LET US DO SOME PROBLEMS In MATHEMATICS-XXV

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JEE (Advanced) 2020 Examination for admissions to various streams of Engineering in IITs was earlier scheduled to be held on August 23, 2020 but due to the spread of Corona it was rescheduled to be held on September 27, 2020.

There were two Question Papers. Here for the readers from both the Question Papers some of the questions have been selected to understand the standard of the questions. Answers are given with the questions. The detailed solutions are not being written here. If some reader needs the detailed solution, he or she may request the Coordinator desk for that.

QUESTIONS FROM PAPER I AND PAPER II

Q1. Suppose *a, b* denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose *c,d* denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$.

Then the value of
$ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$
is

(a)0 (b) 8000
(c) 8080 (d) 16000
Ans. (d)

Q2. If the function $f: R \rightarrow R$ is defined by $f(x) = |x|(x - sinx)$, then which of the following statements is **TRUE ?**
(a) *f* is one-one, but **NOT** onto
(b) *f* is onto, but **NOT** one-one
(c) *f* is **BOTH** one-one and onto
(d) *f* is **NEITHER** one-one **NOR** onto
Ans. (c)

Q3. Let the functions $f: R \rightarrow R$ and $g: R \rightarrow R$ be defined by
$f(x) = e^{x-1} - e^{-|x-1|}$ and
$g(x) = \frac{1}{2}(e^{x-1} + e^{1-x})$.
Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is
(a) $(2 - \sqrt{3}) + \frac{1}{2}(e - e^{-1})$
(b) $(2 + \sqrt{3}) + \frac{1}{2}(e - e^{-1})$
(c) $(2 - \sqrt{3}) + \frac{1}{2}(e + e^{-1})$
(d) $(2 + \sqrt{3}) + \frac{1}{2}(e + e^{-1})$
Ans. (a)

Q4. Let *a, b* and λ be positive real numbers. Suppose *P* is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point P. If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is
(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$
(c) $\frac{1}{3}$ (d) $\frac{2}{5}$
Ans. (a)

Q5. Let C_1 and C_2 be two biased coins such that the probabilities of getting head in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose α is the number of heads that appear when C_1 is tossed twice, independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently, Then probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is
(a) $\frac{40}{81}$ (b) $\frac{20}{81}$
(c) $\frac{1}{2}$ (d) $\frac{1}{4}$
Ans. (b)

Q6. Consider all rectangles lying in the region $\left\{(x, y) \in R{\times}R: 0{\le}x{\le}\frac{\pi}{2} \text{ and } 0 \le y \le 2\sin(2x)\right\}$ and having one side on the *x*-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is
(a) $\frac{3\pi}{2}$ (b) π
(c) $\frac{\pi}{2\sqrt{3}}$ (d) $\frac{\pi\sqrt{3}}{2}$
Ans. (c)

Q7. Let the function $f : R \to R$ be defined by $f(x) = x^3 - x^2 + (x - 1) \sin x$ and let $g : R \to R$ be an arbitrary function.

Let $fg : R \to R$ be the product function defined by $(f g)(x) = f(x) g(x)$.

Then which of the following statements is/are TRUE ?

(a) If g is continuous at $x = 1$, then fg is differentiable at $x = 1$

(b) If fg is differentiable at $x = 1$, then g is continuous at $x = 1$

(c) If g is differentiable at $x = 1$, then fg is differentiable at $x = 1$

(d) If fg is differentiable at $x = 1$, then g is differentiable at $x = 1$

Ans. (a,c)

Q8. Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If $M^{-1} = adj\,(adj\,M)$, then which of the following statement is/are ALWAYS TRUE ?

(a) $M = I$ (b) $\det M = 1$

(c) $M^2 = I$ (d) $(adj\, M)^2 = I$

Ans. (b,c,d)

Q9. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE ?

(a) $\left|z + \frac{1}{2}\right| \le \frac{1}{2}$ for all $z \in S$

(b) $|z| \le 2$ for all $z \in S$

(c) $\left|z + \frac{1}{2}\right| \ge \frac{1}{2}$ for all $z \in S$

(d) The set S has exactly four elements

Ans. (b,c)

Q10. Let x, y and z be positive real numbers. Suppose x, y and z are lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively.

If $\tan\frac{X}{2} + \tan\frac{Z}{2} = \frac{2y}{x+y+z}$, then which of the following statements is/are TRUE?

(a) $2Y = X + Z$ (b) $Y = X + Z$

(c) $\tan\frac{X}{2} = \frac{x}{y+z}$ (d) $x^2+z^2-y^2=xz$

Ans. (b,c)

Q11. Let $L1$ and $L2$ be the following straight line.

$L1$: $\frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3}$

$L2$: $\frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}$

Suppose the straight line

L : $\frac{x-\alpha}{l} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$

lies in the plane containing L1 and L2, and passes through the point of intersection of L1 and L2. If the line L bisects the acute angle between the lines L1 and L2, then which of the following statements is/are TRUE?

(a) $\alpha - \gamma = 3$ (b) $l + m = 2$

(c) $\alpha - \gamma = 1$ (d) $l + m = 0$

Ans. (a,b)

Q12. Which of the following inequalities is/are TRUE?

