LET US DO SOME PROBLEMS-XXXI

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Mathematical Olympiad is organized in different stages like Indian Olympiad Qualifier in Mathematics (IOQM), Indian National Mathematical Olympiad (INMO), Orientation Cum Selection Camps (OCSC), Pre Departure Camp, and International Olympiad.

Mathematical Olympiad in India is jointly organised by the Mathematics Teachers Association of India (MTAI), Indian Association of Physics Teachers (IAPT) and Homi Bhabha Centre for Science Education (Tata Institute of Fundamental Research)–HBCSE (TIFR).

Some problems from Indian Olympiad Qualifier in Mathematics are given here to understand the standard and difficulty level of the questions.

Q1. Ari chooses 7 balls at random from n balls numbered 1 to n. If the probability that no two of the drawn balls have consecutive numbers equals the probability of exactly one pair of consecutive numbers in the chosen balls, find n.

Solution:

Number of ways when no two balls are consecutive = ${}^{n-7+1}C_7 = {}^{n-6}C_7$

To find number of ways for exactly one pair being consecutive, let x_0 be number of balls before 1st selected ball.

x_i (i=1 to 6) denotes number of balls between i^{th} and $(i+1)^{th}$ balls selected and x_7 be number of balls after 7^{th} ball is selected.

Now $x_0+x_1+x_2+x_3+x_4+x_5+x_6+x_7 = n-7$

If 1st and 2nd selected balls are consecutive then $x_1=0$

$\Rightarrow x_0+x_1+x_2+x_3+x_4+x_5+x_6+x_7 = n-7$

$\geq 0 \geq 1 \geq 1 \geq 1 \geq 1 \geq 1 \geq 0$

$\Rightarrow (x_0+1)+x_1+x_2+x_3+x_4+x_5+x_6+(x_7+1)=n-5$

$\Rightarrow$ Number of Solutions= ${}^{n-5-1}C_{7-1} = {}^{n-6}C_6$

So total ways in which exactly one pair is consecutive = $6.\,{}^{n-6}C_6$

Now ${}^{n-6}C_7 = 6.\,{}^{n-6}C_6$

$\Rightarrow n=54$

Q2. Let A and B be two finite sets such that there are exactly 144 sets which are subsets of A or subsets of B. Find the number of elements in $A \cup B$.

Solution:

Let A contains x elements, B contains y elements and Common elements of A and B are z elements

$n(A)=2^x$

$n(B)=2^y$

$n(\text{Common})=2^z$

$\Rightarrow 2^x+2^y-2^z=144$

$\Rightarrow 2^7+2^5-2^4=144$ only possibility

Hence

$x+y-z=8$

Q3. Sita and Geeta are two sisters. If Sita's age is written after Geeta's age a four digit perfect square (number) is obtained. If the same exercise is repeated after 13 years another four digit perfect square (number) will be obtained. What is the sum of the present ages of Sita and Geeta?

Solution:

Let the age of Sita be AB and that of Geeta be CD.

Given CDAB is a perfect square.

$\Rightarrow 100(CD)+AB=X^2$

Also

$(CD+13)(AB+13)=Y^2$

$\Rightarrow 100\{(CD+13)\}+(AB+13)=Y^2$

$\Rightarrow 100CD+1300+AB+13=Y^2$

$\Rightarrow X^2+1313=Y^2$

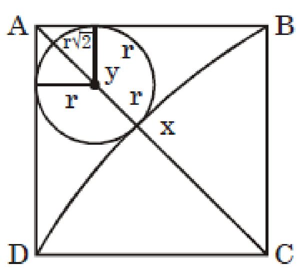
$\Rightarrow Y^2-X^2=1313$
$\Rightarrow (Y-X)(Y+X)=13\times 101$
$\Rightarrow Y-X=13,\ Y+X=101$
$\Rightarrow X=44,\ Y=57$
$\Rightarrow 100CD+AB=X^2=44^2=1936$
$\Rightarrow CD=19,\ AB=36$
$\Rightarrow AB+CD=55$

Q4. Find the number of positive integers n such that the highest power of 7 dividing n! is 8.

