Modern Physics: Part I: Thought Experiment and Special Theory of Relativity

Knowledge is limited, but imagination is not.

- Albert Einstein

*Every innovation is a result of an out-of-box thoughts and Theory of Relativity, propounded by **Albert Einstein** is one of the brilliant examples. It came up at a time when whole school of scientific pursuit was on modernization. It is just an imagination, called thought experiment, without any experimental verification, whose results revolutionized subsequent course of science. This theory in original text of Einstein might be difficult for students at school level, and it is in general with research papers. Yet it is very simple and understandable with the concepts developed at this stage of this Mentors' Manual. This illustration is limited to motion along X-axis, while Einstein developed general purpose equation in XYZ coordinates. It is an effort to take out phobia of theory of relativity from a students, who is already hypnotized with the magnanimity of the theory. Therefore, a supporting illustration of the theory upto most famous equation* $\boldsymbol{E = mc^2}$ *is covered here for familiarization of the concept. It is an effort to open up the thought process, an aim of this endeavor, in which this theory forms a best case study on strength and potential of an imagination and encourage students to think out-of-box. Though, it does not form part of course content of competitive exams for students of 12th class, it likens to an essay recommended for a pleasure time reading, with mathematical alertness.*

This is more of a structured compilation and interlacing of the context from different sources; thanks to Google web resource which was extremely useful in presenting this case study on Thought Experiment - Theory of Relativity.

Thought Experiment: When an idea is taken as an hypothesis, theory or principle to analyze its consequence without either experimental verification or a proposition of experiment which many not be possible to performs is called a ***Thought Experiment***. Main objective of a thought experiment is to explore potential consequence of the idea in question by performing an intentional and structured process of intellectual deliberation in order to speculate potential consequence of a designed conditions within the specified problem domain. History of science, philosophy, psychology, law, mathematics and all fields of abstract knowledge, is full of such though experiments predating to Socrates, 400 BC.

Contribution of every thinker, philosopher and scientist is an outcome of thought experiment, which has continued irrespective of social, theological and political responses. Nearing the end of 19th century, there was a sudden spike in Thought Experiment, and in this contributions of ***Albert Einstein*** in 1905, changed the perspective of scientific community through ***Special Theory of Relativity***.

BACKGROUND: Electromagnetic Wave Equation by James Clerk Marx, little before the birth of Einstein had predicted velocity of light in vacuum $\approx \mathbf{3 \times 10^8}\ \mathrm{ms}^{-1}$. Einstein, an unusual child, at the age of 16 around 1995, had entered into thought experiments to examine relevance of mechanics, assuming himself riding on the light wave, and as he grew, he became increasingly restless to reconcile laws of classical mechanics with the laws of Electromagnetic Field Theory established by Maxwell.

Albert Michelson and Edward Morley in 1887, failed to determine speed of earth's revolution in Luminiferous Aether, an absolute medium, through a specially designed experiment based on principles of *classical mechanics*. Negative results of the experiment became matter of debate and deliberation among contemporary scientist. It shook faith of scientific community on Classical Mechanics, which believed that they were close to a complete description of the universe. This dilemma seems to have encouraged *Though Experiments* of Einstein, to conclude that ***velocity of light (c) is absolute***. In 1887 and 1895, Vogit and Lorentz, respectively, published their early approximation of correlating coordinates of a point, in frame moving with a constant velocity, w.r.t a stationary frame. These transformations were later brought to modern form by Jules Henri Poincaré in 1905, giving it a name Lorentz Transformation. These transformations find place in the landmark paper "Special Theory of Relativity" in 1905 by Einstein. He showed that these transformations follow the principles of relativity; and clarified at the note-1, in the paper, that *he was unaware of transformation propounded by Lorentz* . There are reasons to believe Einstein in light of his pursuance of the Theory of Relativity and multiple cases of identical discoveries, in history of science, concurrently done by different scientists, who were unaware of each other's work. *Despite, failure of the Michelson's experiment, in its objective, it is a classic case of breakthrough of scientific discoveries*. **His theory made velocity of light** c **is a universal constant while time and space are relative to each observer, and thus abandoned idea of absolute time and absolute rest. Stephen Hawking** in his **Brief History of Relativity** has said that **"***The equivalence of mass and energy is summed up in Einstein's famous equation $E = mc^2$, probably the only physics equation to have recognition on the street*".

In the following section, efforts have been made to collate texts and derivations of various associated concepts *starting with Michelson & Morley's Experiment upto Relativistic relation of Energy-Momentum, based on concepts developed upto class 12^th^*, the target of this manual. It is little short of ***General Theory of Relativity***, and would be supplemented separately, at an appropriate time.

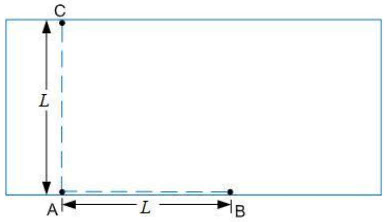
This *Special Theory of Relativity* is based on ***Two Postulates***:

1. *The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion.*

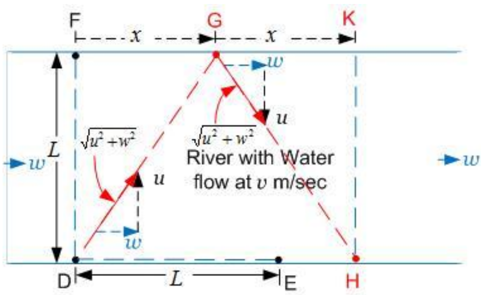
2. *Any ray of light moves in the "stationary" system of co-ordinates with the determined velocity* $\boldsymbol{c}$, *whether the ray be emitted by a stationary or by a moving body.*

After the theory of relativity was accepted, *Einstein recalled his imagination at the age of 16 years and how important role the thought experiment played in establishing Special Relativity*. Here it is important to quote Einstein "***The object of all science, whether natural science or psychology, is to co-ordinate our experiences and to bring them into a logical System***".

