Trigonometry and Coordinate Geometry

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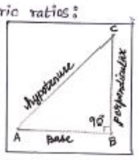
TRIGONOMETRY

Trigonometry (from greek trigōnon, "triangle" & metron, "measure") is a branch of mathematics that studies relationships between side lengths & angles of triangles. The field emerged in the Hellenistic world during 3rd century BC from applications of geometry to astronomical studies. The Greeks focused on the calculations of chords, while mathematicians in India created the earliest-known tables of values for trigonometric ratios (also called trigonometric functions).

Trigonometric ratios:

Figure : 1



sine of $\angle A$ = $\sin\theta = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{BC}{AC}$

cosine of $\angle A$ = $\cos\theta = \frac{\text{Base}}{\text{hypotenuse}} = \frac{AB}{AC}$

tangent of $\angle A$ = $\tan\theta = \frac{\text{perpendicular}}{\text{Base}} = \frac{BC}{AB}$

cosecant of $\angle A$ = $\operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{AC}{BC} = \frac{1}{\sin\theta}$

secant of $\angle A$ = $\sec\theta = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{AC}{AB} = \frac{1}{\cos\theta}$

cotangent of $\angle A$ = $\cot\theta = \frac{\text{Base}}{\text{perpendicular}} = \frac{AB}{BC} = \frac{1}{\tan\theta}$

Also, $\tan\theta = \frac{\sin\theta}{\cos\theta}$

& $\cot\theta = \frac{\cos\theta}{\sin\theta}$

→ The value of trigonometric ratio of an angle does not depend on the size of the triangle (Δ) but depends on the angles only.

Trigonometric Identities:

→ An equation is called an identity if it is true for all values of the variable(s) involved.

→ An equation involving trigonometric ratio of an angle is called a trigonometric identity if it is true for all values of the angles.

In ΔABC, right angled at B, By Pythagoras Theorem,

$$AB^2 + BC^2 = AC^2 \quad \rightarrow (1)$$

Dividing each term of (1) by AC^2

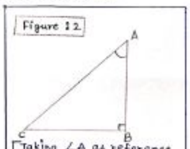
$$\Rightarrow \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

$$\Rightarrow \left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1$$

$$\Rightarrow (\cos A)^2 + (\sin A)^2 = 1$$

$$\Rightarrow \boxed{\cos^2 A + \sin^2 A = 1} \rightarrow (2)$$

Figure : 2



[Taking $\angle A$ as reference angle]

{This is true for all values of A such that $0^\circ \le A \le 90^\circ$. So, this is a trigonometric identity} Now, dividing eq (1) by AB^2.

$$\Rightarrow \frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$

$$\Rightarrow \left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

$$\Rightarrow \boxed{1 + \tan^2 A = \sec^2 A} \rightarrow (3)$$

Q: Is "$\tan^2 A + 1 = \sec^2 A$" true for $A = 0^\circ$?

Ans: Yes, it is.

Q. What about $A = 90^\circ$?

Ans: $\tan A$ & $\sec A$ are not defined for $A = 90^\circ$. So, eq (3) is true for all values of A such that $0^\circ \le A < 90^\circ$

Dividing eq (1) by BC^2,

$$\Rightarrow \frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$$

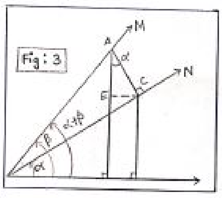
$$\Rightarrow \left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$$

$$\Rightarrow \boxed{\cot^2 A + 1 = \operatorname{cosec}^2 A} \rightarrow (4)$$

NOTE: $\operatorname{cosec} A$ & $\cot A$ are not defined for all $A = 0^\circ$.

Therefore, eq (4) is true for all values of A such that $0^\circ < A \le 90^\circ$

Fig: 3



* **How to:** $\sin(A+B)$; $\sin(A-B)$
$\cos(A+B)$; $\cos(A-B)$
$\tan(A+B)$; $\tan(A-B)$

In ΔOAB, (From fig: 3)

$$\sin(\alpha+\beta) = \frac{AB}{OA}$$

$$= \frac{AE + EB}{OA}$$

$$= \frac{AE + CD}{OA} \quad \{\because EB = CD\}$$

$$= \frac{AE}{OA} \times \frac{AC}{AC} + \frac{CD}{OA} \times \frac{OC}{OC}$$

$$\boxed{\sin(\alpha+\beta) = \cos\alpha \cdot \sin\beta + \sin\alpha \cdot \cos\beta}$$