Solution:
We know that
$\left[\frac{49}{7}\right]=7\ and\ \left[\frac{7}{7}\right]=1$
And
From $\left[\frac{49}{7}\right]\ to\ \left[\frac{55}{7}\right]=7$
Hence only 7 positive numbers will be possible.

Q5. Let ABCD be a square with side length 100. A circle with centre C and radius CD is drawn. Another circle of radius r, lying inside ABCD, is drawn to touch this circle externally and such that the circle also touches AB and AD. If = $\boldsymbol{m+n\sqrt{k}}$, where m, n are integers and k is a prime number, find the value of $\frac{m+n}{k}$.

Solution:



From the figure,
A, y, x and C are collinear
Given
$AC=100\sqrt{2}=AY+yx+xC=r\sqrt{2}+r+100$

$\Rightarrow r=100\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)=100(\sqrt{2}-1)^2=300-200\sqrt{2}$
$\Rightarrow m=300,\ n=200,\ k=2$
Hence $\frac{m+n}{k}=50$

Q6. If a,b,c are positive real numbers such that $a^2+b^2=c^2$ and ab=c. Determine the value of $\left|\frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{c^2}\right|$.

Solution:
The given expression can be re-written as
$$\left|\frac{\{(a+b)^2-c^2\}\{c^2-(a-b)^2\}}{c^2}\right|$$
=
$$\left|\frac{\{2ab\}\{2ab\}}{c^2}\right|=\left|\frac{4a^2b^2}{c^2}\right|=4$$

Q7. Find the largest 2-digit number N which is divisible by 4, such that all integral power of N end with N.

Solution:
The all possible 2-digit numbers are
10,11,12,….,98,99
Out of these the numbers divisible by 4 are
96, 92, 88, 84,…..
Largest N among them whose all integral powers of N ends with 9 is 76
As 76x7=5776
Therefore
76x76x76=438976

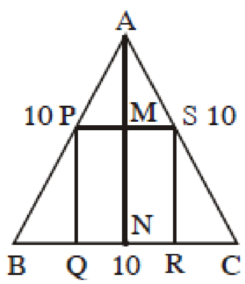
Q8. If a,b,c are real numbers and $(a+b-5)^2+(b+2c+3)^2+(c+3a-10)^2=0$, then find the integer nearest to $a^3+b^3+c^3$.

Solution:
Squares of real numbers is always positive and the sum of positive numbers is never zero until they all are zero. Hence individual squares of the given equation are zero.
$a+b-5=0 \Rightarrow a=5-b$
$b+2c+3=0 \Rightarrow c=\frac{-3-b}{2}$

c+3a-10=0 $\Rightarrow$ c+3a=10 $\Rightarrow \frac{-3-b}{2} + 3(5-b) = 0 \Rightarrow$ b=1, a=4, c=-2
hence the required answer is 57

Q9. Let ABC be an equilateral triangle with side length 10. A square PQRS is inscribed in it, with P on AB. Q, R on BC and S on AC. If the area of the square PQRS is $m + n\sqrt{k}$, where m, n are integers and k is a prime number then determine the value of $\sqrt{\frac{m+n}{k^2}}$.

Solution:



The height of the triangle = AN= $5\sqrt{3}$
Now, triangles APS and ABC are similar
Hence

$$\frac{AM}{AN} = \frac{AP}{AB} = \frac{PS}{BC} = \frac{AS}{AC}$$

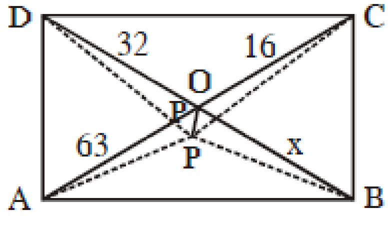
$\Rightarrow \frac{AN-MN}{AN} = 1 - \frac{MN}{5\sqrt{3}} = \frac{PS}{10}$