Michelson & Morely's Experiment: This experiment has played a key role in advancement and establishing theory of relativity. It derives inspiration of an observation of difference in swimming time swimming across and along the river and water pool. It is based on relative velocity of the swimmer with respect to bank of river and pool, and is in accordance with classical mechanics, as shown in the figure. Taking two points B and C, in steady pool, which are equidistant from a swimmer, but in perpendicular direction from a swimmer A. The swimmer separately touches the Two points and return back to original position, swimming at a speed u. In this experiment, time taken to swim for to-&-fro between points A and C shall be $2t_1 = \frac{2L}{u}$. Since conditions are same for to-&-fro swimming, time would be same. Likewise, in another attempt to swim to-&-fro between points A and B shall be $2t_2 = \frac{2L}{u} = 2t_1$. In this direction of swimming is perpendicular to that while swimming along A and C, but neither the velocity of swimmer nor the velocity of water in the pool, which is steady, has changed, and hence both the timing are same. Thus, $\Delta t = t_2 - t_1 = 0$.

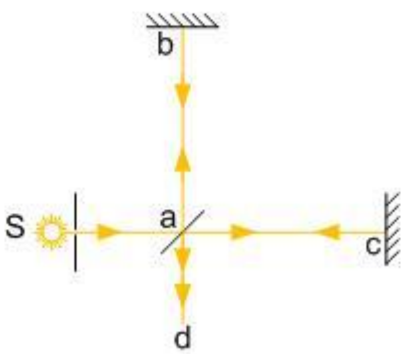


Taking another case of attempt to swim along and across the river, over a same distance L, in a river which is flowing with velocity w. In this case while swimming along DE, the relative velocity of the swimmer w.r.t. to ground shall be $u+w$, and swimmer shall reach faster, but during return along ED, the relative velocity shall be $u-w$. And thus time taken in to-&-fro journey between D and E shall be $t_1' = \frac{L}{u+w} + \frac{L}{u-w} = \frac{2Lu}{u^2-w^2}$.

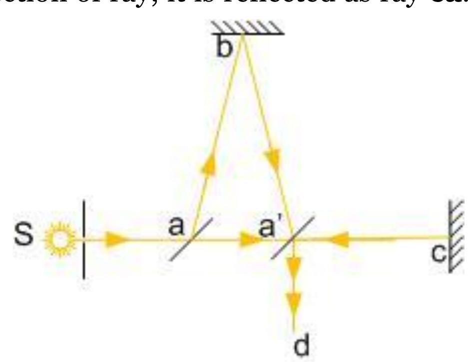


But, in another attempt to swim along DF, due to river current path of swim is DG, maintaining same swimming velocity u during. Thus the swimmer ends up at G a distance x along the bank he is heading to. Let t_2' is time taken by swimmer to cover DG, then $x = v \times t_2'$. Effective velocity of swimmer perpendicular to the river current i.e. along DF is $\sqrt{u^2-w^2}$. Hence, time taken by the swimmer to reach H via G is $2t_2' = \frac{2L}{\sqrt{u^2-w^2}}$.

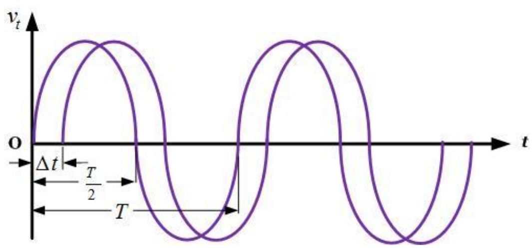
In this experiment $\mathbf{u}$ *and* $\mathbf{v}$ *are comparable. Thus, the relative difference in time of travel* $2\Delta t' = 2(t_1' - t_2')$ *which is perceivable. This experiment was extended by Michelson and Morley in a separately designed experiment as shown in the figure below to verify orbital speed of earth. It was visualized by the duo that velocity of light* ($c = 3 \times 10^8\ \mathrm{ms^{-1}}$)*, is very high as compared with orbital speed of earth* ($v = 3 \times 10^4\ \mathrm{ms^{-1}}$)*. This relative motion between the earth and luminiferous aether*, which was *considered to be a medium of transmission of light, an absolute and stationary frame of reference. It is akin to the river bank, in the above example, and considers earth to be moving at an orbital while rotation at a speed relative to the eather. The experiment with a diagram is illustrated below.*

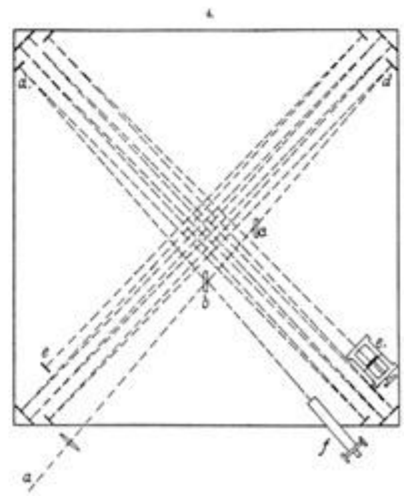


This a conceptual diagram of the experiment based on original paper, with an assumption there is no relative motion between eather and experimental setup on the earth's surface. A ray **Sa** from a monochromatic source of light 'S' is incident on 'a' half-silvered glass played at an angle 45^o to the incident ray. Out of this, part of light passes through the glass as ray **ac**. After reflection of ray ac by mirror at 'c', placed perpendicular to the direction of ray, it is reflected as ray **ca**. This ray **ca** is again reflected at 'a' and returns to the observer as ray **ad**. Another part of the ray reflected at 'a' becomes a ray **ab**, at 90^o to the incident ray **Sa**, and after reflection by mirror at 'b', perpendicular to the incident ray, is returned along line **ba**. Again a part of this ray passes through glass as ray along 'ad' to interfere with the reflected ray **ad**, cited above. This interference is noticed by an observer through a telescope at d. In this distance of mirrors at 'b' and 'c' from the half-silvered glass is at 'a' is equal. Discrepancy in the optical length of the ray **ab**, which before reflection is refracted twice in glass sheet at 'a', with that of the ray **ac** is corrected by placing a glass sheet of same refractive index and thickness as that at 'a', but it is not shown in the diagram for simplicity. If the assumption were true, there would both the rays to the observer at 'd' shall be in the phase and there would not be any interference.

