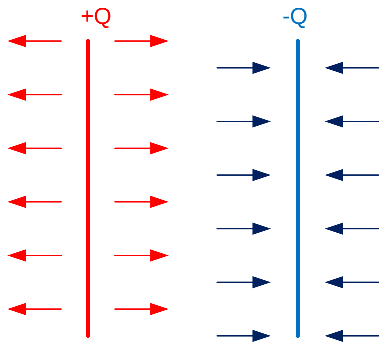
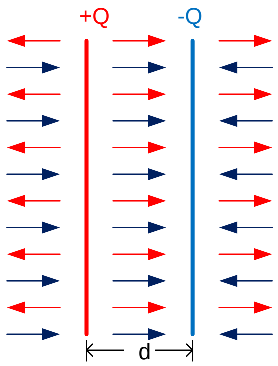
ELETROSTATICS –Part II: BASICS

Presence of charge on any object creates an electrostatic field around it and in turn an electrical potential is experienced around the object. This phenomenon has found application in storage of charge on a pair of electrodes called Capacitors. Analysis of properties of Capacitor has been greatly simplified with the use of Gauss's. as shown below.

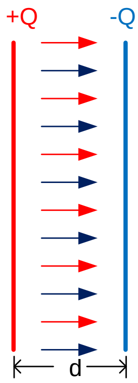
Consider an isolated large conducting plate carrying charge +Q, it will establish an electric field $\oint E.ds = 2EA = \frac{Q}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$; or $E = \frac{\sigma}{2\epsilon_0}$. Here, E is the uniform electric field intensity on both sides of the charged plate, while σ is charge density per unit area on the plate. Likewise, electric field due to an isolated large plate carrying charge –Q would be $E = -\frac{\sigma}{2\epsilon_0}$. When these two plates are brought sufficiently closer to each other, such that their separation is sufficiently smaller than their area, electric field in the inter-plate spaces shall be additive, while beyond he plates they would be subtractive (opposite) in nature. Accordingly, net electric field between equal and oppositely charged plates shall be $E = 2 \times \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$. Such a combination of equal and oppositely charged parallel plates is called ***Parallel Plate Capacitors***.



Electric Field Due to Isolated +ve and –ve Charged Plates



Electric Field When +ve and –ve Charged Plates at Distance d



Net Electric Field Due to Oppositely Charged Plates (Capacitor)

Properties of Capacitors:

Equipotential Surface: Capacitor has two charge carrying surfaces of conducting material. Electric field drives +ve charge to move in the direction of field and –ve charges against the direction of the field. Moreover, +ve charge emanates electric from the surface and –ve charge received electric field into the surface. Accordingly, charge inside a conductor moves onto its surface, which serves as equipotential surface, with uniform charge density.

Voltage Difference between Parallel Plates of Capacitor (*line integral of Electric Field along line of separation between parallel plates of the capacitor***) =** $\boldsymbol{V = \int_0^d -E.dx = -E\int_0^d dx = -Ed}$ Volts

Capacitance: Substituting Value of E in the Voltage $V = \frac{\sigma}{\epsilon_0}d = \frac{\sigma Ad}{\epsilon_0 A} = \frac{Qd}{\epsilon_0 A} = \frac{Q}{C}$; here, $C = \frac{\epsilon_0 A}{d}$; or $Q = CV$ Coulomb. Here, C is called Capacitance, in Farads.

Force Between Plates of Capacitors: Since capacitor is always oppositely charged and hence each of plates will experience a force of attraction due to the electric field created by the other plate Electric field in the inter plate space is uniform and is the combined effect of charges on both the plates, hence charge on one plate will find it self placed in the field $(=\frac{\sigma}{2\epsilon_0})$ of the other plate will experience a force $= \frac{\sigma}{2\epsilon_0}Q = \frac{(\sigma A)}{2\epsilon_0 A}Q = \frac{Q^2}{2\epsilon_0 A}$ Newton.

Energy Stored in Capacitor: There are two approaches to determining energy stored in a capacitor.