$\Rightarrow$PS=$\frac{10\sqrt{3}}{2+\sqrt{3}}$

$\Rightarrow$PS^2=2100-1200$\sqrt{3}$

On Comparing with m+n$\sqrt{k}$
m=2100, n=-1200, k=3
Hence, the required answer is 10

Q10. If ABCD is a rectangle and P is a point inside it such that AP=33, BP=16, DP=63. Find CP.
Solution:



In the figure, O is the mid points of BD and AC.
From triangle APC, by Apollonius Theorem
AP^2+CP^2=2(AO^2+OP^2)
From triangle BPD, by Apollonius Theorem
BP^2+DP^2=2(BO^2+OP^2)
We know, BO=OA $\Rightarrow$
AP^2+CP^2=BP^2+DP$^2\Rightarrow$63^2+16^2=x^2+33$^2\Rightarrow$x= 56

Q11. Find the number of ordered triples (x,y,z) of real numbers that satisfy the system of equation
$x + y + z = 7; x^2 + y^2 + z^2 = 27; xyz = 5.$

Solution:
We know that
$(x+y+z)^2=x^2+y^2+z^2+2\ (xy+yz+zx)$
$\Rightarrow$xy+yz+zx=11 after putting all the values
From the property of Cubic Equation,
x,y,z are the roots of the equation
$a^3-7a^2+11a-5=0$
$\Rightarrow(a-1)^2(a-5)=0$
$\Rightarrow$a=1,1, 5
Hence possible pairs are
(1,1,5), (1,5,1), (5,1,1) i.e. 3 only

Q12. The prime numbers a, b and c are such that $a + b^2 = 4c^2$. Determine the sum of all possible values of a + b + c.

Solution:
Given a,b,c are prime
From given equation
$(2c)^2-(b)^2=a$
$\Rightarrow$(2c+b)(2c-b)=a
The possible sets of pair of values may be
1, a
a, 1
-1, -a
-a, -1
Hence
Case I
2c+b=a and 2c-b=1
On adding both
4c=a+1

$\Rightarrow c = \frac{a+1}{4}$, b=$\frac{a-1}{2}$
Similarly,
Case II
2c+b=1 is not possible as a,b,c are prime and more than 2
Case III
2c+b=-a is also not possible as RHS is negative and LHS is positive
Case IV
2c+b=-1 is also not possible
So only possibility is
a, $\frac{a+1}{4}, \frac{a-1}{2}$
hence
a+b+c=$\frac{7a-1}{4} = k, (k \text{ is an integer})$
$\Rightarrow a = \frac{4k+1}{7}$
When k=5, a=3
But b=1 that is not prime
So these values are neglected

Again when k=12, a=7, b=3 and c =2
Here all are prime
Therefore a+ b+c=12

On putting k=19, a=11, b=5, c=3
This is possible
Therefore a+b+c=19

On putting k=20, 33, 40, 47,… a,b,c are not prime
Hence
The required result is 12+19=31

Q13. Let A = {m : m an integer and the roots of $x^2 + mx + 2020 = 0$ are positive integers.}
And B = {n : n an integer and the roots of $x^2 + 2020x + n = 0$ are negative integers}.
Suppose M is the largest element of A and L is the smallest element of B. Find the sum of digits of M +L.

Solution:
Let roots of x^2+mx+2020=0 be a and b
Hence
a+b=-m, and ab=2020
since a and b are positive, so a+b is positive i.e. m is negative

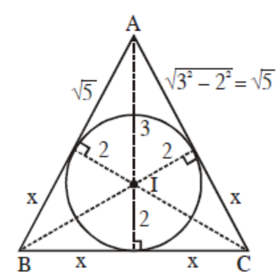
let roots of x2+2020x+n=0 be c and d
hence
c+d=-2020 and cd=n
since c and d are negative hence m should be positive
now,
ab=2020=22x5x101=2020x1=1010x2=504x4=404x5=202x10=101x20

for Maximum –(a+b), take (101,20)
hence m=-(101+20)=-121

for (c,d) to be minimum, it can be (-2019,-1), (-2018,-2),…(-1,-2019)
take n=-2019x-1=2019
therefore,
M+L=-121+2019=1898
Hence the sum of the digits = 1+8+9+8=26

Q14. Let ABC be an isosceles triangle with AB = AC and in-centre I. If AI = 3 and the distance from I to BC is 2, what is the square of the length of BC?