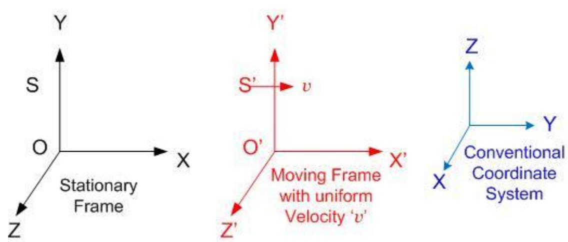


Comparing the two experiments, it leads to the fact that they are identical in nature, but different in context. Accordingly, replacing the variables $u \to c$ and $w \to v$, $\Delta t = t_1' - t_2' = \frac{2Lc}{c^2-v^2} - \frac{2L}{\sqrt{c^2-v^2}} = \frac{2L}{c}\left(\frac{1}{1-\frac{v^2}{c^2}} - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)$. This is simplified using binomial theorem, with an approximation that all terms of higher order of $\frac{v^2}{c^2} \to 0$. This is a valid approximation since $v \ll c$. Accordingly, $\Delta t == \frac{2L}{c}\left(\left(1+\frac{v^2}{c^2}\right)-\left(1+\frac{1}{2}\cdot\frac{v^2}{c^2}\right)\right) = L\frac{v^2}{c^3}$. Thus path difference to the observer is $\Delta l = c\cdot\Delta t$, while $\lambda = c\cdot T = \frac{c}{\upsilon}$, here, λ is wavelength, and υ is the frequency of light. In interference for first fringe to occur $\Delta t = \frac{T}{2}$, first coincidence of (+)ve and (-)ve peaks coincide to create a dark patch. Accordingly, fringe number is expressed as $n = \frac{\Delta t}{\frac{T}{2}} = \frac{2\Delta t}{T}$. Further, wavelength $\lambda = \frac{c}{\upsilon} = c\cdot T$, or, $T = \frac{\lambda}{c}$ therefore, $n = \frac{2c\cdot\Delta t}{\lambda} = \frac{2c}{\lambda}\cdot L\frac{v^2}{c^3} = \frac{2Lv^2}{\lambda c^2}$. Since this fringe number is directly proportional L, perceivable fringe number, length was elongated by multiple reflections creating a folded path as shown in the figure. In experiment for $L = 10\,m$ and $\lambda = 4\times10^{-7}m$, calculated fringe number is arrived at $n = 0.4$, with their experimental having an accuracy of 0.01. This experiment was repeated by rotation the instrument by 90^0, i.e. virtually interchanging the two paths, at different location and in different seasons. But, the experimental results showed null fringe shift. *This resulted into – **a)** negation of the premise of earth, and **b)** orbital velocity of earth aimed at could not be determined.* This controversy was explained by *Einstein through his second postulate according to which **velocity of light in vacuum is a universal constant**.*

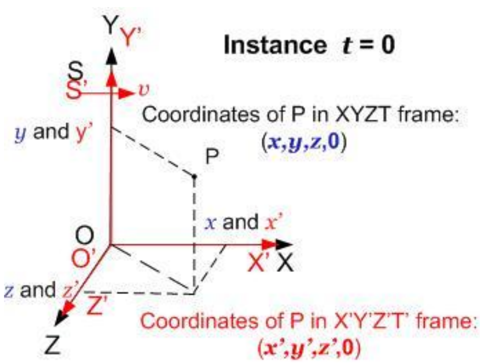


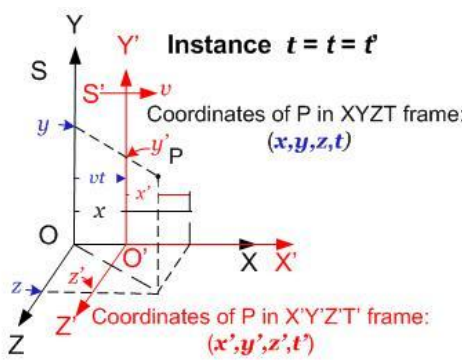


Lorentz Transformation: A beginning of ***Lorentz Transformation*** is made by analyzing coordinates of a fixed point in a stationary frame S, and frame S' which is moving with a constant velocity ($\boldsymbol{v}$) along X axis. The three reference coordinates of S and S' viz-a-viz conventional frame of reference is shown in the figure. Two instances are identified one at ***t*=0,** when **O** and **O'** coincide, and all the three corresponding axes **X-X', Y-Y'** and **Z-Z'** also coincide**.** Time is taken to be absolute and hence corresponding time ***t* = *t'*** for S and S'. Thus as per Galilean **Transformation**: $\boldsymbol{x' = x - vt}$, $\boldsymbol{y' = y}$, $\boldsymbol{z' = z}$ and $\boldsymbol{t' = t}$.

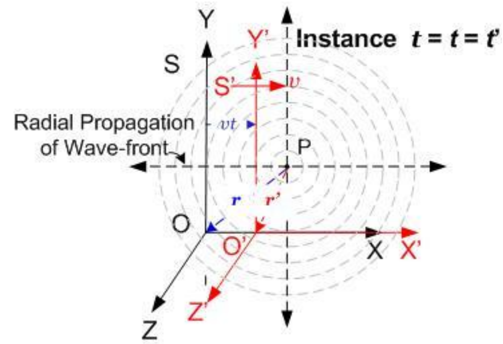


Accordingly, $r^2 = x^2 + y^2 + z^2 = c^2t^2$, in frame S; here, $\boldsymbol{t}$ is the time taken by wave-front of light to reach O, as per *Huygens Wave Theory, and shown in the Figure*. Alternately, it can be written as, $x^2 + y^2 + z^2 - c^2t^2 = 0$. While, in frame S' $r'^2 = x'^2 + y^2 + z'^2 = c^2t'^2$, which works out to $x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$. Here, ***t'*** is the time taken by wave-front to reach O'. Thus, combining the Two equation of spherical wave-fronts leads to $x'^2 + y^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$. It satisfies the Two observers at O and O' in frames S and S'. Using Galilean Transforms, it reduces to m $x^2 - x'^2 = 0$, or $x^2 = x'^2$. It, further, simplifies to $x^2 = (x - vt)^2 => x^2 = x^2 - 2vtx + v^2t^2 => x = vt$. But, from the basic premise, of **classical mechanics** as shown on the figure $x \neq x'$ and hence $x \neq vt$, and it is a contradiction in the premise.