Method 1: Displacement of plate experiencing a force $\left(=\frac{Q^2}{2\epsilon_0 A}\right)$ through a distance d will cause a work which will be stored as energy inside capacitor. It is equal to : $\left(W = \int_0^d -F.dx = \int_0^d \left(-\frac{Q^2}{2\epsilon_0 A}\right).dx = -\frac{Q^2}{2\epsilon_0 A}d = -\frac{1}{2}\frac{Q^2}{C} = -\frac{1}{2}\frac{(CV)^2}{C}\right) = -\frac{1}{2}CV^2$

Method 2: It is based on building of incremental charge and is equal to :

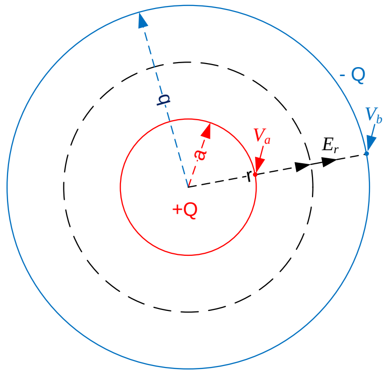
$$W = \int dw = \iint_0^V (-dq)\,(dv) = \iint_0^V (-Cdv)\,(dv) = -C\iint_0^V (dv)\,(dv) = -C\int_0^V vdv = -\frac{1}{2}CV^2$$

Enhancing Properties of Capacitors: Capacitors have extensive utility. Ideal parallel plate capacitor s, which require extremely large surface area and very small separation between plates. These two conditions pose practical limitation in their use to meet its application requirement. Accordingly, there are three approaches to enhance the capability of capacitors viz**: a)** Configuration, **b)** Combination of Capacitors and **c)** use of Dielectric in the inter electrode region. Each of these

A) Determination of Capacitance of Electrode in Different Configurations:

i) Spherical Capacitor

a. Concentric Hollow Spheres : Field outside a inner conducting sphere, of radius a, carrying charge $+Q$, at a distance r from its centre is



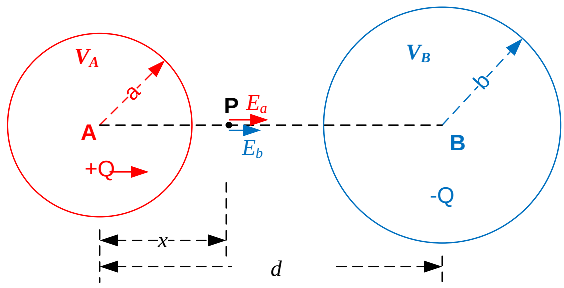
$E = \frac{Q}{4\pi\epsilon_0 r^2}$. All the field emanating from inner sphere would end on –Q charge on the inner surface of the radius b. Therefore, potential difference between Two Sphere is $V = \int_b^a -Edr = -\int_b^a \frac{Q}{4\pi\epsilon_0 r^2}dr = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r}\right]_b^a = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{Q}{4\pi\epsilon_0}\frac{(b-a)}{ab} = \frac{Q}{C}$.

Accordingly, Capacitance of Spherical capacitors is $C = \frac{4\pi\epsilon_0 ab}{(b-a)}$

It leads to Two cases –

- Outer sphere has a radius b>>a then , $C = \frac{4\pi\epsilon_0}{a}$. It behaves like an isolated sphere
- Radius of inner sphere $a \to b$; then, $C = \frac{4\pi\epsilon_0 a^2}{d} = \frac{\epsilon_0 A}{d}$. It behaves like parallel plate capacitor

b. Oppositely Charged Isolated Spheres – This configuration creates non-uniform electric field. And hence analysis of the problem shall be done by determining electric field on a point P on the line joining centre of the Two spherical isolated opposite charges. The net electric field $\bar{E} = \bar{E}_a + \bar{E}_b$. Here, $\bar{E}_a = \frac{Q}{4\pi\epsilon_0 x^2}\hat{x}$ and $\bar{E}_a = \frac{Q}{4\pi\epsilon_0 (d-x)^2}\hat{x}$, where $\hat{x}$ is the unit vector along line AB. Since, the two spheres are of conducting material and hence each, independently, would be equipotential surface, and path for calculating V, can be chosen based on mathematical convenience. Accordingly potential difference between Two Spheres shall be:

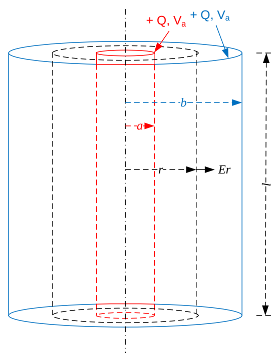


$V = -\int_{d-b}^{a}\bar{E}\cdot d\bar{x} = -\frac{Q}{4\pi\epsilon_0}\int_{d-b}^{a}\left(\frac{1}{x^2}+\frac{1}{(d-x)^2}\right)dx = -\frac{Q}{4\pi\epsilon_0}\left[\int_{d-b}^{a}\frac{1}{x^2}dx + \int_{d-b}^{a}\frac{1}{(d-x)^2}dx\right] = -\frac{Q}{4\pi\epsilon_0}[I_1 + I_2]$. Here, integrating by parts $(I_1 + I_2)$ where, $I_1 = \left[-\frac{1}{x}\right]_{d-b}^{a} = \left[\frac{1}{d-b}-\frac{1}{a}\right]$ and I_2 is done using substitution $u = d - x$. It leads to $dx = -du$ and, further, replacing limits. As $x \to a, u \to d-a$ and $x \to d-b, u \to b$. Accordingly, $I_2 = \int_b^{d-a}\frac{1}{u^2}(-du) = \left[\frac{1}{u}\right]_b^{d-a} = \left[\frac{1}{d-a}-\frac{1}{b}\right]$. Re-substituting the two integrals:

$V = -\frac{Q}{4\pi\epsilon_0}\left[\left[\frac{1}{d-b}-\frac{1}{a}\right]+\left[\frac{1}{d-a}-\frac{1}{b}\right]\right] = \frac{Q}{4\pi\epsilon_0}\left(\left(\frac{1}{a}+\frac{1}{b}\right)-\left(\frac{1}{d-a}+\frac{1}{d-b}\right)\right) = \frac{Q}{4\pi\epsilon_0}\left(\frac{a+b}{ab}+\frac{2d-(a+b)}{(d-a)(d-b)}\right)$. Accordingly, capacitance the Two isolated spheres $C = \frac{4\pi\epsilon_0}{\left(\frac{a+b}{ab}+\frac{2d-(a+b)}{(d-a)(d-b)}\right)}$. If $a = b$, and $d \gg a$, then $C = \frac{4\pi\epsilon_0}{\left(\frac{2}{a}+\frac{2}{d}\right)} = 2\pi\epsilon_0\frac{ad}{a+b}$ Farad.

ii) Cylindrical Capacitor

a. Concentric Hollow Cylinders: In this electric field at distance r from the surface of the inner cylinder is $E_r = \frac{Q}{2\pi\epsilon_0 lr}$. Accordingly potential difference between the Two concentric, oppositely charged cylinders is:



$V = \int_b^a -E_r.dr = -\frac{Q}{2\pi\epsilon_0 l}\int_b^a \frac{1}{r}.dr = -\frac{Q}{2\pi\epsilon_0 l}[\ln r]_b^a = \frac{Q}{2\pi\epsilon_0 l}\ln\left(\frac{b}{a}\right) = \frac{Q}{C}$.

Therefore, capacitance of cylindrical capacitor, C= $\frac{2\pi\epsilon_0 l}{\ln\left(\frac{b}{a}\right)}$.

b. Isolated Parallel Cylinders: In this case cross-sectional diagram of the two isolated cylinders shall be same as that for the Two isolated spheres. The difference is created in mathematical formulation due to electric field at a point having a radial distance from the parallel cylinders. Like the case of isolated spheres, here electric field ($\bar{E}$) and V are calculated based on mathematical convenience. Accordingly, $\bar{E} = \bar{E}_a + \bar{E}_b$. Here, $\bar{E}_a = \frac{Q}{4\pi\epsilon_0 x}\hat{x}$ and $\bar{E}_a = \frac{Q}{2\pi\epsilon_0(d-x)}\hat{x}$, where $\hat{x}$ is the unit vector along line AB. Accordingly potential difference between Two Spheres shall be:

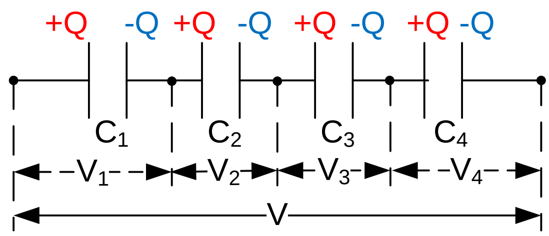
$$V = -\int_{d-b}^{a} \bar{E}\cdot d\bar{x} = -\frac{Q}{2\pi\epsilon_0}\int_{d-b}^{a}\left(\frac{1}{x}+\frac{1}{d-x}\right)dx = -2\frac{Q}{4\pi\epsilon_0}\left[\ln x - \ln(d-x)\right]_{d-b}^{a} = \frac{Q}{2\pi\epsilon_0}\left[\ln\frac{d-x}{x}\right]_{d-b}^{a}$$

