \left(\frac{\omega R^2}{2}\right)T$, or $A \propto T$, and is in accordance with the KTL; and this shall be true for all planetary motion.

Now, imposing condition of circular motion of earth, $\frac{GM_sM_e}{R^2} = \frac{4\pi^2 M_eR}{T^2}$; $T^2 = \left(\frac{4\pi^2}{GM_s}\right)R^3$, or $T^2 \propto R^3$, and is in accordance with KTL, with a moderation radius of earth's orbit is replaced with major axis of elliptical orbit. This is explained with the help of coordinate geometry where, $r = \frac{l}{1+\epsilon\cos\theta}$; here,$l$- is the Semi-latus-rectum of the Ellipse,ϵ- is the eccentricity of polar orbit, r – is radial distance of earth from Sun, θ – is the angle between earth's position and from its closest position such that (R, θ) are the polar coordinates of earth. Since, in case of circular motion $\epsilon = 0$, it leads to $R = l$. Some of the properties of ellipse are as under –

a. Correlation between Major Axis (*a)*, Minor Axis (*b)* and eccentricity (ϵ)is $a^2(1-\epsilon^2) = b^2$
b. Correlation between Major Axis (*a)*, Minor Axis (*b)* and Semi-Latus- Rectum (*l*) is $l = \frac{b^2}{a}$



In circular motion $\epsilon = 0$, a special case of an ellipse, it leads to $a = b = 2R$ or $R = \frac{a}{2}$ i.e. semi-major axis this translates $T^2 = \left(\frac{4\pi^2}{GM_s}\right)R^3 \rightarrow T^2 = \left(\frac{\pi^2}{2GM_s}\right)a^3$; or $T^2 \propto a^3$, without changing basic nature of KTL. Further in an ellipse $l = \frac{b^2}{a}$.

As a phenomenon, proportionality constant involves parameters independent of any planet; it is common for all planets, and so also the relationship.

Next is the obvious curiosity is explanation of elliptical orbit as per postulate of KFL, and relevance of KSL and KTL. In this context ***A simplistic explanation for elliptical motion as per KFL, without going into intricate dynamics of rigid bodies distributed in space, is the basic configuration of solar system which is not a two body system. Thus the resultant of forces on a planet as it revolves around the earth keep***

changing. Accordingly, the resultant in a closed orbit, under influence of net central force towards Sun, is an ellipse. During motion net velocity and acceleration of the object is towards focus (u,α) and that along tangent (v,β) are required to be determined. Here, $OP = r$, and $OQ = r + \Delta r$ and O represents focus of the orbital motion of planet i.e. Sun.

$u = \frac{OM-OM}{\Delta t}\Big|_{\Delta t\to 0} = \frac{(r+\Delta r)\cos\Delta\theta - r}{\Delta t}\Big|_{\Delta t\to 0} = \frac{dr}{dt}$; $v = \frac{QM}{\Delta t} = \frac{(r+\Delta r)\sin\Delta\theta}{\Delta t}\Big|_{\Delta t\to 0} = \frac{(r+\Delta r)\Delta\theta}{\Delta t}\Big|_{\Delta t\to 0} = r\frac{d\theta}{dt}$

Likewise, $\alpha = \frac{((u+\Delta u)\cos\Delta\theta - (v+\Delta v)\sin\Delta\theta) - u}{\Delta t}\Big|_{\Delta t\to 0} = \frac{\Delta u - v\Delta\theta}{\Delta t}\Big|_{\Delta t\to 0} = \frac{du}{dt} - v\frac{d\theta}{dt} = \frac{d^2r}{dt^2} - \left(r\frac{d\theta}{dt}\right)\frac{d\theta}{dt} = \frac{d^2r}{dt^2} - \left(r\frac{d\theta}{dt}\right)^2 = -P$; here P is indicative of centripetal acceleration. Further,

$$\beta = \frac{((u+\Delta u)\sin\Delta\theta + (v+\Delta v)\cos\Delta\theta) - v}{\Delta t}\Big|_{\Delta t\to 0} = \frac{u\Delta\theta + \Delta v}{\Delta t}\Big|_{\Delta t\to 0} = u\frac{d\theta}{dt} + \frac{dv}{dt} = \frac{dr}{dt}.\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right) = 2\frac{dr}{dt}.\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$$= \frac{1}{r}\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right]$$

In planetary motion acceleration is always towards Sun (focus) i.e. $\alpha = \frac{d^2u}{dt^2} \neq 0$and $\beta = \frac{d^2v}{dt^2} = 0$. It implies angular momentum of the revolving planet $(\vec{P} = m(\vec{r}\times\vec{v}) = m(\vec{r}\times(r\omega\hat{v})) = m\left(\vec{r}\times\left(r\frac{d\theta}{dt}\right)\hat{v}\right) = m\vec{h}$; here $h = |\vec{h}| = r^2\frac{d\theta}{dt}$. It leads to $\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2$; here, u is substituted for $\frac{1}{r}$ for mathematical convenience. In definition of h it is to be noted that *if radial distance of revolving body increases the its angular velocity decreases in square-root proportion, and accordingly instantaneous linear velocity* $(= r\frac{d\theta}{dt})$.