Again, In ΔOAB (From fig: 3)

$$\cos(\alpha+\beta) = \frac{OB}{OA} = \frac{OD - BD}{OA}$$

$$= \left(\frac{OD}{OA} \times \frac{OC}{OC}\right) - \left(\frac{EC}{OA} \times \frac{AC}{AC}\right)$$

$$= \left(\frac{OD}{OC} \cdot \frac{OC}{OA}\right) - \left(\frac{EC}{AC} \cdot \frac{AC}{OA}\right)$$

$$\boxed{\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta}$$

We know that, $\sin(-\alpha) = -\sin\alpha$
& $\cos(-\alpha) = -\cos\alpha$

$$\sin(\alpha+\beta) = \sin[\alpha + (-\beta)]$$

$$= \sin\alpha \cdot \cos(-\beta) + \cos\alpha \cdot \cos(-\beta)$$

$$= \sin\alpha \cdot \cos\beta + \cos\alpha \cdot (-\sin\beta)$$

$$\boxed{\sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta}$$

$$\sin 2\alpha = \sin(\alpha+\alpha)$$

$$= \sin\alpha \cdot \cos\alpha + \cos\alpha \cdot \sin\alpha$$

$$\underline{\sin 2\alpha = 2\sin\alpha \cdot \cos\alpha}$$

$$\cos 2\beta = \cos(\beta+\beta)$$

$$= \cos\beta \cdot \cos\beta - \sin\beta \cdot \sin\beta$$

$$\underline{\cos 2\beta = \cos^2\beta - \sin^2\beta}$$

[Using 1st trigonometric identity, $\sin^2\alpha - \cos^2\alpha = 1$]

$$\cos 2\beta = (1 - \sin^2\beta) - \sin^2\beta$$

$$= 1 - 2\sin^2\beta$$

$$= \cos^2\beta - (1 - \cos^2\beta)$$

$$\cos 2\beta = 2\cos^2\beta - 1$$

$$\boxed{\cos 2\alpha = \cos^2\alpha - \sin^2\alpha}$$

(or) $= \underline{2\cos^2\alpha - 1}$

(or) $= \underline{1 - 2\sin^2\alpha}$

since, $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta}{\cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta}$$

Now, we have to express $\tan(\alpha+\beta)$ in terms of $\tan\alpha$ & $\tan\beta$.
By dividing the expression with $\cos\alpha\cdot\cos\beta$ both in numerator & denominator, we get.

$$\tan(\alpha+\beta) = \frac{\frac{\sin\alpha\cdot\cancel{\cos\beta}}{\cos\alpha\cdot\cancel{\cos\beta}} + \frac{\cancel{\cos\alpha}\cdot\sin\beta}{\cancel{\cos\alpha}\cdot\cos\beta}}{\frac{\cos\alpha\cdot\cos\beta}{\cos\alpha\cdot\cos\beta} - \frac{\sin\alpha\cdot\sin\beta}{\cos\alpha\cdot\cos\beta}}$$

$$\boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\cdot\tan\beta}}$$

$\sin(-\alpha) = -\sin\alpha$; $\cos(-\alpha) = \cos\alpha$

$$\therefore \tan(-\alpha) = \frac{\sin(-\alpha)}{\cos(-\alpha)} = \frac{-\sin\alpha}{\cos\alpha} = -\tan\alpha$$

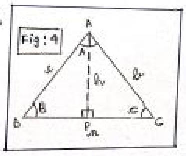
$$\tan(-\alpha) = -\tan\alpha$$

$$\tan(\alpha-\beta) = \tan(\alpha+(-\beta)) = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\cdot\tan(-\beta)}$$

$$\boxed{\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\cdot\tan\beta}}$$

Property : $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Construction : Draw a $\perp$ AP from vertex A on opposite side BC at P.
Let AP = h
In $\triangle$ ABP,
$$\sin B = \frac{AP}{AB} = \frac{h}{c}$$



In $\triangle$ APC, (from fig : 4)

$$\sin C = \frac{AP}{AC} = \frac{h}{b}$$

$h = c\sin B \longrightarrow (1)$

And $h = b\sin C \longrightarrow (2)$

Combining (1) and (2)

$$c\sin B = b\sin C$$

Dividing both sides by "bc", we get

$$\frac{\cancel{c}\sin B}{b\cancel{c}} = \frac{\cancel{b}\sin C}{\cancel{b}c}$$

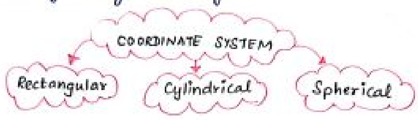
$$\Rightarrow \frac{\sin B}{b} = \frac{\sin C}{c}$$

Similarly, taking a $\perp$ from B and C on sides opposite we can arrive the property,

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$

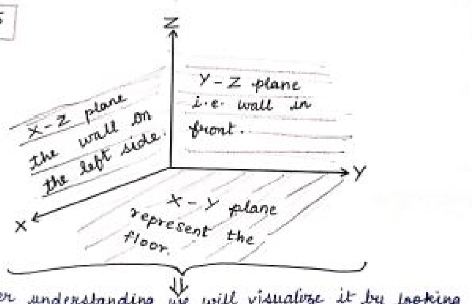
COORDINATE GEOMETRY

The invention of Cartesian coordinates in the 17th century by René Descartes (Latinized name : Cartesius) revolutionized mathematics by providing the first systematic link between Euclidean geometry and Algebra.