$$= \frac{Q}{2\pi\epsilon_0}\left[\ln\frac{d-a}{a} - \ln\frac{b}{d-b}\right]$$

$= \frac{Q}{2\pi\epsilon_0}\ln\left(\frac{(d-a)(d-b)}{ab}\right) = \frac{\sigma l}{2\pi\epsilon_0}\ln\left(\frac{(d-a)(d-b)}{ab}\right)$. Here, σ is the charge density in Coulomb per metre and l is length of the cylinders such that $l \gg d$.Accordingly, for the Two parallel cylinders capacitance $C = \frac{2\pi\epsilon_0}{\ln\left(\frac{(d-a)(d-b)}{ab}\right)}$ Farad/m. If $a = b$, and $d \gg a$, then $C = \frac{2\pi\epsilon_0}{\ln\left(\frac{d^2}{a^2}\right)} = \frac{\pi\epsilon_0}{\ln\left(\frac{d}{a}\right)}$ Farad/m.

B) Combination of Capacitors: Capacitors are used either in parallel, series combination, and/or both. Each of the combination has its specific utility.

i) Series Combination of Capacitors: This combination is shown below. In this, each of the pair of plates of a capacitor carries equal charge, as characteristic to capacitor When a +Q charge is given to the leftmost plates of the capacitor C_1, it induces –Q charge on the opposite plate of the capacitor C_1. This induced charge is caused by the electric field produced by the +Q charge. This induction leaves +Q charge on the left plate of capacitor C_2. Likewise, each of the pair of connected plate also carries equal and opposite charge. This is attributed to the fact that there is no flow or exchange of charge outside the capacitor, except in the extreme leftmost and rightmost plates. These, extreme plates transfer –ve charge through the source. The -ve charges can flow easily due to their inertia being very small, this leaving a +ve charge behind. This will become more clear as journey of the study proceeds into current electricity.



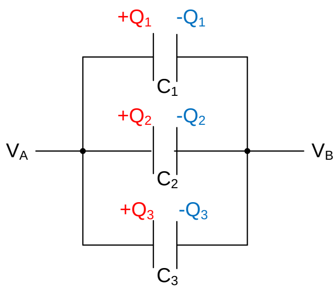
Now according to standard relationship, $Q = C_1V_1 = C_2V_2 = C_3V_3 = C_4V_4$.. Moreover, $V = V_1 + V_2 + V_3 + V_4 = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_2} + \frac{Q}{C_3}$.

Let C is the equivalent capacitance of the series combination of capacitors shown above. Then, $V = \frac{Q}{C} = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \frac{1}{C_4}\right)$.

This can be generalized into an expression, $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \ldots\ldots + \frac{1}{C_n}$; or, $C = \frac{\prod_{i=1}^{i=n} C_i}{\sum_{i=1}^{o=n} C_i}$.

Series capacitors are used when voltage to be applied is larger than capacity of any individual capacitor

ii) Parallel Combination of Capacitors: In this plates connecting +ve Charge and –ve Charge are connected together through a conductor. Accordingly, all plates carrying +ve charge shall be at the same potential, likewise all plates carrying –ve charge shall be at same potential. Nevertheless, charge on each pair of plates of a capacitor shall be regulated by capacitance of each capacitor. Thus,



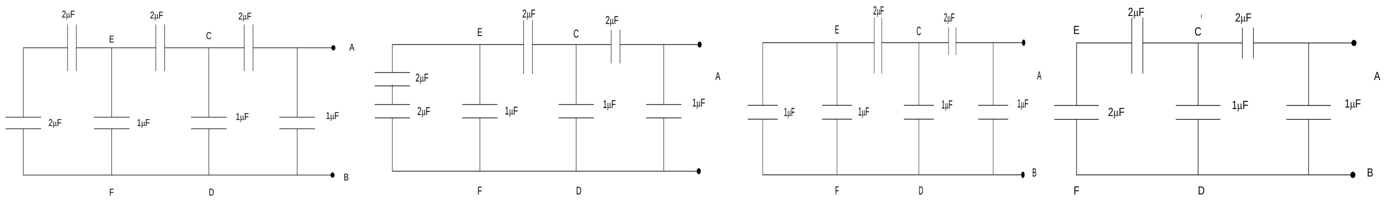
$$V = V_A - V_B = \frac{Q_1}{C_1} = \frac{Q_2}{C_2} = \frac{Q_3}{C_3}.$$

Thus, total +ve charge, on the on the parallel combination of capacitors is:

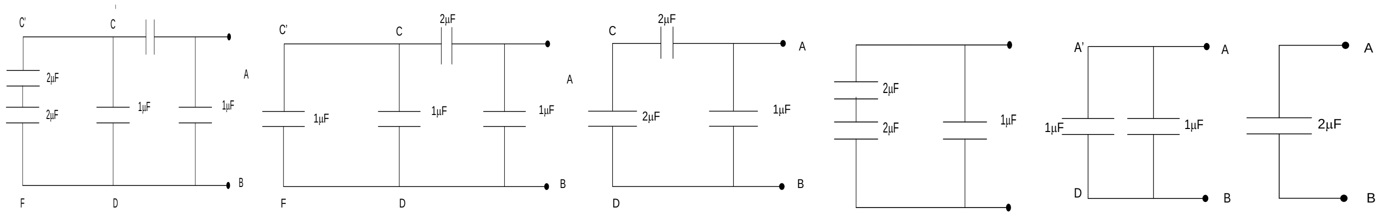
$Q = CV = Q_1 + Q_2 + Q_3 = C_1V + C_2V + C_3V = (C_1 + C_2 + C_3)V$. Thus it can we concluded that an equivalent capacitance of a parallel combination of capacitors is: $C = C_1 + C_2 + \ldots C_n = \sum_{i=1}^{n} C_i$.

Parallel capacitors are used when charge to be handled is greater than the capacity of any of the capacitor.

iii) Series-Parallel Combination: It can be any combination involving capacitor connected in series and parallel in any sequence and order. *This involves decomposition of a complex combination of capacitors into simplest of series and parallel combinations,* *and then integrating these decomposed elements into the mail solution*. To each of these simple combinations, above concepts are applied. It is illustrated through a stage-wise solution in an example below where equivalent capacitance across A-B is to be determined.



Stage 0 (Given Problem) Stage 1(Solution) Stage 2 (Solution) Stage 3 (Solution)

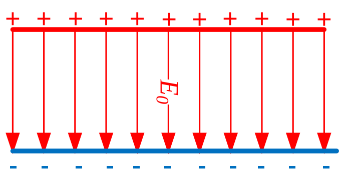


Stage 4 (Soln.) Stage 5 (Soln.) Stage 6 (Soln.) Stage 7 (Soln.) Stage 8 (Soln.) Final (Soln.)

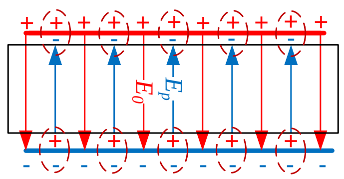
C) Dielectric and Capacitors: Charge carrier conducting surfaces are since at different electric potential they have to be physically separated and electrically insulated. This introduction of insulating material in the interspace of the electrodes of capacitors brings into play a new phenomenon called ***Polarization*** with new possibilities to enhance capability of capacitors. These possibilities are – a) enhancing ***voltage withstand capability***, and b) increasing ***energy density*** of capacitor.

Every substance is composed of electrically neutral atoms (where centre of +ve and -ve charges in an atom are concentric) and molecules (despite bonding are so dispersed that the combined effect is electrically neutral). But, in presence of electric field, these constituents are electrically stressed and thus it produces displacement of charges, without dislodging them from atoms and/or molecules. This kind of displacement of charges, within the inter-molecular spaces is called polarization. In case of pure atomic substance, polarization can be due to drift of electron orbits under the influence of electric field. While, in case of molecules it can be due to reorientation of molecular bond along the electric field.

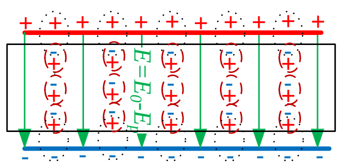
A typical behaviour of dielectric inside a capacitor is shown in the figure below.



A Capacitor with Vacuum Between Plates



Constituent Electric Fields In Presence of Dielectric



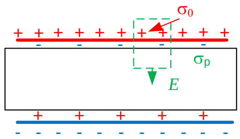
A Capacitor With Reorientation of Molecules of Dielectric in Inter-Plate Region

Let, a capacitor with inter-plate space filled with vacuum holds charge Q at a potential difference between plates V_0, and the same capacitor when filled with a dielectric in the inter-plate region require voltage V to hold the same charge Q. Since, potential difference between the plates is different and hence presence of dielectric must be changing the capacitance from original value C_0 to some value C. Accordingly, $Q = C_0V_0 = CV$; or $\frac{C}{C_0} = \frac{V_0}{V} = K$. In the above figures E_0 is the electric field caused by the potential difference between plates of capacitor, while E_p is the field caused by induced polarization in dielectric. Thus net field in the region filled by dielectric is $E = E_0 - E_p$.

