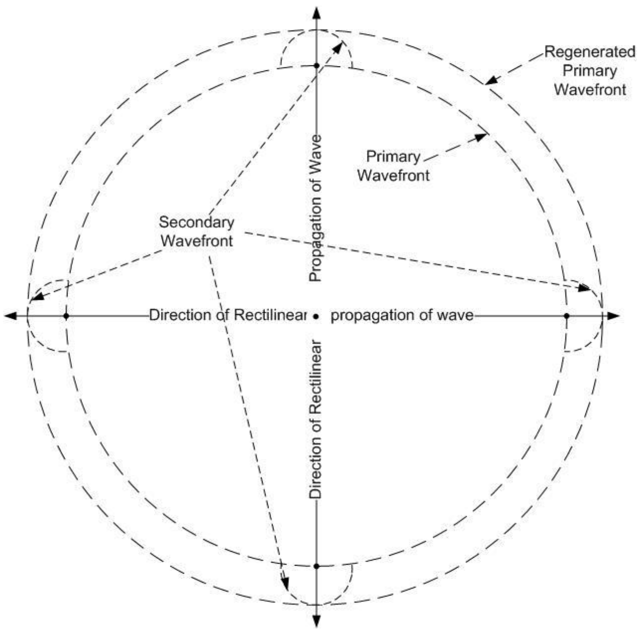
Chapter-VI: Waves and Motion

Discovery of correlation between matter and energy has revolutionized understanding of nature. Waves form an indispensable coupling between matter and energy and define boundaries of classical mechanics vis-à-vis relativistic and quantum mechanics. This chapter is, accordingly, positioned in this manual after classical mechanics and heat where existence of waves was introduced. Starting with concept of waves, this chapter is intended to integrate SHM, with concepts of sound and light, which are manifestation of waves in different frequency domain. While elaborating the subject matter electro-magnetic nature of waves is left untouched; it would be incomprehensible without knowledge of electro-magnetism and vector calculus.

Propagation of sound and light through a medium was initially considered to be motion of particles from source to destination. It was **Christian Huygens** in 1678 who proposed that rectilinear propagation of light, which was substantiated by **Augustine-Jean Fresnel** in 1816 with his own theory to explain phenomenon of interference in light. The ***Huygens Wave Theory*** **(HWT)** is explained with a set of postulates that – **a)** light travels like propagation of wave away from the source, **b)** the propagation is in the forms of a spherical wave-front in three dimensions in space. Wave travels with a uniform velocity in a homogenous medium, **c)** every point on the wave-front acts like a secondary source of wave and it perpetuates secondary wave-front, **d)** Envelop of Secondary wave-fronts regenerates new wave-front which propagates like the primary wave-front.



The concept of wave-front can be best visualized by throwing a stone in a pond or lake and then observing waves so generated propagate towards its bank. Postulates of wave propagation propounded by HWT successfully explains phenomenon of reflection and refraction. Further, concept of superimposition of waves is used to explain interference and diffraction phenomenon.

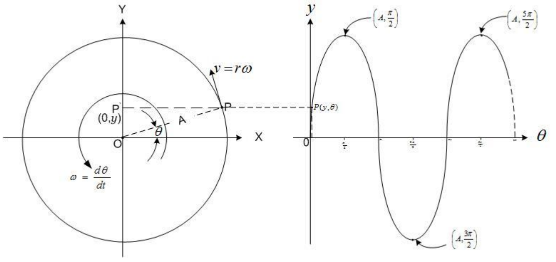
Generation of secondary wave-fronts is explained by taking Four points on a primary wave-front that has travelled some distance as shown in the figure. As we proceed into the journey, use of HWT shall be made while elaborating the above phenomenon.

Simple Harmonic Motion (SHM): This is a simple extension of mechanics and very useful in analysis of waves. It would be no exaggeration to state that the SHM is fundamental and most natural motion. *Any periodic motion or vibration, which is also called oscillation, can be explained with its constituent SHMs*. This was established by a mathematician ***Joseph Fourier*** *in 1807*, known as ***Fourier Analysis***. It is a subject matter of higher studies, and hence its elaboration at this point is limited to mere reference without much of details. Nevertheless, keen readers are welcome to connect through ***Contact Us***.

The SHM is best explained with trace of a particle, along a diameter of a circle, which is performing a uniform circular motion along perimeter of the circle i.e. with a constant angular velocity $\boldsymbol{\omega}$. It will be seen that in that **– a)** acceleration on the the particle performing SHM is always directed towards its mean position i.e. of equilibrium, and **b)** the acceleration is always proportional to the displacement of particle from its mean position. These two considerations form premises of SHM and are departure from Galileo's Equation of motion where acceleration was considered to be uniform.

This can be compared with motion of a pendulum or vibration in a spring for a real life visualization so as to appreciate SHM. In respect of oscillation of pendulum and spring, certain assumptions are involved, while motion of trace of a particle performing circular motion is an ideal SHM and trace of a particle corresponding to time is represented mathematically as $y = A\sin\omega t$, here y is displacement of particle from mean position at any instant of time t, A is amplitude i.e,. maximum displacement from mean position of particle during motion and ω is uniform angular velocity of the particle. Graphically SHM is with sinusoidal waveform which will be elaborated a little later. Accordingly the three types of motions are compared for analysis in the table below.

Motion of Trace of a Particle Performing Circular Motion



A particle is performing uniform circular motion of radius $\boldsymbol{A}$, at $\boldsymbol{\omega}$ rad/sec. If a parallel source of light is placed such that its rays cast a shadow on Y-axis. Thus at any instance displacement of the shadow from origin O, the mean position, on y-axis is : $y = A\sin\theta$

Velocity of the trace: $v = \frac{dy}{dt} = \frac{d}{dt}(\mathrm{A}\cdot\sin\theta)$

⇨ $A\cdot\frac{d}{d\theta}\sin\theta\cdot\frac{d\theta}{dr} = A\omega\cos\theta$

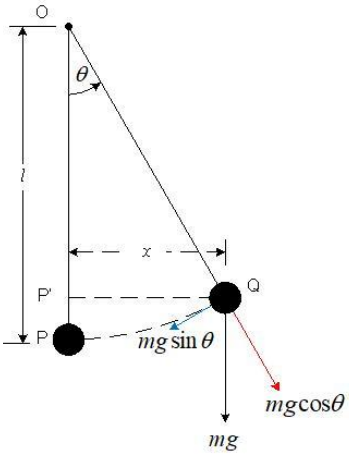
Acceleration of the particle: $a = \frac{dv}{dt} = \frac{d}{dt}(A\omega\cos\theta)$

⇨ $-A\omega^2\sin\theta = -\omega^2 y;\ a \propto y$

This too complies with both the premises of SHM i.e. (i) acceleration on the mass is proportional to the displacement, and (ii) acceleration is in direction opposite to the displacement.

Since this analysis of motion in this form does not involve any assumption, as done in case of pendulum and spring-mass system, it is ideal representation of SHM.

Oscillation of Pendulum



Length of Pendulum = l , and that of arc traced by it during oscillation is $PQ = l\theta$. When θ is small, or $x \ll l$,

$$\sin\theta \to \theta = \frac{\text{Arc } PQ}{l} \cong \frac{P'Q}{l}$$

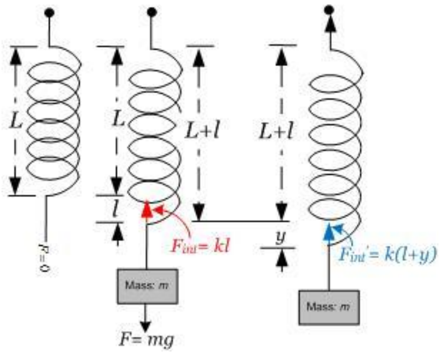
Force $\bar{F} = -mg\sin\theta = -mg\,\frac{P'Q}{l} = -mg\,\frac{x}{l}$

Acceleration of Mass m is $a = -\left(\frac{g}{l}\right)x$. It indicates that - **(i)** $a \propto x$ and **(ii)** (-)ve sign indicates that acceleration is in a direction opposite to the displacement

Both of these observations are in conformance with the premises of SHM

*This involves an **assumption** that* θ is small.

Oscillation of a Mass attached to Spring



Here for a spring of length L, its spring constant is k.

Expansion (l) of a spring caused by a mass m suspended from it is: $kl = mg$;

Internal force on stretching of the spring by additional length (y) : $F' = -(l+y)k$;

Net force on the mass m is :

$$F'' = F' + mg = ma$$

⇨ $ma = -(l+y)k + kl$

⇨ $ma = -ky;\ a = -\left(\frac{k}{m}\right)y;\ a \propto x$

This too complies with both the premises of SHM i.e. (i) acceleration on the mass is proportional to the displacement, and (ii) acceleration is in direction opposite to the displacement.