With this illustration sectorial area OPQ swept by the radial shall be $\Delta A = \frac{1}{2}OP\cdot OQ\sin\Delta\theta\Big|_{\Delta\theta\to 0} = \frac{1}{2}r\cdot(r+\Delta r)\sin\Delta\theta\Big|_{\Delta\theta\to 0}$.

Accordingly, rate of describing the sectorial area by the radial shall be $\frac{dA}{dt} = \frac{\Delta A}{\Delta t} = \frac{1}{2}\frac{r\cdot(r+\Delta r)\sin\Delta\theta}{\Delta t}\Big|_{\Delta t\to 0} = \frac{1}{2}\left(\frac{r\cdot(r+\Delta r)\sin\Delta\theta}{\Delta\theta}\right)\frac{\Delta\theta}{\Delta t}\Big|_{\Delta t\to 0}$.

Thus, $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}h = Const$. ***This is the mathematical justification of postulate contained in KSL.***

Properties of h are important in illustration of elliptical orbits and are –

a. Since, orbital motion has only central force, tangential acceleration is Zero,
b. The (a) above leads to Angular moment of a mass is constant,
c. The proportionality constant of angular momentum is mass of the body moving in an elliptical orbit,
d. The value of h is equal to the twice the rate of describing area the radial of the elliptical orbit.
e. Thus area described by radial in completing one orbit is $= 2hT$, here T is the time period of the orbital motion.

Now remains KTL and for elliptical motion and it involves redefining. This is done by first solving its Two addends.

Initially, $\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\left(\frac{du}{d\theta}\right)\left(\frac{d\theta}{dt}\right) = -\frac{1}{u^2}\left(\frac{du}{d\theta}\right)hu^2 = -h\frac{du}{d\theta}$. Accordingly, first addend is -

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\left(\frac{d\theta}{dt}\right) = -h\left(\frac{d^2u}{d\theta^2}\right)\left(\frac{d\theta}{dt}\right) = -h\left(\frac{d^2u}{d\theta^2}\right)(hu^2) = -h^2u^2\frac{d^2u}{d\theta^2}$$

The second addend solves to $-r\left(\frac{d\theta}{dt}\right)^2 = -\frac{1}{u}(hu^2)^2 = -h^2u^3$.

Further, equation of an ellipse in polar form leads to $r = \frac{l}{1+\epsilon\cos\theta}$ and it leads to $u = \frac{1}{r} = \frac{1}{l} + \frac{\epsilon}{l}\cos\theta$ Taking second differential of u w.r.t θ it resolves into :$\frac{d^2u}{d\theta^2} = -\frac{\epsilon}{l}\cos\theta$. Now combing equation of ellipse with α leads to -

$$\alpha = -h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = -h^2u^2\left(\left(-\frac{\epsilon}{l}\cos\theta\right) + \left(\frac{1}{l} + \frac{\epsilon}{l}\cos\theta\right)\right) = -\frac{h^2}{l}u^2.$$

Now that central acceleration varies inversely as square of the distance of revolving planet from the focus Sun, and itis $= -\frac{\mu}{r^2} = -\mu u^2$; here, μ is constant determined by all celestial bodies influencing net gravitational pull.. Equating this central acceleration to α leads to $\mu u^2 = \frac{h^2}{l}u^2$, or $h^2 = \mu l$, or $h = \sqrt{\mu l}$. Here, l is Semi-Latus-rectum of the elliptical orbit.

Moreover area of an ellipse is $= \pi ab$, here while a is major axis of the ellipse, b is the minor axis of the orbit. Thus equating the Two representation of the area of the elliptical orbit, together with Second property of ellipse, it leads to –

$$\frac{1}{2}hT = \pi ab; \text{ or} \frac{1}{2}\sqrt{\mu l}T = \pi ab \Rightarrow \frac{1}{2}\sqrt{\mu\left(\frac{b^2}{a}\right)}T = \pi ab \Rightarrow \frac{1}{2}\left(b\sqrt{\frac{\mu}{a}}\right)T = \pi ab \Rightarrow \frac{1}{2}\left(\sqrt{\frac{\mu}{a}}\right)T = \pi a \Rightarrow T^2 = \frac{4\pi^2}{\mu}a^3 \Rightarrow T^2 \propto a^3.$$

This is the mathematical justification of KTL.

It is pertinent to highlight academic requirement at school level is the knowledge of Kepler's Laws and not its derivation. Likewise, each concept has been deliberately stretched a bit beyond, but within the realm the knowledge of mathematics and physics of the target audience, and thus ignite a desire to understand nature in an out-of-box manner, a pre-requisite of an inquisitive mind and an aim of education.

Summary: *Analysis of varieties of problems, representing different situation involve concepts of Newton's Laws of Motion, Circular Motion, Work-power-energy, and conservation of energy and momentum. Many such situations are observed in real life. Examples drawn from real life experiences is to build a visualization and an insight into the phenomenon occurring around. A deeper journey into the problem solving would make integration and application of concepts intuitive. This is absolutely true for any real life situation which requires multi-disciplinary knowledge in skill for evolving solution. Thus, problem solving process is more a conditioning of the thought process, rather than just learning of a subject. Practice with wide range of problems is the only pre-requisite to develop proficiency and speed of problem solving, and making formulations more intuitive rather than a burden on memory. This intuitive way of thinking is an inseparable attribute of overall personality of a person.*

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