Solution:



From the figure,
$$S = \frac{AB + BC + CA}{2} = \sqrt{5} + 2x$$
Area of triangle=
$\sqrt{S(S-AB)(S-BC)(S-CA)}$
$= \sqrt{(\sqrt{5}+2x)(x)(\sqrt{5})(x)}$
Area of triangle is also = $\frac{1}{2}\ Base \times Height$
$= \frac{1}{2}(2x)(5) = 5x$

Hence,

$$5x = \sqrt{(\sqrt{5}+2x)(x)(\sqrt{5})(x)}$$

$\Rightarrow 25x^2 = \sqrt{5}\,(\sqrt{5}+2x)x^2$

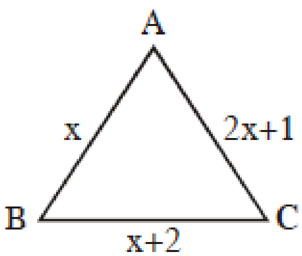
$\Rightarrow 25 = 5 + 2\sqrt{5}x$

$\Rightarrow x = 2\sqrt{5}$

Hence $BC^2=(2x)^2=(4\sqrt{5})^2 = 80$

Q15. The sides of a triangle arc x, 2x + 1 and x + 2 for some positive rational number x. If one angle of the triangle is 60°, what is the perimeter of the triangle?

Solution:



Cos A= $\cos 60^0 = \frac{AC^2+AB^2-BC^2}{2AC.AB} = \frac{(2x+1)^2+(x)^2-(x+2)^2}{2x.(2x+1)} = \frac{1}{2}$

$\Rightarrow x=\frac{3}{2}$

Therefore, the perimeter is 4x+3= 9

Q16. Ria has 4 green marbles and 8 red marbles. She arranges them in a circle randomly, if the probability that no two green marbles are adjacent is $\frac{p}{q}$ where the positive integers p,q have no common factors other than 1, what is p + q?

Solution:

Green marbles= 4

Red marbels= 8

Total ways to put in circle= (12-1)!=11!

Number of ways no two green are together=$^8P_4$ (7!)

Probability=$\frac{8P4\ (7!)}{11!} = \frac{7}{33} = \frac{p}{q}$

Hence p+q=40

Q17. If x and y are positive integers such that (x – 4)(x – 10) = 2^y , find the maximum possible value of x + y.

Solution:

Given

$(x-4)(x-10) = 2^y$

Let $2^y=2^a.2^b$

Hence

$(x-4)(x-10) = 2^y=2^a.2^b$

$\Rightarrow x-4=2^a$ and

$x-10=2^b$

on subtraction

$6=2^a-2^b$

$\Rightarrow$a=3, b=1

On adding

$2x-14=2^a+2^b=2^3+2^1=10$

$\Rightarrow$x=12

$\Rightarrow 2^y=2^a.2^b=2^3.2^1=16=2^4$

$\Rightarrow$y=4

Hence x+y=12+4=16

Q18. Two sides of a regular polygon with n sides, when extended, meet at an angle of 28°. What is the smallest possible value of n ?

Solution:

For 28^0 angle, a turn of $28^0+180^0=208^0$ is needed

For 2080 turn, say we turned k times. In n-gon, we know that on turn in n-gon is

$\left(\frac{360}{n}\right) degrees$

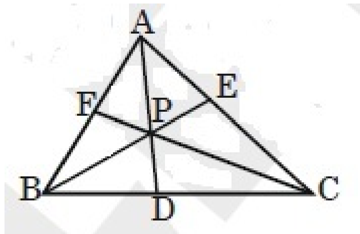
$\Rightarrow 208=\left(\frac{360}{n}\right)k$

$\Rightarrow \left(\frac{k}{n}\right)=\left(\frac{208}{360}\right) = \left(\frac{26}{45}\right)$

$\Rightarrow$n=45

Q19. Let D,E,F be points on the sides BC,CA,AB of a triangle ABC, respectively. Suppose AD, BE,CF are concurrent at P. If PF/PC = 2/3, PE/PB = 2/7 and PD/PA = m/n, where m,n are positive integers with gcd(m,n) = 1. find m + n.