These contradictions were used to modify Galilean Transformation in to a set of linear equations, contemplating Time-Space coordinate system. Accordingly, a new set of transformations are: $\boldsymbol{x'} = \boldsymbol{a_1 x} + \boldsymbol{a_2 t}$, $\boldsymbol{y'} = \boldsymbol{y}$, $\boldsymbol{z'} = \boldsymbol{z}$ and $\boldsymbol{t'} = \boldsymbol{b_1 x} + \boldsymbol{b_2 t}$, where values of coefficients $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ are to be determined. From equation of $\boldsymbol{x'}$, it works out to $\boldsymbol{x'} = \boldsymbol{a_1 x} + \boldsymbol{a_2 t} = \boldsymbol{0}$, it leads to $x = -\frac{a_2}{a_1}t$. In this case, $x = vt$. Equivalence between these Two values of x leads to $v = -\frac{a_2}{a_1}$. Further, a sequence of mathematical manipulation are carried out, starting with $\boldsymbol{x'}$. Thus, $\boldsymbol{x'} = \boldsymbol{a_1}\left(\boldsymbol{x} + \frac{\boldsymbol{a_2}}{\boldsymbol{a_1}}\boldsymbol{t}\right) = \boldsymbol{a_1}(\boldsymbol{x} - \boldsymbol{vt})$. Substituting, this $\boldsymbol{x'}$ together with the $\boldsymbol{t'}$, in combined equation of spherical of wave-fronts can be rewritten as $a_1^2(x-vt)^2 + y'^2 + z'^2 - c^2(b_1x + b_2t)^2 = x^2 + y^2 + z^2 - c^2t^2$. This expression leads to $a_1^2(x-vt)^2 - c^2(b_1x+b_2t)^2 = x^2 - c^2t^2 \Rightarrow (a_1^2 - c^2b_1^2 - 1)x^2 - (2a_1^2v - 2c^2b_1b_2)tx + (a_1^2v^2 - b_2^2c^2 - c^2)t^2 = 0$.



Equating coefficients of x^2, t^2 and tx, it leads to a set of Three equations : **i)** $a_1^2 - c^2b_1^2 = 1 \Rightarrow b_1^2c^2 = a_1^2 - 1$, **ii)** $b_2^2c^2 = c^2 + a_1^2v^2$, and **iii)** $2a_1^2v - 2c^2b_1b_2 = 0 \Rightarrow c^2b_1b_2 = a_1^2v$. Further, multiplying equations (i) and (ii) together, $b_1^2b_2^2c^4 = (a_1^2-1)(c^2 + a_1^2v^2)$. The Left Hand Side of this product equation is square of the LHS of equation **iii).** Thus these three equations combine into $a_1^4v^2 = a_1^4v^2 - a_1^2v^2 + a_1^2c^2 - c^2$. This resolves into $a_1^2c^2 - a_1^2v^2 = c^2$, it further leads to $a_1^2 = \frac{c^2}{c^2-v^2}$, and in turn one of the coefficients $a_1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. This together with value of $v\left(= -\frac{a_2}{a_1}\right)$ leads to value of another coefficient $a_2 = -\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}$. Likewise from (i) $\boldsymbol{b_1^2} = \frac{a_1^2-1}{c^2} \Rightarrow$ $b_1^2 = \frac{\frac{c^2}{c^2-v^2}-1}{c^2} = \frac{\frac{v^2}{c^2-v^2}}{c^2} = \frac{v^2}{c^2}\cdot\frac{1}{c^2-v^2} = \frac{\boldsymbol{v^2}}{\boldsymbol{c^4}}\cdot\frac{\boldsymbol{1}}{\boldsymbol{1-\frac{v^2}{c^2}}}$. Taking square-root of the final form, $b_1 = \pm\frac{v}{c^2}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, and choice was made of -ve value as $b_1 = -\frac{v}{c^2}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ to arrive at uniformity of pattern in the new set of transformations, and thus Third coefficient I determined. In respect if Fourth coefficient, using (ii), $\boldsymbol{b_2^2} = \frac{c^2+a_1^2v^2}{c^2}$. It, further, leads to $\boldsymbol{b_2^2} = 1 + a_1^2\frac{v^2}{c^2} = 1 + \frac{c^2}{c^2-v^2}\cdot\frac{v^2}{c^2} = 1 + \frac{v^2}{c^2-v^2} = \frac{c^2}{c^2-v^2} = \frac{\boldsymbol{1}}{\boldsymbol{1-\frac{v^2}{c^2}}}$. This closes on $b_2 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = a_1$. Use of a factor $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ in all the Four coefficients, has be summarized as Lorentz Factor $\lambda = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Accordingly, the set of coefficients in the new transformations work out to : $\boldsymbol{a_1} = \boldsymbol{\lambda}$, $\boldsymbol{a_1} = \boldsymbol{-\lambda v}$, $\boldsymbol{b_1} = -\frac{\boldsymbol{v}}{\boldsymbol{c^2}}\cdot\lambda$ and $b_2 = \lambda$. Thus there is a new set of transformations are : **a)** $\boldsymbol{x'} = \lambda(\boldsymbol{x} - \boldsymbol{vt})$, **b)** $\boldsymbol{y'} = \boldsymbol{y}$, **c)** $\boldsymbol{z'} = \boldsymbol{z}$, and **d)** $\boldsymbol{t'} = \lambda\left(\boldsymbol{t} - \frac{\boldsymbol{v}}{\boldsymbol{c^2}}\boldsymbol{x}\right)$. These were propounded by Hendrik Lorentz in 1889 and are known as Lorentz Transformation. Effect of Lorentz Transformation is strange and yet not realizable in real life since fastest travelling object that can be realized has $v \ll c$ and this leads to $\lambda \rightarrow 1$, as much as, $\frac{v}{c^2} \rightarrow 1$where Gailean Transformation is valid. This is where Theory of Realtivity becomes essential to look beyond classical mechanics.