*This involves **assumption that** spring constant k remains uniform while elongation.*

*This analysis is being extended to determine Time Period **T**, and velocity of particle **v** at a particular displacement **y**, for circular motion and applied to oscillation of pendulum and spring*

*Displacement of particle from mean position is since a **Sine** function of angular displacement θ, it is also called a **Sinusoidal Wave**.*

Parameters of Oscillation

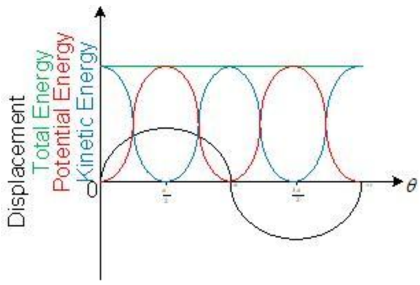
Circular Motion	Pendulum	Spring
Amplitude *-Maximum Displacement* : A	***Amplitude*** *-Max^m Disp.* : $A = l\sin\theta_{\max}$ *Max^m angular disp.:* $\theta_{\max}$ $\theta_{\max} << \Rightarrow \theta_{\max} \approx \sin\theta_{\max} \Rightarrow \theta_{\max} = \frac{A}{l}$	***Amplitude*** *-Max^m Disp.* : $A = y\sin\theta_{\max}$ *Max^m corredng angular disp.:* $\theta_{\max} = \frac{\pi}{2}$ $F = ky\sin\theta = ky\sin\omega t \Rightarrow F_{\max} = ky = kA$
Max^m acceleration *is at Max^m Disp.* : $\lvert a_{\max}\rvert = A\omega^2$	***Max^m accel^n*** *is at Max^m Disp.* : $\lvert a_{\max}\rvert = \frac{mg\sin\theta_{\max}}{m} = g\sin\theta_{\max} = \frac{gA}{l} = A\frac{g}{l}$	***Max^m accel^n*** *is at Max^m Disp.* : $\lvert a_{\max}\rvert = \frac{F_{\max}}{m} = \frac{k}{m}A = A\omega^2$
Max^m velocity *is at Mean position (y=0)* : $v_{\max} = A\omega$	***Max^m vel.*** *at Mean postn.* $(\theta = 0): v_{\max} = A\omega$	***Max^m vel.*** *at Mean postn.* $(\theta = 0): v_{\max} = A\omega$
Angular velocity : $\omega = \frac{\text{Acceleration}}{\text{Displacement}} = \sqrt{\frac{a_{max}}{A}}$; *Also,* $\omega = 2\pi f = \frac{2\pi}{T}$.	***Angular velocity*** :$\omega = \sqrt{\frac{g}{l}}$;	***Angular velocity*** : $\omega = \sqrt{\frac{k}{m}}$
Frequency of oscillation *(Cycles/Sec)*: $f = \frac{\omega}{2\pi}$	***Frequency of oscillation*** *(Cycles/Sec)*: $f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$	***Frequency of oscillation*** *(Cycles/Sec)*: $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$
Time period of oscillation : $T = \frac{1}{f} = \frac{2\pi}{\omega}$	***Time period of oscil^n***: $T = \frac{1}{f} = 2\pi\sqrt{\frac{l}{g}}$	***Time period of oscil^n***: $T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$
Angular displacement corresponding to Max Dis.: $\frac{\pi}{4}$	*Angular disp. corrpndg. to Max Dis.*: $\frac{\pi}{4}$	*Angular disp. corrpndg. to Max Dis.*: $\frac{\pi}{4}$

In the above table motion of pendulum and spring-mass system has been qualitatively compared with ideal SHM. It is possible the analytically derive expression of form of SHM from force-displacement relation $F_x = ma = m\frac{d^2}{dt^2}x = -kx \Rightarrow \frac{d^2}{dt^2}x = -\omega^2 x$ of the two systems, a linear differential equation of Second order, and is being derived. Multiply both sides by $2\frac{dx}{dt}$, the equation becomes $2\frac{dx}{dt}\cdot\frac{d^2}{dt^2}x = -2\omega^2 x\cdot\frac{dx}{dt} \Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -2\omega^2 x\cdot\frac{dx}{dt} \Rightarrow d\left(\frac{dx}{dt}\right)^2 = -2\omega^2 x\cdot dx$. On integrating the equation it leads to $\int d\left(\frac{dx}{dt}\right)^2 = -2\omega^2\int x\cdot dx \Rightarrow \left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + C$. At $x = 0 \Rightarrow v = \frac{dx}{dt} \to V_{\max} = A\omega$. Using this limiting condition $C = A^2\omega^2$. Accordingly, $\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + A^2\omega^2 \Rightarrow \left(\frac{dx}{dt}\right)^2 = \omega^2\left(A^2 - x^2\right) \Rightarrow \frac{dx}{dt} = \pm\omega\sqrt{A^2 - x^2}$. Thus the problem of Second Order Linear Differential Equation (SOLDE) has been reduced to First Order Linear Differential Equation (FOLDE). Integration of the latest form of equation is $\int\frac{dx}{\sqrt{A^2 - x^2}} = \pm\omega\int dt$. On substituting $x = A\sin\theta \Rightarrow dx = A\cos\theta d\theta$, the integration becomes $\int\frac{A\cos\theta}{A\sqrt{1-\sin\theta^2}}d\theta = \pm\omega t + \alpha \Rightarrow \theta = \pm\omega t + \alpha$. Reverting back to the original variable, $\sin^{-1}\frac{x}{A} = \pm\omega t + \alpha \Rightarrow x = A\sin\left(\omega t + \alpha\right)$, here α defines initial condition. Reduction of SOLDE to FOLDE has been made simple by mathematician using *D* operator, a subject matter of mathematics, an integral part of physics. Readers more keen to know about D operator to solve higher order differential equations may refer to books on Differential Equations or are welcome to reach us through ***Contact Us***.

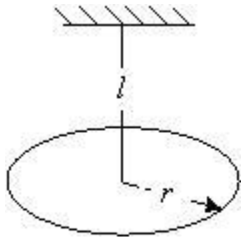
It is important to note that at mean position of particle performing SHM all forces are in equilibrium, yet the particle is at maximum velocity as per principle of conservation of energy. But, the particle around equilibrium position either while approaching or separating from it, experiences acceleration or retardation, respectively in accordance with Newton' Second Law of Motion.

Composition of Energy of a particle performing SHM: Taking that particle is performing SHM in frictionless environment, where there is exchange of energy with external systems. In such a situation energy of particle shall comprise of *Potential Energy (PE)* and *Kinetic Energy (KE),* and the two together shall constitute *Total Energy (TE)* of the Particle.

As per definition, $KE = \frac{1}{2}mv^2 = \frac{1}{2}m(A\omega\cos\theta)^2 = \frac{1}{2}mA^2\omega^2\cos^2\theta$, here $v = A\omega.\cos\theta$ is the instantaneous velocity of the particle of mass ***m,*** $PE = -\int_0^y m\cdot a\cdot dy$. It leads to $PE = -\int_0^y m\cdot(A\omega^2\sin\theta)\cdot dy$ since $y = A\sin\theta$, hence $dy = A\cos\theta d\theta$. For convenience limits shall be managed at the last step. Accordingly, $PE = \int_0^y m\cdot(A\omega^2\sin\theta)\cdot(A\cos\theta d\theta) = mA^2\omega^2\int_0^y \sin\theta\cos\theta d\theta = \frac{mA^2\omega^2}{2}\int_0^y \sin 2\theta\, d\theta = \frac{mA^2\omega^2}{4}\int_0^\theta \sin u\, du\Big|_{u=2\theta;d\theta=\frac{du}{2}}$. It leads to $PE = \frac{mA^2\omega^2}{4}[-\cos u]_0^\theta = -\frac{mA^2\omega^2}{4}[\cos 2\theta - 1] = -\frac{mA^2\omega^2}{4}[(1 - 2\sin^2\theta) - 1]$, where $PE = \frac{mA^2\omega^2}{2}\sin^2\theta$. Thus, $TE = PE + KE = \frac{mA^2\omega^2}{2}\sin^2\theta + \frac{mA^2\omega^2}{2}\cos^2\theta = \frac{mA^2\omega^2}{2}$. Accordingly, $TE = \frac{1}{2}mV_{max}{}^2$ here, $V_{max} = A\omega$, which corresponds to tangential velocity of particle performing uniform circular motion, while in case of oscillation of simple pendulum and spring it is velocity of particle at the mean position corresponding to $\theta = n\pi$, where, $n \in Z$ *n*, an element of set of Whole Numbers, PE is ZERO.

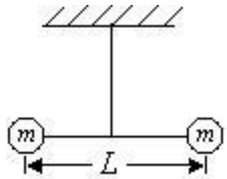


Oscillations are of various kinds: **a)** ***Free Oscillation***, where no external force is applied, e.g. trace of trajectory of motion of planets, satellites electrons in their orbit, along one of its axes **b)** ***Damped Oscillations***, whose amplitude depletes with passage of time e.g. a swing left unattended, **c)** ***Forced Oscillations***, like swing or clock where regular at regular interval, extra energy is supplied to make up energy lost in each oscillation, **d)** ***Resonant Oscillations***, these occur in a system when its natural frequency is an integral multiple of an oscillation present in the environment. This finds extensive application in musical instruments. **e)** ***Coupled Oscillation***, occur in a system which communicates, exchanges, energy with an external system when it is set into oscillation. This principle is widely used in sound box, speakers.



