LET US DO SOME PROBLEMS – XXXII

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Some mathematics problems from *Chinese Math Olympiad* are given here for the readers who are aspiring for Mathematics Olympiad. These problems are really very-very hard. The composition of the problems shows how responsibly teachers are taking their works.

Every one of us knows that China stands first in scoring Gold medals so far and we Indians stand at 31st rank. Certainly, it is the matter of great concern to think being a teacher why we and our students are lagging so behind.

No solutions of these problems are being written here. Only the answers are given. If any reader solves these problems and wants to check his/her solution, he or she may contact the Coordinator's desk for the solution or solutions.

PROBLEMS

1. The interval on which the function $f(x) = log_{\frac{1}{2}}(x^2 - 2x - 3)$ is monotone increasing is

(a)$(-\infty,-1)$ (b)$(-\infty,1)$
(c)$(1,+\infty)$ (d)$(3,+\infty)$

Ans.(a)

2. If real numbers x and y satisfy $(x+5)^2+(y-12)^2=14^2$, then the minimum value of x^2+y^2 is

(a)2 (b)1
(c)$\sqrt{3}$ (d) $\sqrt{2}$

Ans.(b)

3. The function $f(x) = \frac{x}{1-2^x} - \frac{x}{2}$ is

(a)an even but not odd function
(b)an odd but not even function
(c)a both even and odd function
(d)a neither even nor odd function

Ans.(a)

4. It is given that complex numbers z_1 and z_2 satisfy $|z_1| = 2$ and $|z_2| = 3$. If the included angle of their corresponding vectors is 60^0, then is $\left|\frac{z_1+z_2}{z_1-z_2}\right|$

Ans. $\frac{\sqrt{133}}{7}$

5. It is given that $f(x)$ is a function defined on R, satisfying $f(1)=1$, and for any $x\in R$, $f(x+5) \geq f(x)+5$, and $f(x+1) \leq f(x)+1$. If $g(x)=f(x)+1-x$, then $g(2021)$ is

Ans.1

6. A new sequence is obtained from the sequence of the positive integers {1,2,3,….} by deleting all the perfect squares. Then the 2003rd term of the new sequence is

(a)2046 (b)2047
(c)2048 (d)2049

Ans.(c)

7. Let $x \in \left[-\frac{5\pi}{12}, -\frac{\pi}{3}\right]$, then the maximum value of $y = tan\left(x + \frac{2\pi}{3}\right) - tan\left(x + \frac{\pi}{6}\right) + cos\left(x + \frac{\pi}{6}\right)$ is

(a)$\frac{12\sqrt{2}}{5}$ (b) $\frac{11\sqrt{2}}{6}$
(c) $\frac{11\sqrt{3}}{6}$ (d) $\frac{12\sqrt{3}}{5}$

Ans.(c)

8. Suppose in the tetrahedron ABCD, AB=1, CD=$\sqrt{3}$, the distance and angle between the lines AB and CD are 2 and $\frac{\pi}{3}$respectively. Then the volume of the tetrahedron is

(a)$\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$

(c) $\frac{1}{3}$ (d) $\frac{\sqrt{3}}{3}$

Ans.(b)

9. Let $M_n = \{0, a_{1,} a_{2,} \ldots . a_n : a_i = 0 \; or \; 1, 1 \le i \le n-1, a_n = 1\}$ be a set of decimal fractions, T_n and S_n be the number and the sum of the elements in M_n respectively. Then $\lim_{n \to \infty} \frac{S_n}{T_n}$ is

Ans. $\mathbf{\frac{1}{18}}$

10. Let O be an interior point of ΔABC such that $\overrightarrow{OA} + 2\overrightarrow{OB} + 3\overrightarrow{OC} = 0$. then the ratio of the area of ΔABC to the area of ΔAOC is

(a)2 (b)4

(c)3 (d)5

Ans.(c)

11. The curve represented by the equation is $\frac{x^2}{sin\sqrt{2} - sin\sqrt{3}} + \frac{y^2}{cos\sqrt{2} - cos\sqrt{3}} = 1$