It is obvious to question that *if transformation of coordinates of S → S' is it exists there should also be a correspondence between coordinates of S' → S, and* ***it does****, which is called* ***Inverse Lorentz Transformation*** and is as under.

Taking transformation of $t' \rightarrow t, x$, it comes to $\frac{t'}{\lambda} = t - \frac{v}{c^2} \cdot x \Rightarrow t = \frac{t'}{\lambda} + \frac{v}{c^2} \cdot x$, likewise from transformation of $x' \rightarrow x, t$, it arrives at $x' = \lambda\left(x - v\left(\frac{t'}{\lambda} + \frac{v}{c^2} \cdot x\right)\right) = \lambda\left(1 - \frac{v^2}{c^2}\right)x - vt' \Rightarrow x = \frac{x'+vt'}{\lambda\left(1-\frac{v^2}{c^2}\right)} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot (x' + vt')$. It eventually comes to $\boldsymbol{x = \lambda(x' + vt')}$.

In a nested manner, using this equation of $\boldsymbol{x}$, into equation of $\boldsymbol{t}$ used above $t = \frac{t'}{\lambda} + \frac{v}{c^2} \cdot \lambda(\boldsymbol{x' + vt'})$. It further resolves into $\boldsymbol{t} = \frac{t'}{\lambda} + \frac{v}{c^2} \cdot \lambda(x' + vt') = \left(\frac{1}{\lambda} + \frac{\lambda v^2}{c^2}\right)t' + \frac{v}{c^2} \cdot \lambda x' = \left(\frac{1+\lambda^2 \cdot \frac{v^2}{c^2}}{\lambda}\right)\boldsymbol{t'} + \frac{v}{c^2} \cdot \boldsymbol{\lambda x'}$. In order to simplify the solution $\frac{1+\lambda^2 \cdot \frac{v^2}{c^2}}{\lambda} = \frac{1+\frac{1}{1-\frac{v^2}{c^2}} \cdot \frac{v^2}{c^2}}{\lambda} = \frac{1-\frac{v^2}{c^2}}{\lambda} = \frac{\lambda^2}{\lambda} = \lambda$, Thus, $\boldsymbol{t = \lambda\left(t' + \frac{v}{c^2} \cdot x\right)}$. The Lorentz Transformations and their inverses are summarized in a table here and are extremely useful in pursuit of *Theory of Relativity*.

Lorentz Transformation	Inverse Lorentz Transformation
$x' = \lambda(x - vt)$	$x = \lambda(x' \mp vt')$
$y' = y$	$y' = y$
$z' = z$	$z' = z$
$t' = \lambda\left(t - \frac{vx}{c^2}\right)$	$t = \lambda\left(t' + \frac{vx'}{c^2}\right)$
Here, $\lambda = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is called **Lorentz Factor**	

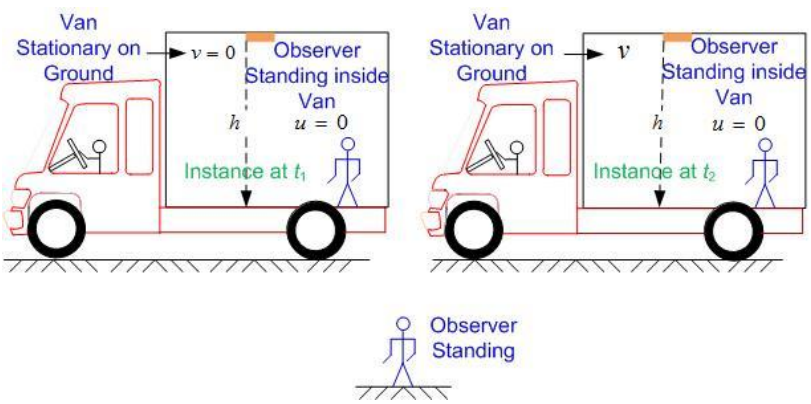
Implication of Lorentz Transformation: These implication are being analyzed, through a set of ***Thought Experiments***, taking Two non-inertial frames of reference as **S** and **S'**, such that at time measured $\boldsymbol{t = 0}$ origins of both the frames coincide and that **S** is stationary while **S'** is moving with a uniform velocity $\boldsymbol{v}$ along X-axis. This Model shall be used throughout the illustration. Alignment of X-axis of both the frames is for convenience. In this sequence of coordinates is similar to the convention with only one difference that coordinates have been rotated by one position in the same sequence, and is shown in the figure.

Principle of Simultaneity: Two events, simultaneous for one observer, may not be simultaneous for another observer if the observers are in uniform relative motion. But, it is no longer satisfactory when events are connected in time series occurring at different places.

Relativity of Space – Length Contraction: In this **Thought Experiment** a rod is taken, in frame S, having its two ends at x_1, y_1, z_1, t and x_2, y_2, z_2, t. Thus length of the rod observed is a state of rest in S shall be $l_0 = x_2 - x_1$. The same rod observed by an observer in frame S' which is moving with a velocity v will depend upon position coordinates of ends of the rod $x_1', y_1', z_1', t'$and x_2', y_1', z_1', t' observed in S'. According to Inverse Lorentz Transformation, for an observer w.r.t. S, $x_1' = \frac{(x_1-vt)}{\sqrt{1-\frac{v^2}{c^2}}}$ and $x_2' = \frac{(x_2-vt)}{\sqrt{1-\frac{v^2}{c^2}}}$. Thus, length of the rod observed in frame S' shall be $l_0' = x_2' - x_1' = \frac{(x_2-vt)}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{(x_1-vt)}{\sqrt{1-\frac{v^2}{c^2}}}$. This reduces to $l_0' = \frac{(x_2 - x_1)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{l_0}{\sqrt{1-\frac{v^2}{c^2}}} = \lambda l_0$, where λ is Lorentz Factor. It is independent of direction wither along X-axis or in a direction on negative of the axis, since this term appears as square. Thus, $l_0 = l_0'\sqrt{1 - \frac{v^2}{c^2}}$. As long as $v \neq 0$, the factor $0 < \sqrt{1 - \frac{v^2}{c^2}} < 1$ and hence $l_0 < l_0'$, length observed in S is less than that observed in S' and it is ***Relativistic Length Contraction***.