Torsional Pendulum: A shaft or a taught string or wire when twisted axially, it undergoes an elastic deformation which beyond yield point leading to shear of the material at its cross-section perpendicular to the axis. This deformation can be viewed as deformation of diametric-axial-plane twisted such that fixed end remains stationary and twist gradually increases towards free-end to its maximum value. Below this limiting condition for a small angular twist undergoes Simple Harmonic Motion. Such a system is called Torsional Pendulum as shown in the figure. In this system one end is fixed at ceiling and at its free-end hanging below the fixed end a uniform-rigid disc of radius r and mass m is suspended. Moment of inertia of the disc is $I = \frac{mr^2}{2}$. Let, k is the torsional constant of the suspension material. i.e. torque per unit angle deformation. Therefore, for an angular displacement of the disc by an angle θ the restoring torque exerted on the free end of the suspension varies linearly as $\tau = -k\theta$ and the torque in terms of rotational dynamics would create an angular acceleration such that $\tau = I\alpha$. Equating these two expressions of torques it leads to $I\alpha = -k\theta \Rightarrow \alpha = -\frac{k}{I}\theta$. This equation is comparable to that of SHM where $\omega^2 = \frac{k}{I} \Rightarrow \omega = \sqrt{\frac{k}{I}}$. Since, we have $\omega = \frac{2\pi}{T}$ and hence, $\frac{2\pi}{T} = \sqrt{\frac{K}{I}} \Rightarrow T = 2\pi\sqrt{\frac{I}{K}}$

An example below illustrates application of the concept of torsional pendulum:



Question: Two small balls, each of mass m, are connected by a light rigid rod of length L as shown in the figure. The system is suspended from its centre by a thin wire of torsional constant k. The rod is rotated about the wire through an angle θ_0 and released. Find the tension in the rod as the system passes through the mean position.

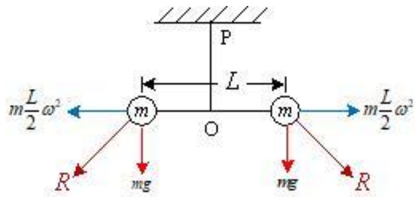
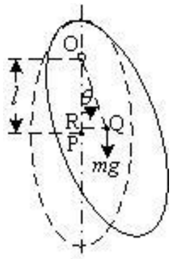


Illustration: Moment of Inertia of two small balls of mass m separated by a light rigid rod of length about its centre O is $I = m\left(\frac{L}{2}\right)^2 + m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{2}$…(1) Torsional energy stored in the suspension wire $TE = \frac{1}{2}k\theta^2$…(2) and when the rod passes through its mean position it will be converted into Kinetic Energy $KE = \frac{1}{2}I\omega^2$…(3) such that $\frac{1}{2}I\omega^2 = \frac{1}{2}k\theta^2 \rightarrow \omega = \sqrt{\frac{k}{I}}\theta$…(4).

The centripetal force on the rod would be $F_c = m\left(\frac{L}{2}\right)\omega^2 \rightarrow F_c = m\left(\frac{L}{2}\right)\left(\frac{k}{I}\right)\theta^2 = m\left(\frac{L}{2}\right)\left(k \times \frac{2}{mL^2}\right)\theta^2 = \frac{k}{L}\theta^2$. In addition gravitational force of the balls is $F_g = mg$. It is seen from the figure that both F_c and F_g are orthogonal and both the balls are attached to the rod, while being symmetrical to the wire. Hence, resultant force on the rod that supports the balls is $R = \sqrt{F_c^2 + F_g^2} = \sqrt{\left(\frac{k}{L}\theta^2\right)^2 + (mg)^2} = \sqrt{\frac{k^2}{L^2}\theta^4 + m^2g^2}$. **Thus answer is** $\sqrt{\frac{k^2}{L^2}\theta^4 + m^2g^2}$

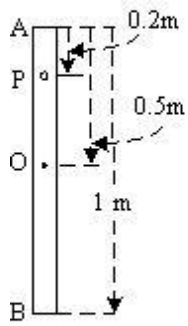
N.B.: Since magnitude of resultant motion is always (+) ve hence correct representation in radical form and not in exponential form.



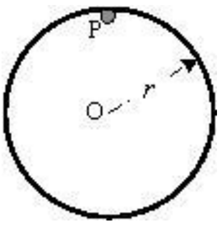
Physical Pendulum: This is case of oscillation of physical bodies and can be conceptualized from the figure shown here. Let a body of mass m and moment of inertia (MOI) I is hanging from a point O such that it is above P, the centre of gravity (CG), by a length l . When it is hanging in a steady state its centre of gravity (CG) is at point P. When body is set into oscillation and is inclined by an angle θ with the vertical line passing through O, it will experience a torque $\tau = RQ_{length} \times mg = -mgl\sin\vartheta$.

Considering the situation from the point of view of rotational dynamics $\tau = I\alpha$. Combining the two expressions if torque τ we get $I\alpha = -mgl\sin\vartheta$. Since for small amplitude oscillation when $\sin\theta \to \theta\big|_{\theta<<}$. Accordingly, $I\alpha = -mgl\theta$, it leads to $\alpha = -\frac{mgl}{I}\theta$. The angular acceleration of the physical body in its final form is comparable to translational SHM where $a = -\omega^2 x$, where $a \to \alpha$ and $x \to \theta$ and, therefore, $\omega^2 = \frac{mgl}{I} \to \omega = \sqrt{\frac{mgl}{I}} = \frac{2\pi}{T}$. It leads to $= 2\pi\sqrt{\frac{I}{mgl}}$. Accordingly, for a physical body $\alpha = \omega^2\theta$ …(5) and time period of SHM of any physical body is $T = 2\pi\sqrt{\frac{I}{mgl}}$. …(6).

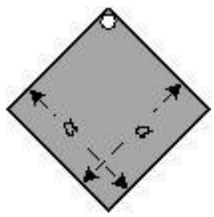
This concept has been applied to find time period of oscillation of a physical pendulum in typical cases as shown below :



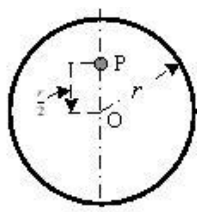
Case (a) Uniform Bar: The bar of mass m and length $L = 1$m is supported at point P. Moment of inertia about its centre O is $I = \frac{mL^2}{12} = \frac{m}{12}$. Since the bar is hanging from point P, above by $l = 0.5 - 0.2 = 0.3$m and hence moment of inertia of the bar about P, by parallel axis theorem is $I_P = I + ml^2$, it simplifies into $I_O = \frac{m}{12} + \frac{9m}{100} = \frac{52m}{300}$, therefore time period would be $T = 2\pi\sqrt{\frac{\frac{52m}{300}}{m\times10\times0.3}}$. It reduces to $T = 2\pi\sqrt{\frac{\frac{52m}{300}}{m\times10\times0.3}} = 2\pi\sqrt{\frac{52}{900}} = 1.51$s. **Hence $\boldsymbol{T = 1.51}$ is 1.51 s.**



Case (b) Circular Ring: A circular uniform ring of radius r and mass m shall have moment of inertia about its centre O is $I = mr^2$ and therefore its MI about point of hanging P is $I_P = I + mr^2 = 2mr^2$. Therefore, its time period would be $T = 2\pi\sqrt{\frac{2mr^2}{mgr}} = 2\pi\sqrt{\frac{2r}{g}}$. **Hence answer of part (b) is $\boldsymbol{T = 2\pi\sqrt{\frac{2r}{g}}}$ s.**



Case (c) Square Plate: In case of a square plate of dimension $a \times a$ having mass m hung from one of its corner. Using perpendicular axis theorem MI about its center O is $\boldsymbol{I = \frac{ma^2}{12} + \frac{ma^2}{12} = \frac{ma^2}{6}}$. Accordingly, MA about the point P shall be $I_P = \frac{ma^2}{6} + m\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{2}{3}ma^2$. Therefore, time period of SHM shall be $T = 2\pi\sqrt{\frac{\frac{2}{3}ma^2}{mg\left(\frac{a}{\sqrt{2}}\right)}}$. **Accordingly, $\boldsymbol{T = 2\pi\sqrt{\frac{\sqrt{8}a}{3g}}}$ s.**



Case (d) Uniform Circular Disc: The uniform circular disc is taken to be of mass m and radius r. The disc is hung at point P on its surface such that the disc will swing like a pan across its surface unlike that in case (b) above. Accordingly, MI about point P is $I_P = \frac{I_O}{2} + m\left(\frac{r}{2}\right)^2 = \frac{mr^2}{2} + \frac{mr^2}{4} = \frac{3mr^2}{4}$. And hence time period $T = 2\pi\sqrt{\frac{\frac{3mr^2}{4}}{mg\frac{r}{2}}} = 2\pi\sqrt{\frac{3r}{2g}}$. **Thus, $\boldsymbol{T = 2\pi\sqrt{\frac{3r}{3g}}}$ s.**

N.B.: In this problem MI of an object about different points of its plane have been very nicely articulated.

Composite SHM: This case of composite SHM is explained through an example given below -

Example: Assume that a tunnel is dug across the earth (radius = R) passing through its centre. Find the time a particle takes a cover the length of the tunnel if-

(a) It is projected into the tunnel with a speed of $\sqrt{gR}$,

(b) It is released from a height R above the tunnel

(c) It is thrown vertically upward along the length of tunnel with a speed of $\sqrt{gR}$.