As per Gauss's Law $E_0A = \frac{\sigma_0 A}{\epsilon_0}$; $E_0 = \frac{\sigma_0}{\epsilon_0}$; Likewise, $E = \frac{\sigma_0 - \sigma_p}{\epsilon_0} = \frac{\sigma}{\epsilon_0}$; here, $(\sigma = \sigma_0 - \sigma_p)$ is the net charge density. Accordingly, for a separation d between plates, $\frac{V_0}{V} = \frac{\left(V_0/d\right)}{\left(V/d\right)} = \frac{E_0}{E} = K = \frac{\epsilon}{\epsilon_0} = \frac{\sigma_0}{\sigma_0 - \sigma_p}$; or $\epsilon = K\epsilon_0$; here and ϵ is the ***absolute permittivity or simply permittivity of the dielectric, while, K is the relative permittivity and is always >1.***

Thus capacitance in presence of dielectric of permittivity ϵ is $C = \frac{\epsilon A}{d} = \frac{K\epsilon_0 A}{d}$; i.e. increase in capacitance for same dimensions by an order of K. Likewise, in presence of dielectric voltage across dielectric, for holding same charge is $\frac{V_0}{K}$ is reduced by the order of its relative permittivity. The ***energy density*** (capacity to store energy per unit volume) is: $\frac{\frac{1}{2}CV^2}{Ad} = \frac{\frac{1}{2}\frac{\epsilon A}{d}V^2}{Ad} = \frac{1}{2}\epsilon\left(\frac{V}{d}\right)^2 = \frac{1}{2}\epsilon E^2 =.\frac{1}{2}K\epsilon_0 E^2$.

Gauss's Law in Dielectric: Revisiting the dielectric filled capacitor in the perspective of Gaussian surface as shown in the figure below leads to $EA = \frac{\sigma_0 - \sigma_p}{\epsilon_0}A = \left(\frac{\sigma_0 - \sigma_p}{\sigma_0}\right)\frac{\sigma_0 A}{\epsilon_0} = \frac{1}{K}\cdot\frac{\sigma_0 A}{\epsilon_0} = \frac{\sigma_0 A}{K\epsilon_0}$. This can be written in the another form of ***Gaussian Surface integral*** in a dielectric as : $\oint \boldsymbol{K\bar{E}\cdot d\bar{A}} = \frac{\boldsymbol{\sigma_0 A}}{\boldsymbol{\epsilon_0}} = \frac{\textbf{Free Charge}}{\boldsymbol{\epsilon_0}}$. Here, $\boldsymbol{\sigma_0}$A is called enclosed ***Free Charge***, since it exists on the plates of capacitors freely; while $\boldsymbol{\sigma_p}$A is the called the **Bound Charge**, which is induced inside the dielectric. This bound charge is caused due to electric field produced within dielectric by Free Charge. ***It is also known as Gauss's Law in a Dielectric.***



Displacement Vector: Pictorial analysis of electric field in dielectric is shown above. It is seen that vectors $\bar{E}_0$ and $\bar{E}_p$ are in opposite directions. Accordingly, effective field inside dielectric is $\bar{E} = \bar{E}_0 + \bar{E}_p$. But, by convention polarization vector $\bar{P}$ is along the +ve charge created by polarization and hence, $\bar{E} = \bar{E}_0 - \frac{\bar{P}}{\epsilon_0}$; or $\epsilon_0\bar{E}_p = -\bar{P}$. Accordingly, expression of Gauss's law using polarization vector becomes $\oint(\epsilon_0\bar{E} + \bar{P})\cdot d\bar{A} = \oint \bar{D}\cdot d\bar{A} = Q_{\text{free}}$. Here, $\bar{D} = \epsilon_0\bar{E} + \bar{P}$ is called ***Displacement Vector***, which takes into account effective charge and charge created by polarization.

References: *(Making these concepts intuitive requires practicing with problem solving for which following references are advised.)*

1. *NCERT; PHYSICS, Text Book for Class XI and XII (Part I and II),*
2. *2. H.C. Verma; Concepts of Physics, (Vil 1 & 2).*
3. *Resnick, Halliday, Resnick and Krane; Physics (Vol I and II).*
4. *4. Sears & Zemansky; University Physics with Modern Physics.*