Relativity of Time – Time Dilation: It refers to the time duration of a physical process in which time taken by object in a moving frame appears to be longer than that when viewed from the stationary reference frame. This is another consequence of relativistic mechanics which is important to arrive at relativistic mass, in furtherance of Special Theory of relativity. This can be proved using **Inverse Lorentz Transformation,** by taking Two instances $\boldsymbol{t'_1}$ and $\boldsymbol{t'_2}$, when an object is stationary in a moving frame, it implies $\boldsymbol{x'_1} = \boldsymbol{x'_2}$. Thus corresponding time in stationary frame of reference shall be $t_1 = \frac{t'_1 + vx'_1}{\sqrt{1-\frac{v^2}{c^2}}}$ and likewise, $t_2 = \frac{t'_2 + vx'_1}{\sqrt{1-\frac{v^2}{c^2}}}$. Therefore, time duration in stationary frame $\Delta \boldsymbol{t} = t_{21} - t_1 = \frac{t'_2 + vx'_1}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t'_1 + vx'_1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t'_2 - t'_1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} = \lambda \Delta \boldsymbol{t'}$. Since, the range of *Lorentz Factor* (λ) is $0 \le \lambda \to \infty$, as velocity of the moving frame of reference ranges $0 \ll v \to c$, therefore, $\Delta \boldsymbol{t'}$ to an observer in a moving frame would appear to be longer by a factor λ to a stationary observer in a stationary frame.

This is proved with another **thought experiment** in Two stages. In First Stage -van having a light source at its roof, has its walls are transparent to an observer standing on the ground; ground is a stationary frame of reference. This generates Two observation brought as Set 1 below. In second stage of experiment, the van is moving with constant velocity $\boldsymbol{v}$ w.r.t ground, and an observer is standing on the ground. In this stage also Two Observations brought out in Set 2, below. This makes visible to the observer Four instances, Two sets of Two instances each as brought out here under –



Set 1:

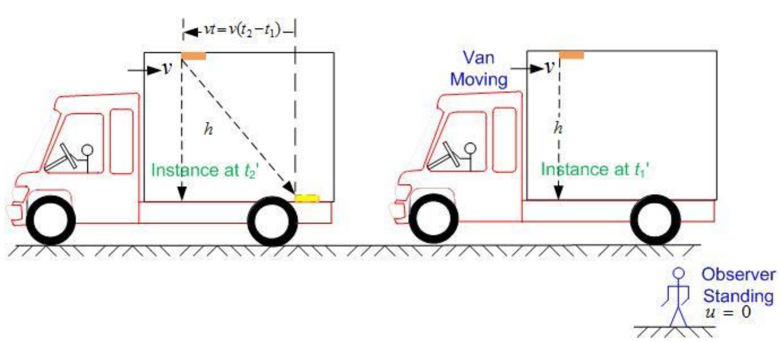
At instance $\boldsymbol{t_1}$: A light beam emanates from the source in the roof of the van.

At instance $\boldsymbol{t_2}$: The light beam emanated at t_1 the light beam reached floor of the van.

Set 2:

At instance $\boldsymbol{t'_1}$: A light beam emanates from the source in the roof of the van already moving with a velocity $\boldsymbol{v}$.

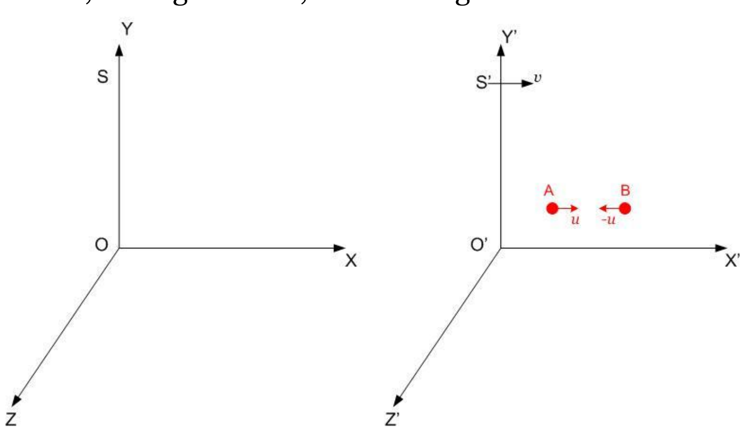
At instance $\boldsymbol{t'_2}$: The light beam emanated at $\boldsymbol{t'_1}$ the light beam reached floor of the van, which continues to move with velocity $\boldsymbol{v}$.



Thus time taken by the light beam to cover a distance $\boldsymbol{h}$, height of the van, at its velocity $\boldsymbol{c}$ is $\boldsymbol{t} = \boldsymbol{t_2} - \boldsymbol{t_1} = \frac{\boldsymbol{h}}{\boldsymbol{c}}$. Whereas, time taken by the light beam in the van which continues to move from time 0^- with a constant velocity $\boldsymbol{v}$ is $(\boldsymbol{t' = t'_2 - t'_1})$**.** But, in this case the light beam covers a diagonal distance $ct' = \sqrt{(vt')^2 + h^2}$. This with algebraic manipulations lead to $c^2t'^2 = v^2t'^2 + h^2$, or $(c^2 - v^2)t'^2 = c^2t^2 \Rightarrow t'^2 = \frac{c^2t^2}{c^2-v^2}$. In the standard form it is $t' = \frac{t}{\sqrt{1-\frac{v^2}{c^2}}} = \lambda t$, where λ is Lorentz Factor. This can be produced in the standard form of relativistic equations $t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$, where $t_0 = t$, and relativistic time is represented as t and not t'. Since, magnitude of velocity can be anything in the range $0 \le |v| \le c$, therefore, corresponding value of time observed by an stationary shall be such that $\boldsymbol{t' > t}$, i.e. time elongates and is called **Dilation of Time.**