Illustration: Each case is separately illustrated below :

In case (a) particle is projected into the tunnel with a speed of $v_a = \sqrt{gR}$.

In case (b) the particle when released from a height R above the tunnel i.e, at a distance $2R$ from the COM of the earth reduction in potential energy is $\Delta PE = \frac{GMm}{R} - \frac{GMm}{2R} = \frac{GMm}{2R} = \frac{mgR}{2}$. Let v is the velocity of the particle when it reaches tunnel then change of kinetic energy would be $\Delta KE = \frac{1}{2}mv_b^2 - 0 = \frac{1}{2}mv_b^2$. Accordingly, As per Law of Conservation of Energy $\frac{1}{2}mv_b^2 = \frac{mgR}{2} \rightarrow v_b = \sqrt{gR}$.

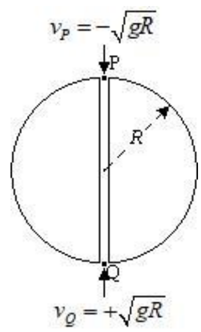
In case (c) the particle is projected vertically upward with velocity$\sqrt{gR}$ and therefore, as per Law of Conservation of Energy when it enters the tunnel it will have a downward velocity $v_c = \sqrt{gR}$, it is similar to that of the case (a).

Thus it is seen that velocity of the particle entering the tunnel in each case $v_a = v_b = v_c = v = \sqrt{gR}$…(1) Rest of the problem in each case is same as the tunnel is same.

Acceleration due to gravity at earth's surface $= \frac{GM}{R^2} = \frac{G}{R^2}\left(\frac{4}{3}\pi R^3 \rho\right) = \frac{4}{3}\pi G\rho R$. Since acceleration due to gravity at any point inside earth at a distance x from the earth's center is due to mass inside the sphere of radius x and not the shell outside it and hence on similar lines $g_x = \frac{4}{3}\pi G\rho x$. Accordingly,$\frac{g_x}{g} = \frac{\frac{4}{3}\pi G\rho x}{\frac{4}{3}\pi G\rho R} = \frac{x}{R} \rightarrow$ $g_x = \frac{g}{R}x$.

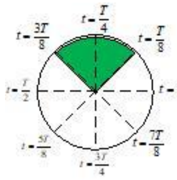
This quantitative relationship can be written with directional sense as $g_x = -\frac{g}{R}x$…(2). Since acceleration vector g_x is toward the centre of the earth while vector x is radially outward. This equation can be compared with characteristic equation of SHM where $a_x = -\omega^2 x$…(3). Comparing equation (2) and (3) $\omega^2 = \frac{g}{R} \rightarrow \omega = \sqrt{\frac{g}{R}}$ and time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}$…(4)

This is an interesting case of composite motion in Two parts – ***(a)*** *motion inside the tunnel where acceleration is directly proportional to the displacement from mean position and is always directed toward the mean position and therefore SHM,* ***(b)*** *motion above the earth's surface where acceleration is inversely proportional to the distance from mean position and is always directed toward the mean position and therefore it follows inverse square law. This situation is converted into an* ***equivalent SHM*** *by evolving* ***velocity-displacement equation (VDE)*** *using the data of motion of the particle along the diametric tunnel.*



In SHM $x = A\sin\omega t$, and velocity $v_x = \frac{dx}{dt} = A\omega\cos\omega t = A\omega\sqrt{1-\sin^2\omega t} \rightarrow v_x = A\omega\sqrt{1-\left(\frac{x}{A}\right)^2} = \omega\sqrt{A^2 - x^2}$…(5); this is being called **VDE** in simple harmonic motion.

At $x = R$, equation (5) leads to $v = \omega\sqrt{A^2 - R^2}$ accordingly using values of v and ω derived above it leads to $\sqrt{gR} = \sqrt{\frac{g}{R}}\left(\sqrt{A^2 - R^2}\right) \rightarrow R^2 = A^2 - R^2 \rightarrow A = \sqrt{2}R$. Applying VDE in the instant case $v = A\omega\cos\omega t \rightarrow v = \left(\sqrt{2}R\sqrt{\frac{g}{R}}\right)\cos\left(\sqrt{\frac{g}{R}}t\right) = \sqrt{2gR}\cos\left(\sqrt{\frac{g}{R}}t\right)$.

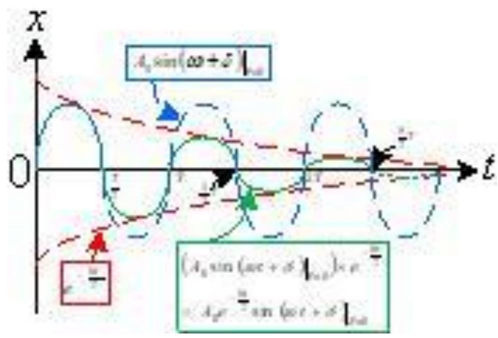


Accordingly, we have $v_p = A\omega\cos\omega t_p \rightarrow -\sqrt{gR} = \sqrt{2gR}\cos\omega t_p \rightarrow \cos\omega t_p = -\frac{1}{\sqrt{2}}$ .or $\omega t_p = \pi \mp \frac{\pi}{4} \rightarrow \frac{2\pi}{T}t_p = \frac{5}{4}\pi, \frac{3\pi}{4} \rightarrow t_p = \frac{3}{8}T, \frac{5}{8}T$. Likewise, $v_q = A\omega\cos\omega t_q$, it leads to $\sqrt{gR} = \sqrt{2gR}\cos\omega t_q \rightarrow \cos\omega t_q = \frac{1}{\sqrt{2}}$, or $\omega t_p = \mp\frac{\pi}{4} \Rightarrow \frac{2\pi}{T}t_p = \pm\frac{\pi}{4} \Rightarrow t_p = \pm\frac{\pi}{8}$.

Thus, $\Delta t = t_q - t_p = \frac{3}{8}T - \frac{1}{8}T = \frac{1}{4}T \rightarrow \Delta t = \frac{1}{4}\left(2\pi\sqrt{\frac{R}{g}}\right) = \frac{\pi}{2}\sqrt{\frac{R}{g}}$., has been depicted graphically. **Hence answer is** $\frac{\pi}{2}\sqrt{\frac{R}{g}}$.

Damped Harmonic Oscillation: Simple harmonic motions with constant amplitude discussed above are realized in conservative systems. In these systems loss of energy in the process is zero. Such systems are ideal. Yet, experiences of an oscillating pendulum coming to rest, unless it is powered, and likewise any oscillating or vibrating object coming to rest is not uncommon. This process of an oscillating object is called **damping**, and such oscillations are called **Damped Harmonic Oscillations.** It may also be observed that faster the damping higher is velocity. Accordingly, this damping, i.e. retardation, effect is represented in equation of SHM by a $-bv$ which is proportional to velocity, here b is damping constant and $v = \frac{dx}{dt}$ is velocity of the oscillating mass. Thus equation of SHM in its modified form is becomes $\frac{d^2x}{dt^2} = -kx - bv$ and its solution requires handling second order linear differential equation, which is deferred for the present, yet readers inquisitive to know about its mathematical solution $x = A_0 e^{-\frac{bt}{2m}}\sin(\omega t + \delta)$where $\omega' = \sqrt{\left(\frac{k}{m}\right) - \left(\frac{b}{2}\right)^2} \Rightarrow \omega' = \sqrt{(\omega_0)^2 - \left(\frac{b}{2}\right)^2}$. Here, m is the oscillating mass and $\omega_0 = \sqrt{\frac{k}{m}}$ is same and as that in ideal SHM. Damped oscillation is represented graphically. Readers inquisitive to know more about this derivation are requested to *Contact Us*.

Typical observations of damped oscillations are as under –



(a) Time period of each oscillation remain unchanged and determined by $\omega_0 = \sqrt{\frac{k}{m}}$ which leads to $\omega_0 = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega_0} \Rightarrow T = 2\pi_0\sqrt{\frac{m}{k}}$, since k and m are characteristic parameters of the system.

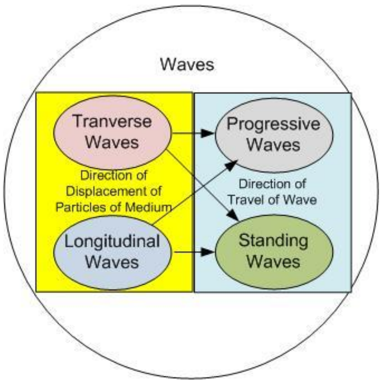
(b) It is different than superimposition of two waves. The resultant wave is product of SHM $x = A_0\sin(\omega t + \delta)$and damping factor $e^{-\frac{bt}{2m}}$ and is represented as $x = A_0 e^{-\frac{bt}{2m}}\sin(\omega t + \delta) \Rightarrow x = [A_0\sin(\omega t + \delta)] \times e^{-\frac{bt}{2m}}$.