Solution:



From the figure

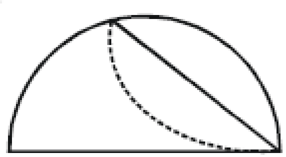
$\frac{PF}{PC} = \frac{2}{3},$

$\frac{PE}{PB} = \frac{2}{7},$

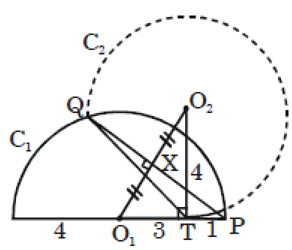
$\frac{PF}{CF} + \frac{PE}{BE} + \frac{PD}{AD} = 1 \Rightarrow \frac{PD}{AD} = \frac{17}{45} \Rightarrow \frac{PD}{PA} = \frac{17}{28}$

Hence m+n = 45

Q20. A semicircular paper is folded along a chord such that the folded circular arc is tangent to the diameter of the semicircle. The radius of the semicircle is 4 units and the point of tangency divides the diameter in the ratio 7 : 1. If the length of the crease (the dotted line segment in the figure.



Solution:



O_2 is the reflection of O_1 over X.

C_1 and C_2 have the same radius and the same chord is equidistant from both centres.

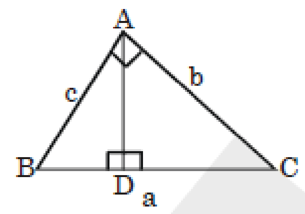
$O_2T = r = 4 \Rightarrow O_1O_2 = 5 \Rightarrow O_2X = \frac{5}{4}$

$\Rightarrow PX^2 = O_2P^2 - O_2X^2 = 16 - \frac{25}{4} = 9.75$

$\Rightarrow PQ^2 = 4PX^2 = 39$

Q21. Let ABC be a triangle with ÐBAC = 90° and D be the point on the side BC such that AD ^ BC. Let r, r_1 and r_2 be the inradii of triangles ABC, ABD, and ACD respectively. If r, r_1, and r_2 are positive integers and one of them is 5. find the largest possible value of r + r_1+ r_2.

Solution:



We know that

$\frac{r_1}{r} = \frac{c}{a},$

$\frac{r_2}{r} = \frac{b}{a},$

$\Rightarrow r_1^2 + r_2^2 = r^2$

One of the r, r_1, and r_2 is 5

Therefore, $(r+r_1+r_2)$ maximum=5+12+13=30

Q22. Find the largest positive integer N such that the number of integers in the set {1,2,3,...,N} which are divisible by 3 is equal to the number of integers which are divisible by 5 or 7 (or both).

Solution:

Given Set of Positive Integers={1,2,3,….,N}

No. of integers divisible by 3=$\left[\frac{N}{3}\right]$

No. of integers divisible by 5=$\left[\frac{N}{5}\right]$

No. of integers divisible by 7=$\left[\frac{N}{7}\right]$

No. of integers divisible by 35=$\left[\frac{N}{35}\right]$

Hence,

No of integers divisible by 5 or 7=$n(5 \cup 7)$

$= n(5)+n(7)-n(5 \cap 7) = \left[\frac{N}{5}\right] + \left[\frac{N}{7}\right] - \left[\frac{N}{35}\right]$

Given

$\left[\frac{N}{3}\right] = \left[\frac{N}{5}\right] + \left[\frac{N}{7}\right] - \left[\frac{N}{35}\right] \Rightarrow \left[\frac{N}{5}\right] + \left[\frac{N}{7}\right] = \left[\frac{N}{35}\right] + \left[\frac{N}{3}\right]$

For N=65, both sides are satisfied.