Relativistic Composition of Velocity – Velocity Addition: In this **Thought Experiment c**onsider an object moving in S' such that $u' = \frac{dx'}{dt'}$. Here, x' is coordinate of the object in S' and t' is the instance at which object is observed by an observer in S'. As per Galilean Mechanics velocity of the object in S should be $u = \frac{dx}{dt} = v + u'$. Here, , x and t are the coordinate and the instance at which the object is observed by an observer in S. Therefore, $u = v + \frac{dx'}{dt'}$. Since, u is in S and. therefore, x' and t' in shall have to be transformed in x and t pertaining to the frame. Using partial derivatives of inverse Lorentz Transformation of x, $\boldsymbol{dx} = \boldsymbol{\lambda(dx' \mp vdt')}$ and likewise, $\boldsymbol{dt = \lambda\left(dt' \mp \frac{v}{c^2}dx'\right)}$**.** Therefore, $\boldsymbol{u = \frac{dx}{dt} = \frac{\lambda(dx' \mp vdt')}{\lambda\left(dt' \mp \frac{v}{c^2}dx'\right)} = \frac{dx' \mp vdt'}{dt' \mp \frac{v}{c^2}dx'}}$**.** Dividing, numerator and denominator by $\boldsymbol{dt'}$**,** it leads to $u = \frac{\frac{dx'}{dt'}+v}{1+\frac{v}{c^2}\cdot\frac{dx'}{dt'}}$. It finally resolves into $u = \frac{v+u'}{1+\frac{vu'}{c^2}}$. This equation is called relativistic addition of velocities where $< (v + u')$, since denominator. In limiting condition of $u' \to c$, it leads to $u \to \frac{v+c}{1+\frac{v}{c}}$, or $u \to c$. It leads to an important conclusion that any velocity added to **c** tends to be **c**, and is a ***mathematical confirmation of the postulate that no object can be seen to be travelling faster than velocity of light. In other words c is the maximum velocity.***

Relativity of Mass – Variation of Mass with Velocity: In this **Thought Experiment** there be two balls A and B, having mass **m**, are moving towards each other in frame S', parallel to X'-axis, as shown in the figure. In collision the two masses coalesce into one body. Therefore, in frame S' as per law of conservation of momentum $mu + m(-u) = 0$. Accordingly velocity of coalesced mass $(2m)$ is Zero, i.e. in state of rest, in frame S'.



Now analyzing the collision phenomenon in frame S, where velocities of the ball A would be $u_1 = \frac{v+u}{1+\frac{vu}{c^2}}$ and that of ball B would be $u_2 = \frac{v-u}{1-\frac{vu}{c^2}}$. Likewise, masses of the balls A and B w.r.t frame S be m_1 and m_2. Then Law of conservation of momentum in S would lead to $m_1u_1 + m_2u_2 = (m_1 + m_2)v$.

Using values of m_1 and m_2, obtained above, $m_1 \cdot \frac{v+u}{1+\frac{vu}{c^2}} + m_2 \cdot \frac{v-u}{1-\frac{vu}{c^2}} = (m_1 + m_2)v$. This on separating variables m_1 and m_2 leads to $m_1 \cdot \left(\frac{v+u}{1+\frac{vu}{c^2}} - v\right) = m_2 \cdot \left(v - \frac{v-u}{1-\frac{vu}{c^2}}\right) => m_1 \cdot \left(\frac{v+u-v-\frac{v^2u}{c^2}}{1+\frac{vu}{c^2}}\right) = m_2 \cdot \left(\frac{v-\frac{v^2u}{c^2}-v+u}{1+\frac{vu}{c^2}}\right)$. It resolves into a form where, $m_1 \cdot \left(\frac{u-\frac{v^2u}{c^2}}{1+\frac{vu}{c^2}}\right) = m_2 \cdot \left(\frac{u-\frac{v^2u}{c^2}}{1-\frac{vu}{c^2}}\right) \to m_1 \cdot u\left(\frac{1-\frac{v^2}{c^2}}{1+\frac{vu}{c^2}}\right) = m_2 \cdot u\left(\frac{1-\frac{v^2}{c^2}}{1-\frac{vu}{c^2}}\right)$, or a ratio $\frac{m_1}{m_2} = \left(\frac{1-\frac{v^2}{c^2}}{1-\frac{vu}{c^2}}\right)\left(\frac{1+\frac{vu}{c^2}}{1-\frac{v^2}{c^2}}\right) = \frac{1+\frac{vu}{c^2}}{1-\frac{vu}{c^2}}$.

At this point a close examination of value of variables u_1 and u_2 derived earlier in this section, leads to a simpler yet effective algebraic manipulation n algebraic manipulation as under –

Manipulation of u_1:

$$1-\frac{u_1^2}{c^2} = 1-\frac{(v+u)^2}{c^2\cdot\left(1+\frac{vu}{c^2}\right)^2} = 1-\frac{\frac{(v+u)^2}{c^2}}{\left(1+\frac{vu}{c^2}\right)^2} = \frac{1+2\cdot\frac{vu}{c^2}+\frac{u^2v^2}{c^4}-\frac{v^2}{c^2}-2\cdot\frac{uv}{c^2}-\frac{u^2}{c^2}}{\left(1+\frac{vu}{c^2}\right)^2}$$

$$=> \frac{1+\frac{u^2v^2}{c^4}-\frac{v^2}{c^2}-\frac{u^2}{c^2}}{\left(1+\frac{vu}{c^2}\right)^2} => \frac{\left(1-\frac{v^2}{c^2}\right)-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{vu}{c^2}\right)^2}$$

$$=> \frac{\left(1-\frac{v^2}{c^2}\right)\cdot\left(1-\frac{u^2}{c^2}\right)}{\left(1+\frac{vu}{c^2}\right)^2}$$