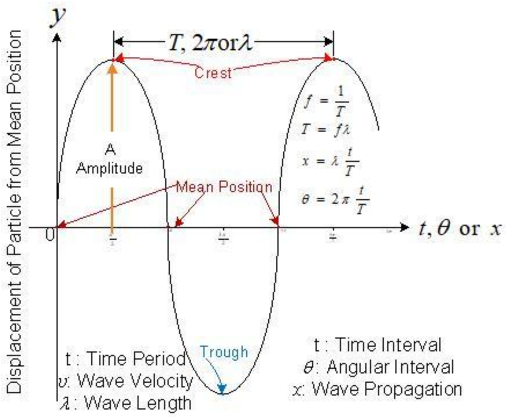
(c) Displacement in each oscillation, both within an oscillation and across successive oscillations decreases with factor with passage of time by a factor $e^{-\frac{bt}{2m}}$.

(d) Rate of reduction in displacement is dependent on velocity $v = \frac{dx}{dt}$ of the damping mass $e^{-\frac{bt}{2m}}$ and thus it corroborates inclusion of $-bv$ in equation of damping harmonic oscillations.

Damping oscillations is a reality, and study of SHM is incomplete without understanding of damping effect.

Waves: Understanding of the SHM is the study of oscillation of a single particle, and is elemental in elaboration of wave, which involves medium to either for its existence or its propagation. *Electromagnetic waves* are exception as they can travel in vacuum, and shall be dealt with separately. Classification of waves, based on direction of motion of particles, is in two categories: **a) *Longitudinal Wave*** - where displacement of particles of medium is about their mean position are in the direction of the wave. And **b) *Transverse Wave*** - where displacement of particles of the medium is perpendicular to the direction of the wave. Further, consideration in classification of the waves is propagation of wave, which can occur both of the categories based on displacement of particles of the medium. According to this classification: **i) *Travelling Waves*** - in which motion of every particle is perpetuated to the adjoining particle of the medium along the direction of propagation, **ii) *Standing waves*** – it is a result of interaction of forward and backward travelling wave, such that all particles of the medium, at any point of time, *are in same phase*, but their amplitude depends upon their position along the wave. While, each of these types of wave is characteristically different in respect of motion of particles of the medium of propagation, and shall be studied with its mathematical and graphical representation in the form of SHM. Basic concepts of waves are common to Sound wave and Light wave. Accordingly, these concepts are considered a prerequisite to the understanding of phenomenon of Sound Waves and Light Waves, and shall be elaborated before going into Sound and Light waves, to develop an integrated perspective of the two. Accordingly, journey in the subject matter has been structured and is in line with the approach of this Mentors' Manual.





Time period (*T*): *It is the time taken to complete One Cycle.*

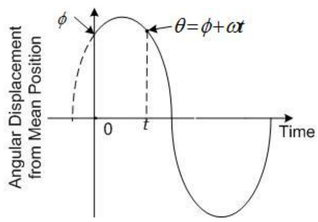
Frequency (*f or* υ) : *It is number of cycles in One Second, it is related to* $T = \frac{1}{f}$.

Crest : *It is the point on wave where displacement of particle from mean position is maximum*.

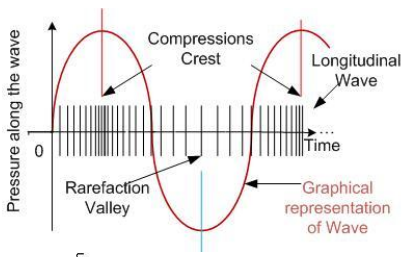
Trough : *It is point on wave where displacement of particle from mean position is minimum i.e. maximum in opposite direction.*

Wavelength (λ) : *It is distance covered by wave in one cycle. Most conveniently is recorded as distance between two consecutive peaks, as shown in the figure. The latter definition goes equally well with standing waves.*

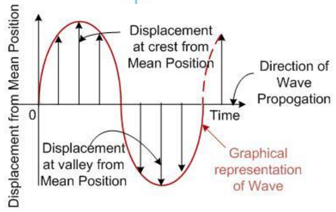
Velocity of wave (*v*) : *It is distance covered by wave in One second, and* $v = \lambda f = \frac{\lambda}{T}$.



Phase (θ): *It is the angular displacement of a particle in a wave from its initial mean position, and mathematically* $\theta = \omega t$*. This repeats after every* 2π *angular displacement and corresponds to time duration T. In case a particle in a wave, initially displaced from its mean-position by an angle* ϕ*, is set into SHM then its phase after a lapse of time t is* $\theta = \emptyset + \omega t$ *and is shown in the figure.*

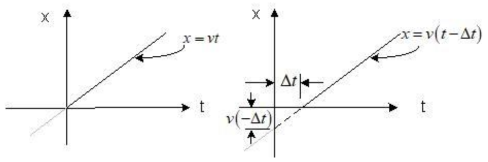


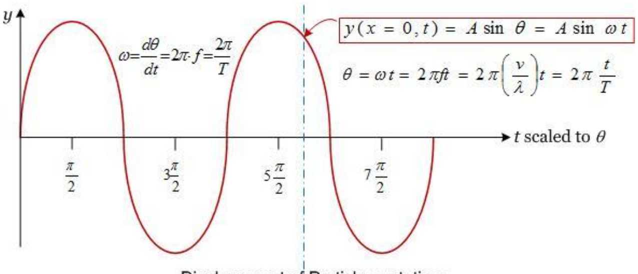
Longitudinal Wave: In a wave if *particles of medium oscillate, about their mean position, along the direction of wave then it is called longitudinal wave. These oscillating particles create compression ad rarefaction as shown in the figure*. These waves are also represented graphically as Sinusoidal Wave as shown in the figure. These waves are realized in rattling sound of doors and windows during a thunderstorm. Sound waves are basically Longitudinal Waves.



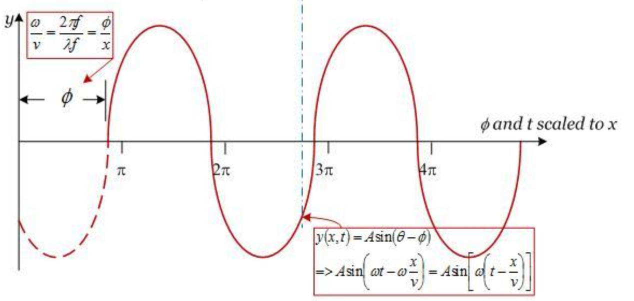
Transverse Wave: In *this type of wave particles of medium oscillate about their mean position, in a direction perpendicular to the direction of wave*. Waves generated in a water pond by dropping a stone are transverse waves. Likewise, all string-based musical instruments produce transverse waves; so are the Light waves also.

Travelling Waves: During elaboration of SHM consider oscillation of particles about its mean position and mathematically represented with a sinusoidal function. Though it is enough to explain the oscillation, but it is insufficient to represent oscillations that travel from one point to other, called waves, which carries or transfer energy from source to receiver or destination. This is elaborated with a simple straight line function $x = vt$. Here, ***x*** is the displacement of particle, from its initial position, at any instance ***t***, and rate of change of displacement $v = \frac{dx}{dt}$ which corresponds to the slope of the line shown in the X-t graph . This line of displacement represents distance travelled by wave at any point of time ***t***. In relativistic terms it is identical to the source moved through a distance $\Delta x = m \cdot \Delta t = v \cdot \Delta t$. In this new situation displacement of the particle from its mean position is identical to that having started from initial displacement $-\Delta t$. Accordingly, as per knowledge of Coordinate geometry displacement of particle at any instance is analogous to that at an instance $(t - \Delta t)$.

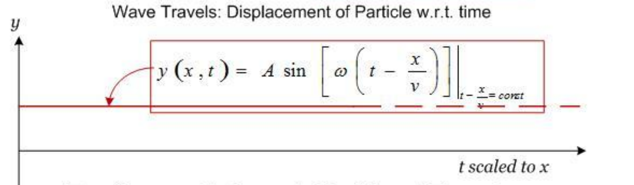




Displacement of Particle w.r.t. time



Wave Travels: Displacement of Particle w.r.t. time



Travelling wave: Displacement of Particle w.r.t. Time and Distance, Mutually Variable such that: $t - \frac{x}{v} = const$

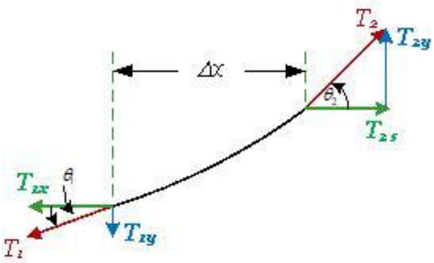
This logic shall be extended to elaborate travelling SHM called ***travelling or progressive waves*** *expressed as* $y = f(x,t)$ and is elaborated in the figure. First graph shows displacement (y) during oscillation from its mean position $y = A\sin\theta$, here $\theta = \omega t = 2\pi f t = \frac{2\pi}{T}t$ at any instant of time. Next is the wave taken to be moving along X-axis through a distance x corresponding to a phase angle ϕ in time t. Accordingly, displacement of a particle from its mean position, at a distance from the reference point, sat source, from its mean position, in accordance with the above example can be represented with a graph below where $y = A\sin[\omega(t - \Delta t)] = A\sin\left[\omega\left(t - \frac{x}{v}\right)\right]$. Taking, variables ***x*** and ***t*** such that $\left(t - \frac{x}{v}\right)$ remain constant, the displacement ***y*** shall also remain constant. This implies that with passage of time displacement is travelling forward, while particles of medium keep oscillating about their mean position. This is shown in the second graph. The third inference is about progressive displacement of a particle of medium from its mean position while both ***x*** and ***t*** are changing. It will be seen that when $\left(t - \frac{x}{v}\right)$ remains constant, displacement remains constant, that is with passage of time ***t***, displacement moves forward along ***x*** with velocity ***v***. This nature of travelling wave is shown in the third graph.