(a)An ellipse with the foci on the *x*-axis

(b)A hyperbola with the foci on the *x*-axis

(c)An ellipse with the foci on the *y*-axis

(d)A hyperbola with the foci on the *y*-axis

Ans.(c)

12. If one side of square *ABCD* is on the line $y=2x-17$, and the other two vertices lie on parabola $y=x^2$, then the minimum area of the square is

Ans. 80

13. A natural number a is called a “lucky number” if the sum of its digits is 7. Arrange all “lucky numbers” in an ascending order, and we get a sequence $a_1, a_2, a_3, \ldots$. If a_n=2005, then a_{5n} is

Ans. 52000

14. There are real numbers *a*,*b*, and *c* and a positive number λ such that $f(x)=x^3+ax^2+bx+c$ has three real roots x_1, x_2, and x_3 satisfying

(1)$x_2-x_1=\lambda$

(2)$x_3 > \frac{1}{2}(x_1 + x_2)$

Find the maximum value of $\frac{2a^3+27c-9ab}{\lambda^3}$

Ans.$\mathbf{\frac{3\sqrt{3}}{2}}$

15. Let three sides of a triangle be integers *l*,*m*,*n* respectively, satisfying $l > m > n$ and $\left\{\frac{3^l}{10^4}\right\} = \left\{\frac{3^m}{10^4}\right\} = \left\{\frac{3^n}{10^4}\right\}$where {x}=x-[x] and [x] denotes the integral part of the number x. find the minimum perimeter of such a triangle.

Ans.3003

16. In an acute triangle ABC, point H is the intersection point of altitude CE to AB and altitude BD to AC. A circle eit DE as its diameter intersects AB and AC at points F and G respectively. FG and AH intersect at point K. if BC=25, BD=20 and BE=7, then find the length of AK.

Ans.8.64

17. Consider the sequence $a_1, a_2, \ldots$.defined by $a_n=2^n+3^n+6^n-1$; (n=1,2,…). Determine all positive integers that are relatively prime to every term of the sequence.

Ans.1

18. Let x,y, and z be positive numbers such that $xyz \ge 1$. Prove that $\frac{x^5-x^2}{x^5+y^2+z^2} + \frac{y^5-y^2}{y^5+z^2+x^2} + \frac{z^5-z^2}{z^5+x^2+y^2} \ge 0.$

19. There are n new students. Suppose that there are two students who know each other in every three students and there are two students who do not know each other in every four students. Find the maximum

value of n.
Ans. 8

20. Assume that $\alpha^{2005}+\beta^{2005}$ can be expressed as a polynomial in $\alpha+\beta$ and $\alpha\beta$. Find the sum of the coefficients of the polynomial.
Ans.1

21. Find all real numbers k, such that the inequality $a^3+b^3+c^3+d^3+1 \geq k(a+b+c+d)$ holds for any $a,b,c,d \in[-1,+\infty)$.
Ans. $\boldsymbol{k=\frac{3}{4}}$

22. Find all positive integers n such that $n^4-4n^3+22n^2-36n+18$ is a perfect square.
Ans. 1 and 3.

23. Let a,b,c be positive real numbers. Determine the minimum value of $\frac{a+3c}{a+2b+c}+\frac{4b}{a+b+2c}-\frac{8c}{a+b+3c}$
Ans. $\boldsymbol{12\sqrt{2}-17}$

24. Find all positive integers n such that $20n+2$ can divide $2003n+2002$.
Ans. There is no positive integer

25. Let $\theta_i \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right), i = 1,2,3,4.$ Prove that there exists $x \in R$ such that the following two inequalities

$$cos^2\theta_1 cos^2\theta_2 - (sin\theta_1 sin\theta_2 - x)^2 \geq 0$$

$$cos^2\theta_3 cos^2\theta_4 - (sin\theta_3 sin\theta_4 - x)^2 \geq 0$$

Hold simultaneously if and only if

$$\sum_{i=1}^{4} sin^2\theta_i \leq 2\left(1+\prod_{i=1}^{4} sin\theta_i + \prod_{i=1}^{4} cos\theta_i\right).$$