(a) $\int_0^1 x\cos x\, dx \ge \frac{3}{8}$

(b) $\int_0^1 x \sin x\, dx \ge \frac{3}{10}$

(c) $\int_0^1 x^2 \cos x\, dx \ge \frac{1}{2}$

(d) $\int_0^1 x^2 \sin x\, dx \ge \frac{2}{9}$

Ans. (a,b,d)

Q13. Let m be the minimum possible value of $\log_3(3^{y_1}+3^{y_2}+)$ where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2 , x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2 (m^3) + \log_3(M^2)$ is

Ans. (8.00)

Q14. For a complex number z, let Re(z) denote the real part of z. Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $\mathrm{Re}(z_1) > 0$ and $\mathrm{Re}(z_2) < 0$, is

Ans. 8

Q15. The probability that a missile hits a target successfully is 0.75. In order to destroy the target completely, at least three successful hits are required. Then the minimum number of missiles that have to be fired so that the probability of completely destroying the target is **NOT** less than 0.95, is

Ans. 6

Q16. Let O be the centre of the circle $x^2 + y^2 = r^2$, where $r > \frac{\sqrt{5}}{2}$.

Suppose PQ is a chord of this circle and the equation of the line passing through P and Q is $2x + 4y = 5$. If the centre of the circumcircle of the triangle OPQ lies on the line $x+2y = 4$, then the value of r is

Ans. 2

Q17**.** The trace of a square matrix is defined to be the sum of its diagonal entries. If A is a 2 ×2 matrix such that the trace of A is 3 and the trace of A^3 is −18, then the value of the determinant of A is

Ans. 5

Q18. Let the functions : $(-1,1)\to\mathbb{R}$ and $g:(-1,1)\to(-1,1)$ be defined by
$f(x) = |2x-1|+|2x+1|$ and $g(x) = x-[x]$,
where $[x]$ denotes the greatest integer less than or equal to x. Let $f\circ:(-1,1)\to\mathbb{R}$ be the composite function defined by $(fog)(x) = f(g(x))$.
Suppose c is the number of points in the interval (−1,1) at which *fog* is **NOT** continuous, and suppose d is the number of points in the interval (−1,1) at
which fog is **NOT** differentiable. Then the value of $c + d$ is

Ans. 4

Q19. The value of the limit

$$\lim_{x\to\frac{\pi}{2}}\frac{4\sqrt{2}(sin3x+sinx)}{\left(2\,sin2xsin\frac{3x}{2}+cos\frac{5x}{2}\right)-\left(\sqrt{2}+\sqrt{2}cos2x+cos\frac{3x}{2}\right)}$$

Is

Ans. 8

Q20**.** Let b be a non-zero real number. Suppose $f:R\to R$ is a differentiable function such that $f(0)=1$.
If the derivative f' of f satisfies the equation $f'(x)=\frac{f(x)}{b^2+x^2}$ for all $x\in R$, then which of the following statements is/are TRUE?

(a) If $b > 0$, then f is an increasing function
(b) If $b < 0$, then f is a decreasing function
(c) $(x)\ (-x)=1$ for all $x\in R$
(d) $(x)-f(-x)=0$ for all $x\in R$

Ans. a, c

Q21**.** Let f: $R\to R$ and g : $R\to R$ be functions satisfying $f(x+y)=f(x)+f(y)+f(x)f(y)$ and $f(x)=xg(x)$ for all $x, y\in R$.

If $\lim_{x\to 0} g(x) = 1$, then which of the following statements is/are TRUE?

(a) f is differentiable at every x ∈R
(b) If $g(0)=1$, then g is differentiable at every x∈R
(c) The derivative f′(1) is equal to 1
(d) The derivative f′(0) is equal to 1

Ans. a,b,d

Q22. Let α, β, γ, δ be real numbers such that $\alpha^2+\beta^2+\gamma^2\neq0$ and $\alpha+\gamma=1$. Suppose the point (3,2,−1) is the mirror image of the point (1,0,−1) with respect to the plane $\alpha x+\beta y+\gamma z=\delta$. Then which of the following statements is/are TRUE?

(a) $\alpha+\beta=2$ (b) $\delta-\gamma=3$
(c) $\delta+\beta=4$ (d) $\alpha+\beta+\gamma=\delta$

Ans. a,b,c

Q23**.** An engineer is required to visit a factory for exactly four days during the first 15 days of every month and it is mandatory that **no** two visits take place on consecutive days. Then the number of all possible ways in which such visits to the factory can be made by the engineer during 1-15 June 2021 is

Ans. 495.00

Q24. In a hotel, four rooms are available. Six persons are to be accommodated in these four rooms in such a way that each of these rooms contains at least one person and at most two persons. Then the number of all possible ways in which this can be done is

Ans. 1080.00

Q25**.** Let the function f:[0,1]→R be defined by $f(x)=\frac{4^x}{4^x+2}$. Then the value of $f\left(\frac{1}{40}\right)+f\left(\frac{2}{40}\right)+f\left(\frac{3}{40}\right)+\cdots.+f\left(\frac{39}{40}\right)-f\left(\frac{1}{2}\right)$ is

Ans. 19.0