Fourth characteristic of travelling wave comes from its periodicity. At any point on the passage of the displacement of a particle of medium from mean position repeats at an interval $T = \frac{2\pi}{\omega} = \frac{1}{f}$, here T - is called ***Time Period*** and it corresponds to angular displacement 2π to complete one oscillation, characteristic to sine function. Likewise, at any instance of time along the passage of time the displacement of particle from its mean position repeats at an interval of $\lambda = \frac{v}{f} = vT$, here λ - is called ***Wavelength***, i.e. distance covered in one Oscillation.

In a wave travelling forward i.e. along ***x***-axis –ve sign appears with $\frac{x}{v}$ and accordingly a general expression of a travelling/progressive wave is $y(x,t) = A\sin\left[\omega\left(t - \frac{x}{v}x\right)\right] = A\sin\left[\omega\left(t - \frac{x}{v}\right)\right]$. If the wave is travelling in a direction along (**–*x***) axis, automatically the equation shall take the form $y(x,t) = A\sin\left[\omega\left(t - \frac{(-x)}{v}x\right)\right] = A\sin\left[\omega\left(t + \frac{x}{v}\right)\right]$. Thus, progression of wave perpetuates with time. **This is general expression of a progressive wave** which represents displacement of a particle in the medium as a function of time and position from the source.

This expression is being extended to a differential equation of with time and position varying phenomenon known as Wave Equation. Accordingly, rate of change of displacement w.r.t. time of particle of the medium from its mean position i.e. velocity of particles of the medium performing SHM is expressed as $v_{y-t} = \frac{\partial}{\partial t} y(x,t) = \omega A \cos\left[\omega(t - \frac{x}{v})\right]$ it is at any position x . Likewise, rate of change of displacement w.r.t. position of the particle of the medium on the line of propagation of the wave, at any instance of point of time t is as $p_{y-x} = \frac{\partial}{\partial x} y(x,t) = \frac{\omega}{v} A \cos\left[\omega(t - \frac{x}{v})\right]$. It is to be noted with a caution that this velocity of the particle $\left(v_{y-t}\right)$ and the velocity of travelling wave (v) are different. Taking it forward, second order of both the derivatives, acceleration of the particle w.r.t time and position works out to $a_{y-t} = \frac{\partial^2}{\partial t^2} y(x,t) = -\omega^2 A \sin\left[\omega(t - \frac{x}{v})\right]$and $q_{y-x} = \frac{\partial^2}{\partial x^2} y(x,t) = -\frac{\omega^2}{v^2} A \sin\left[\omega(t - \frac{x}{v})\right]$, respectively. This leads to ratio of the two accelerations $\frac{a_{y-t}}{q_{y-x}} = \frac{\frac{\partial^2}{\partial t^2} y(x,t)}{\frac{\partial^2}{\partial x^2} y(x,t)} = \frac{=-\omega^2 A \sin\left[\omega(t - \frac{x}{v})\right]}{-\frac{\omega^2}{v^2} A \sin\left[\omega(t - \frac{x}{v})\right]} = v^2$. It is most convenient to express a dynamic process in the form of a differential equation. Accordingly, $\boldsymbol{\frac{\partial^2}{\partial t^2} y(x,t) = v^2 \left(\frac{\partial^2}{\partial x^2} y(x,t)\right)}$, and in its complementary form as $\boldsymbol{\frac{\partial^2}{\partial x^2} y(x,t) = \frac{1}{v^2} \left(\frac{\partial^2}{\partial t^2} y(x,t)\right)}$, called ***Wave Equation***. *Discovery of One Dimensional Wave Equation by* **Jean le Rond D' Alembert**, *in 1746, followed by* **Leonhard Euler** *Three Dimensional Wave Equation within a decade, was a great leap in discovery of physical systems and processes. Discovery of wave equation later helped to generalize transfer of energy through wave right from mechanical vibrations to sound and electromagnetic radiation.* ***Here, analysis is confined to One Dimensional waves.***

Velocity of Wave: In this wave equation velocity of wave is a parameter which rationalizes acceleration of particles of medium w.r.t. time and position from the source. In strings based musical instruments transverse waves are established. While in gases, waves are where longitudinal. In both the cases, velocity of waves is governed by different phenomenon and are being elaborated separately.



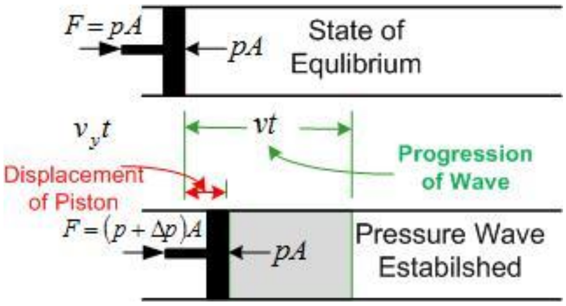
Velocity of Wave in String: Strings are so made that their mass per unit length (μ) is uniform and is valid in normal state of rest. When string is set to transverse wave, along its length, non-uniform extension will take place and this influences uniformity of μ. Looking at graphical representations of waves it might be perceived that theoretically rigidity of a metal string is significant, but in reality it is quite small and for all practical purposes it is considered to be uniform. Now it needs to be explored as to how Tension (T) and μ play role in velocity of wave. Consider an infinitesimal element of string Δx having tensions T_1 and T_2 at its Two ends, which goes in to decide shape of the waveform. Since, the wave propagation is transverse and hence lateral displacement, velocity and accelerations of the particles of the element of string along length shall not exist, Accordingly, T_{1x}=T_{2x} and it complies with Newton's Third Law of Motion. But, the nature of wave demanding transverse motion of particles of string will utilize difference in transverse components of tension to cause an acceleration such that $\Delta T = T_{2y} - T_{1y} = (\mu \cdot \Delta x) \frac{\partial^2 y}{\partial t^2} \dots (1)$, in accordance with the Newton's Second Law of Motion. Looking at the tensions over element Δx of the string, $T_{2y} - T_{1y} = T_2 \sin\theta_2 - T_1 \sin\theta_1$. For the infinitesimal length approximation $T_1 = T_2 = T$ and $\sin\theta_1 = \tan\theta_1 = \frac{T_{y1}}{T_{x1}}$ likewise $\sin\theta_2 = \tan\theta_2 = \frac{T_{2y}}{T_{2x}}$, being $\theta_1 <<$ and so also $\theta_2 <<$. Thus,, $\Delta T = T_{2y} - T_{1y} = T\left(\frac{T_{2y}}{T_{2x}} - \frac{T_{1y}}{T_{1x}}\right) \rightarrow \Delta T = T\left(\frac{\left(\frac{\partial y}{\partial x}\Big|_{x\to x+\Delta x} - \frac{\partial y}{\partial x}\Big|_{x\to x}\right)}{\Delta x}\right)\Delta x = T \cdot \Delta x \frac{\partial^2 y}{\partial x^2} \dots (2)$

Combining equation (1 & 2) we get $\mu \Delta x \frac{\partial^2 y}{\partial t^2} = T \Delta x \frac{\partial^2 y}{\partial x^2} \Rightarrow \frac{\frac{\partial^2 y}{\partial t^2}}{\frac{\partial^2 y}{\partial x^2}} = \frac{T}{\mu} \dots (3)$, While, as per Wave Equation, $\frac{\frac{\partial^2}{\partial t^2} y(x,t)}{\frac{\partial^2}{\partial x^2} y(x,t)} = v^2 \dots (\mathbf{4})$,

Now, combining equation (3 & 4) we get $v^2 = \frac{T}{\mu} \Rightarrow v = \sqrt{\frac{T}{\mu}}$ is the equation of velocity of wave in a string. This is also expressed as $v = \sqrt{\frac{Y}{\rho}}$ based on dimensional equality of $\sqrt{\frac{T}{\mu}} = \sqrt{\frac{Y}{\rho}}$.

Understanding of waves in strings, as seen in musical instruments, is with its ends fixed called ***Node***, which has no motion, irrespective of the type of wave be it transverse or longitudinal. Therefore, definition of wave where $v = \lambda f$. The number of Nodes between the fixed ends having length of wire (L) would decide pitch length and in turn frequency of wave. In case there are no nodes between the fixed ends, the length of wires it constitutes half pitch length $L = \frac{\lambda}{2}$. Accordingly,$v = 2L \cdot f = \sqrt{\frac{T}{\mu}}$,. It leads to natural frequency of vibration of string as $f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$.