Manipulation of u_2:

$$1-\frac{u_2^2}{c^2} = 1-\frac{(v-u)^2}{c^2\cdot\left(1-\frac{vu}{c^2}\right)^2} = 1-\frac{\frac{(v-u)^2}{c^2}}{\left(1-\frac{vu}{c^2}\right)^2} = \frac{1-2\cdot\frac{vu}{c^2}+\frac{u^2v^2}{c^4}-\frac{v^2}{c^2}+2\cdot\frac{uv}{c^2}-\frac{u^2}{c^2}}{\left(1-\frac{vu}{c^2}\right)^2}$$

$$=> \frac{1+\frac{u^2v^2}{c^4}-\frac{v^2}{c^2}-\frac{u^2}{c^2}}{\left(1-\frac{vu}{c^2}\right)^2} => \frac{\left(1-\frac{v^2}{c^2}\right)-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{vu}{c^2}\right)^2}$$

$$=> \frac{\left(1-\frac{v^2}{c^2}\right)\cdot\left(1-\frac{u^2}{c^2}\right)}{\left(1-\frac{vu}{c^2}\right)^2}$$

Taking ratios of the above two manipulations leads to $\frac{1-\frac{u_1^2}{c^2}}{1-\frac{u_2^2}{c^2}} = \frac{\frac{\left(1-\frac{v^2}{c^2}\right)\cdot\left(1-\frac{u^2}{c^2}\right)}{\left(1+\frac{vu}{c^2}\right)^2}}{\frac{\left(1-\frac{v^2}{c^2}\right)\cdot\left(1-\frac{u^2}{c^2}\right)}{\left(1-\frac{vu}{c^2}\right)^2}} => \frac{\left(1-\frac{vu}{c^2}\right)^2}{\left(1+\frac{vu}{c^2}\right)^2}$. Taking square roots of the inverse of both the sides $\frac{1+\frac{vu}{c^2}}{1-\frac{vu}{c^2}} = \frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}}$. This equation together with ratio $\frac{m_1}{m_2}$ has consecutive equivalence and hence, $\frac{m_1}{m_2} = \frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}} => m_1 = m_2 \cdot \frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}}$. This result is simplified by taking Ball B in a state of rest w.r.t S before collision i.e. as $u_2 \to 0$, $\sqrt{1-\frac{u_2^2}{c^2}} \to 1$, and $m_2 \to m_0$, here, m_0 is the rest mass of the ball B. Thus a generic formula is arrived where $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$.

Another *interesting conclusion* of this relativistic mass is that **as $v \to c$, mass of the object $m \to \infty$,** i.e. a body travelling at velocity of light shall have infinite mass which is improbable. Therefore, for an object to attain velocity of light greater than c, which can happen only if velocity is continuously increased, it shall have to pass through a state where $v = c$, which itself is impossible, and hence no object can travel at velocity greater than velocity of light ($v > c$).

Mass Energy Equivalence: As per classical mechanics momentum of a moving object is $\boldsymbol{p} = \boldsymbol{mv}$, here $\boldsymbol{m}, \boldsymbol{v}$ and $\boldsymbol{p}$ are mass, velocity and momentum of an object in a stationary frame S . Since, force on an object as per laws of mechanics is $F = \frac{dp}{dt} = m\frac{dv}{dt} + \frac{dm}{dt}v$, and work done by the force in moving the object through a distance dx is $dW = Fdx = m\frac{dv}{dt}dx + \frac{dm}{dt}vdx$. This equation can be manipulated as $dW = m \cdot dv \cdot \frac{dx}{dt} + dm \cdot v \cdot \frac{dx}{dt}$. Since, $v = \frac{dx}{dt}$ it can be written as $dE = mv \cdot dv + v^2 dm$. Using relativistic mass $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} => m^2\left(1 - \frac{v^2}{c^2}\right) = m_0^2$. It leads to $m^2c^2 = m_0^2c^2 + m^2v^2$. Taking partial derivative of this equation $c^2 \cdot 2mdm = m^2 \cdot 2vdv + v^2 \cdot 2mdm$, since both c and m_0 are constant. It leads to $2mc^2dm = 2m(mvdv + v^2dm) => c^2dm = mvdv + v^2dm$. Using equation derived from classical mechanics, and as per Einstein's First Postulate in Special Theory of Relativity, all laws of mechanics are equally valid in any frame which is in a state of rest or uniform motion. Thus $c^2dm = dE$, and on integration it leads to $\boldsymbol{E = mc^2}$, regarded as ***world's most important equation*** and *was contributed by Einstein through his thought experiments leading to the Theory of Relativity.*

Energy and Momentum Relation: Extending relativistic definition of mass $\left(m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\right)$ to momentum in classical mechanics ($p = mv$) and, mass-energy equivalence classical mechanics ($E = mc^2$) , another interesting result is obtained. Taking square of mass-energy equivalence equation and subtracting from it c^2p^2, another mathematical acumen, using above equations it leads to $E^2 - c^2p^2 = m^2c^4 - c^2m^2v^2 => \frac{m_0^2}{1-\frac{v^2}{c^2}}c^4 - c^2 \cdot \frac{m_0^2}{1-\frac{v^2}{c^2}} \cdot v^2$. It leads to $E^2 - c^2p^2 = m_0^2c^4\left(\frac{1}{1-\frac{v^2}{c^2}} - \frac{1}{1-\frac{v^2}{c^2}} \cdot \frac{v^2}{c^2}\right) = m_0^2c^4\left(\frac{1-\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right) = m_0^2c^4$. Accordingly, $\boldsymbol{E^2 = m_0^2c^4 + c^2p^2}$ it implies that ***total energy of a mass (m) moving with a velocity (v) mass is equal to equal square-root of the sum of square of equivalent energy of the rest mass*** $(\boldsymbol{m_0})$ ***and square of product momentum with velocity of light***.

Summary: Einstein after having stirred the scientific community with his Special Theory of relativity, for inertial frames of reference, did not stop at that. He extended his imagination to accelerated frames of reference and contributed to General Theory of Relativity in 2015. He continued his imagination till his last breadth to discover a single theory which could explain the entire phenomenon governing universe. In this pursuit. imagination and mathematics have crossed boundaries of physical observations which were verifiable. At this point it is worth sharing that -

> ***There is no idea which is obscure, trivial, ridiculous or obnoxious. All that is needed is to think, imagine and meditate. Pursue the idea relentlessly. In the process, it shall undergo refinement and auto correction and then emerge in a final form, the NEED.***

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