Velocity of Wave in Fluids: Progression of wave in fluid is conceptualized in One Dimension in the figure, where travel of piston at velocity $\boldsymbol{v_y}$ in time $\boldsymbol{t}$ establishes longitudinal pressure wave in fluid which travels a distance vt, in corresponding time, such that velocity of wave is v. Beyond the distance of travel of wave, medium remains at equilibrium state. Considering, bulk elasticity of the medium ($\boldsymbol{B}$), it leads to $B = \frac{\Delta p}{\frac{v_y t}{vt}} = \frac{v\Delta p}{v_y}$. Accordingly, it leads to $\Delta p = B\frac{v_y}{v} => \Delta F = A\Delta p = B\frac{Av_y}{v} \dots (1)$, and per Newton's Second Law of Motion impulse $\Delta F = m\Delta v_y = \rho(Av)v_y \dots (2)$, Combining equation (1) and (2) we get $B\frac{Av_y}{v} = \rho(Av)v_y$, or $\boldsymbol{v} = \sqrt{\frac{\boldsymbol{B}}{\boldsymbol{\rho}}}$.



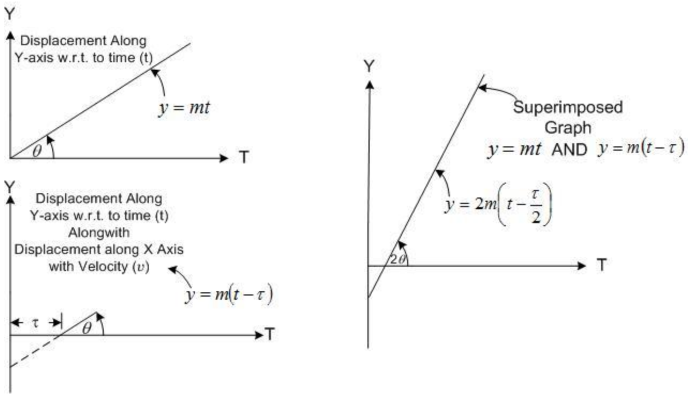
Velocity of Wave in Gas: Gases are highly compressible as compared to liquids**.** Therefore, *Newton* assumed that temperature of gas remains constant and accordingly used Boyle's Law pv = Const. to investigate velocity of wave in gases. Differentiating Boyle's Equation as :$p\partial v + V\partial p = 0$, or $B = -\frac{\partial p}{\left(\partial V/_V\right)} = p$. Using this value of B, Newton redefined velocity of wave in gases as $= \sqrt{\frac{p}{\rho}}$**;** this is known as ***Newton's Equation of Velocity of Wave in Gases.*** This equation is comparable to velocity of wave in strings.

In case of gases which are highly compressible, and the velocity of wave is quite high, the progression of pressure wave is an Adiabatic process where medium has no time to exchange heat with the environment, either during compression or rarefaction. Accordingly, in this case instead of Boyle's Law equation, and comply with $\boldsymbol{pV^{\gamma} = \mathrm{Constant}}$ as per ***Poisson's Law***, covered in Heat and Thermodynamics. Thus Laplace, suggested a correction based on Poisson's Law whose logarithm is $\log p + \gamma \log V = Const.$ Differentiating this log-equation w.r.t. $\boldsymbol{t}$ leads to $\frac{1}{p}\frac{dp}{dt} + \frac{\gamma}{V}\frac{dV}{dt} = 0$, or $B = \frac{dp}{dV/_V} = \gamma p$. Accordingly, the corrected equation $v = \sqrt{\frac{\gamma P}{\rho}}$ is known as **Newton-Laplace Equation** ***of velocity of wave in gases***.

Energy and Power in Wave: In wave represented by $y(x,t) = A\sin\left(\omega t - \frac{\omega}{v}x\right)$. In a string tension is always along the its length and string is considered to be flexible enough to shape its every infinitesimal length Δx corresponding to tensions T_x and T_y. Accordingly, $\frac{\partial y}{\partial x} = \frac{T_y}{T_x} = -\frac{A\omega}{v}\cos\left(\omega t - \frac{\omega}{v}x\right)$. Accordingly, $T_y = -\frac{TA\omega}{V}\cos\left(\omega t - \frac{\omega}{v}x\right)$, here $T_x = T$ which is uniform along the length of string. Thus, instantaneous power $P(x,t) = \frac{dW}{dt} = T_y\frac{\partial y}{\partial t} = T\left[A\left(\frac{\omega}{v}\right)\cos\left(\omega t - \frac{\omega}{v}x\right)\right] \cdot \left[A\omega\cos\left(\omega t - \frac{\omega}{v}x\right)\right]$. It leads to $P(x,t) = \frac{TA^2\omega^2}{v}\cos^2\left(\omega t - \frac{\omega}{v}x\right) = \frac{TA^2\omega^2}{v}\left[\frac{1 - \cos 2\left(\omega t - \frac{\omega}{v}x\right)}{2}\right]$. It comprises of Two components, One is $\frac{1}{2} \cdot \frac{TA^2\omega^2}{v}$ a constant and it contains parameters characteristic to wave and independent of variables $\boldsymbol{x}$ and $\boldsymbol{t}$. And, the other is $\frac{1}{2} \cdot \frac{TA^2\omega^2}{v}\left[\cos 2\left(\omega t - \frac{\omega}{v}x\right)\right]$ having a cosidal trigonometric function as a coefficient of constant component; it is time and place variant which averages to Zero over a cycle. Thus, average power of wave $\boldsymbol{P_{av}}$ is represented as $P_{av} = \frac{1}{2} \cdot \frac{TA^2\omega^2}{v} = \frac{\mathbf{1}}{\mathbf{2}} \cdot \frac{\boldsymbol{TvA^2\omega^2}}{\boldsymbol{v^2}} = \frac{\mathbf{1}}{\mathbf{2}} \cdot \boldsymbol{\mu v A^2 \omega^2}\Big|_{v=\sqrt{\frac{T}{\mu}}}$. This expression of average power is be represented in terms of frequency (ν) as $\boldsymbol{P = 2\pi^2 \mu v A^2 \nu^2}$.

Principle of Superimposition of Waves: A simple case of an object moving vertically with a constant velocity in an inertial frame of reference is shown in the figure as (y,t) graph where $y_1 = mt$. In the graph below, another object moves vertically with the same constant velocity, but after a lapse of time *τ*. Considering motion of second object as a wave, a continuous function $y_2 = m(t - \Delta t)$ in (y,t)graph and it is plotted for $t > 0$. Summation of the two functions $y = y_1 + y_2 = mt + m(t - \tau) = 2m\left(t - \frac{\tau}{2}\right)$,in another (y,t) graph is superimposition of two functions and is best represented mathematically.

On similar lines two wave functions $y_1 = A_1 \sin\left[\frac{2\pi}{\lambda_1}(v_1 t - x)\right]$ and $y_2 = A_2 \sin\left[\frac{2\pi}{\lambda_2}(v_2 t - x)\right]$ are considered for superimposition of waves. It could lead to multiple cases where in the Two wave functions with different – **a)** Amplitudes (A), **b)** Wave lengths (λ), **c)** velocity (v), and **d)**wave travel during initial phase shift. In most of the problems of wave superimposition that are encountered at this stage are for two waves with identical, Amplitude, velocity and frequency or wave length and accordingly simplistic mathematical analysis of waves travelling in opposite directions such as: $y = A\sin\frac{2\pi}{\lambda}(vt + x) + A\sin\frac{2\pi}{\lambda}(vt - x) =>$



$\left[A\sin\frac{2\pi}{\lambda}(vt + x) + \sin\frac{2\pi}{\lambda}(vt - x)\right]$. Using the trigonometric identities it reduces into $y = 2A\cos\left(\frac{2\pi x}{\lambda}\right)\sin\left(\frac{2\pi vt}{\lambda}\right) = A_x \sin\left(\frac{2\pi vt}{\lambda}\right)$. Here, Amplitude of wave function at every point along the pitch is $A_x = 2A\cos\left(\frac{2\pi x}{\lambda}\right)$. And, displacement of each particle (y) from its mean position at any instant (t) is in same phase $y \propto \sin\left(\frac{2\pi vt}{\lambda}\right)$, where proportionality constant is A_x. This is a special case of ***Standing wave or Stationary wave*** and finds extensive application in *Sound Waves*. Further, *analysis of superimposition shall be dealt with as resonance of sound waves in strings and air column. This is also applicable in analyzing reflection, refraction, interference and diffraction of waves; this is common to both sound and light waves*; it is elaborated in Part II of this section of Mentors' Manual..

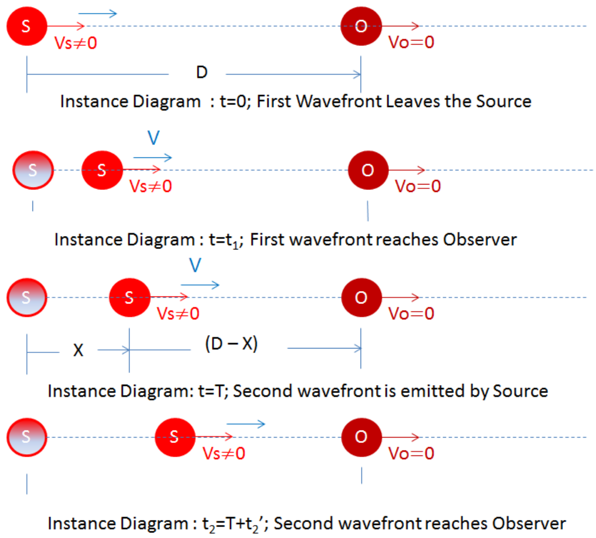
A generic analysis of periodic wave function was suggested by **Joseph Fourier**, in 1807, in the form of a series of sinusoidal functions :$A_x = \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cdot \sin\left(\frac{2\pi nx}{T} + \varphi_n\right)$. Here, parameters of the waveform are, A_0 – is the bias from mean position of the periodic waveform, A_1- is the amplitude of the sinusoidal waveform of frequency as that of the periodic waveform; this is called fundamental frequency (f),A_n – is the amplitude of sinusoidal wave form of frequencies multiple of fundamental frequency (nf) and are called harmonics, n- is called the order of harmonic, and φ_n – is the phase shift from the initial in respect of each harmonic. Determination of these parameters of the frequencies constituting a non-sinusoidal periodic function was suggested by Fourier and is known as ***Fourier Analysis***; this is inverse of superposition of sinusoidal waveforms. Elaboration of Fourier Analysis is outside the scope of this document, nevertheless, *readers are welcome to raise their inquisitiveness through **Contact Us***.

Doppler Effect in an Inertial Frame : Shrill of a train while arriving at platform and while leaving is different. Likewise, shrill of a horn of a Train being chased by a vehicle is different than that experienced on a platform. This is being analysed in three different cases; **a)** Source moving towards a stationary Observer, **b)** Observer moving away from a stationary source, **c)** Both Source and Observer moving in one direction, with Observer ahead of Source. The results of the analysis in three cases have been generalized, at the end.

Case 1: Source moving towards a stationary Observer

Standard notations that are being used in the analysis, elaborated in figure, are as under-

V- Velocity of Sound ;*Vs*– Velocity of Source ≠ 0; *Vo* – Velocity of Observer =0,
T – Time period of Sound ; *t* - an instance of the analysis,
t_1– an instance when **First wave-front** emitted by source at *t*=0 reaches observer at a distance $= \frac{D}{V}$.
X – distance moved by Source during $= T \cdot Vs$, when Source emits **Second wave-front**
t_2'- is the time taken **Second wave-front** to reach the Observer $= \frac{D-X}{V}$
$t_2^{''}$ is the instance when second wave-front emitted by source at $t = T$ reaches observer at a distance X and is $t_2^{''} = T + t_2^{'}$
f- Frequency of Sound; *f'* – Apparent Frequency of Sound



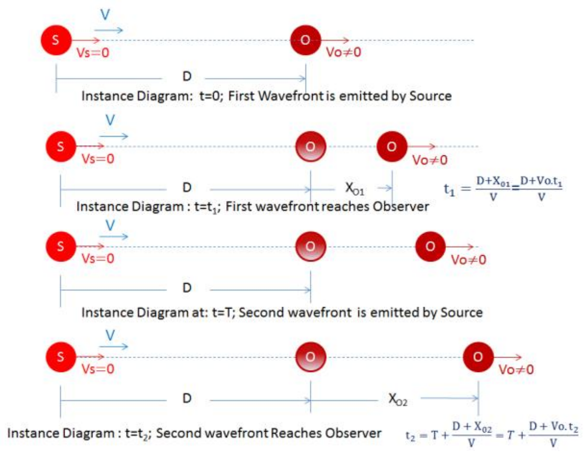
Therefore, effective Time Period for the Observer-

$$T' = t_2'' - t_1 = (T + t_2') - t_1 = \left(T + \frac{D-X}{V}\right) - \frac{D}{V} = \left(T + \frac{D - TVs}{V}\right) - \frac{D}{V} = T\left(1 - \frac{Vs}{V}\right)$$

Hence, apparent frequency : $f' = \frac{1}{T'} = \frac{1}{T\left(1 - \frac{Vs}{V}\right)} = f \cdot \frac{V}{V - Vs}$

Inference: *f* ' is equal to (*f*)x(Ratio of Velocity of sound w.r.t. Observer to Velocity of Sound w.r.t Source

Case 2: An Observer moving away from a stationary source:



Vs=0 and *Vo*≠ *0*, and is elaborated in figure.

From the above-

$t_1\left(1 - \frac{Vo}{V}\right) = \frac{D}{V}$; $t_1 = \frac{D}{V - Vo}$; and $t_2\left(1 - \frac{Vo}{V}\right) = T + \frac{D}{V}$; $T_2 = \frac{TV + D}{V - Vo}$

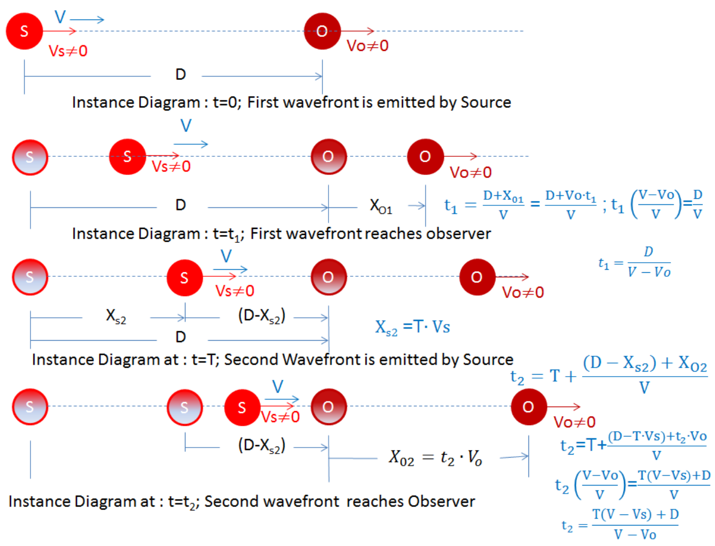
In this case Apparent Time Period for the Observer is –

$t' = t_2 - t_1 = \frac{TV}{V - Vo}$; or $f' = \frac{1}{t'} = \frac{1}{T}\left(\frac{V - Vo}{V}\right) = f\left(\frac{V - Vo}{V}\right)$

Inference: *f* ' is equal to (f)x(Ratio of Velocity of sound w.r.t. Observer to Velocity of Sound w.r.t Source ***(Same as in case 1)***

Case 3: Both Source and Observer moving in one direction, with Observer ahead of Source

Figure below specifies each instance and specially t1 and t2 when First and Second Wave-front emitted by Source reach Observer, respectively alongwith relationships of related variables, as shown in the figure. Accordingly, apparent Time Period would be -



$$T' = t_2 - t_1 = \frac{T(V - Vs) + D}{(V - Vo)} - \frac{D}{(V - Vo)} = T\frac{(V - Vs)}{(V - Vo)}$$

Or, $f' = \frac{1}{T'} = \frac{1}{T} \cdot \left(\frac{V - Vo}{V - Vs}\right) = f . \left(\frac{V - Vo}{V - Vs}\right)$

Inference : *f* ' is equal to (*f*) multiplied by (Ratio of Velocity of sound w.r.t. Observer to Velocity of Sound w.r.t. Source ***(Same as in case 1).*** *It is to be noted that for all velocities reference direction is from source towards the observer; accordingly all velocities on reference direction are (+)ve and all velocities against the reference are (-)ve.*

Effect of Wind Velocity in Doppler's Effect: Since propagation of sound requires a medium. Hence if wind is blowing in reference direction i.e. from source towards the observer then $V \to V' = V + V_w$ and if it in reverse direction then $V \to V' = V - V_w$. The reason behind this is that wind, the medium, being carrier of the sound wave and thus effective velocity of sound is $V' = V \pm V_w$, depending upon the direction of the wind, Accordingly, frequency of sound perceived by the observer is $f' = f.\left(\frac{V' - Vo}{V' - Vs}\right)$.

General Inference on Doppler's Effect : *In case of source and/or observer moving, apparent frequency to the observer is natural frequency of source multiplied by ratio of relative velocity of sound w.r.t. Observer to the relative velocity of sound w.r.t. the Source.* Manifestation of Doppler Effect in Light is change of Colour, called Doppler Shift and shall be elaborated in Part-II of this Chapter in Mentors' Manual.

Summary: *Initially the concepts of waves discussed above are applicable in analysis of Sound Waves and Light Waves, with distinct boundary of frequencies. Accordingly, these concepts shall be used to elaborate commonalities in respect of various phenomena like reflection, refraction, interference, diffraction and polarization common to light and shall be included in later Parts of this Chapter. Light waves are a narrow part of electromagnetic waves, which is outside scope of this manual. Nevertheless, readers are welcome to raise their inquisitiveness through **Contact Us**.*

Examples have been drawn from real life experiences which help to build visualization and an insight into the phenomenon occurring around. A deeper journey into the problem solving would make integration and application of concepts intuitive. This is absolutely true for any real life situations, which requires multi-disciplinary knowledge, in skill for evolving solution. Thus, problem solving process is more a conditioning of the thought process, rather than just learning the subject. Practice with wide range of problems is the only pre-requisite to develop proficiency and speed of problem solving, and making formulations more intuitive rather than a burden on memory, as much as overall personality of a person. References cited below provide an excellent repository of problems. Readers are welcome to pose their difficulties to solve any-problem from anywhere, but only after two attempts have been made to solve it. It is our endeavour to stand by upcoming student in their journey to become a scientist, engineer and professional, whatever they choose to be.

Going forward, these typical problems from contemporary text books and Question papers from various sources are being progressively developed and shall be uploaded as supplement to the respective Chapters of Mentors' Manual. It is a dynamic exercise to catalyse the conceptual thought process.

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