Rectangular coordinate system :

Figure : 5



For better understanding we will visualize it by looking at the left corner of a room in front of Y.

X - axis is edge of wall on left & floor.

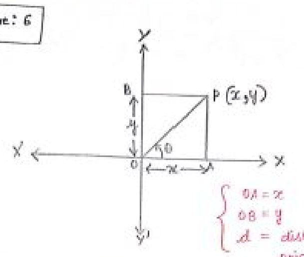
Y - axis is edge of front wall & floor.

Z - axis is edge of the two walls i.e wall on left & two front wall.

This is known a 3-D i.e. Three dimensional coordinate system.

Before we go to understand 3D, we will consider 2D i.e plane surface compared with floor.

Figure : 6



OA = x
OB = y
d = distance of P from origin

$$OP = d = \sqrt{OA^2 + OB^2}$$
$$= \sqrt{x^2 + y^2}$$

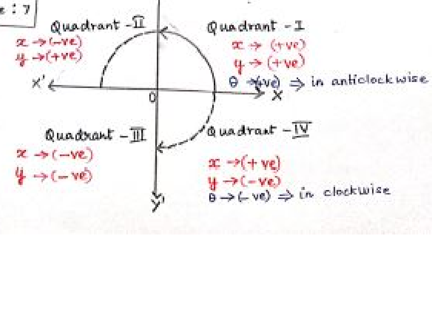
$$\tan\theta = \frac{AP}{OP} = \frac{y}{x}$$

$$\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

If plane is vertical,
X - axis can be horizonal
Y - axis can be vertical
both are perpendicular to each other.

Both can be horizontal but $\perp$ to each other.
{If plane is horizontal}

Figure : 7

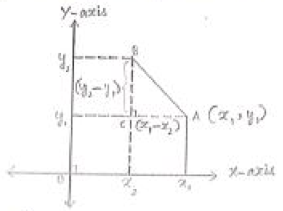


From Figure : 7

X - coordinate is projection (image) of point on x-axis.

Y - coordinate is projection of point on Y-axis.

Figure : 8



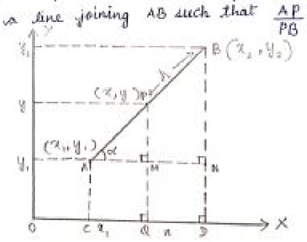
$$AB^2 = AC^2 + BC^2$$
$$= (x_1 - x_2)^2 + (y_2 - y_1)^2$$
$$= [-(x_2 - x_1)^2 + (y_2 - y_1)^2]$$
$$\boxed{AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

{This is as per the Euclid's postulates: distance between two points is the length of line joining}

Previously, we saw about coordinates & distance between two points where coordinates are known.

Now, there can be another situation where two points $A(x_1, y_1)$ & $B(x_2, y_2)$ are given & we have to find coordinates of a point P on a line joining AB such that $\frac{AP}{PB} = \frac{m}{n}$

Figure : 9



$\Delta APM \approx \Delta ABN$

Using property of similar Δs {From Figure : 9}

ratio of corresponding sides, $\frac{AM}{AN} = \frac{AP}{AB} = \frac{PM}{BN}$

Using coordinates

$AM = (x - x_1)$

$AN = (x_2 - x_1)$

$PM = (y - y_1)$

$BN = (y_2 - y_1)$

AP corresponds i.e. adding 1 on both sides,

$$\frac{AP}{PB} = \frac{m}{n} \Rightarrow \frac{PB}{AP} = \frac{n}{m} \Rightarrow \frac{PB}{AP} + 1 = \frac{n}{m} + 1$$

$$\Rightarrow \frac{AP + PB}{AP} = \frac{m+n}{m} \Rightarrow \frac{AB}{AP} = \frac{m+n}{n}$$

Combining these ratios,

$$\frac{AM}{AN} = \frac{AP}{AB} \Rightarrow \frac{(x - x_1)}{(x_2 - x_1)} = \frac{m}{m+n}$$

$$\Rightarrow (x - x_1)(m+n) = (x_2 - x_1)\, m$$
$$x(m+n) = x_2 m - x_1 m + x_1 (m+n)$$
$$x = \frac{x_2 m + x_1 n}{m+n}$$
$$\boxed{x = \frac{x_1 n + x_2 m}{m+n}} \longrightarrow (1)$$

On similar lines we can derive,

$$\boxed{y = \frac{y_1 n + y_2 m}{m+n}} \longrightarrow (2)$$

Similarly, we can find value of any variable (two at a time) out of x_1, y_1, x_2, y_2, m & n provided/remaining values for the variables known.

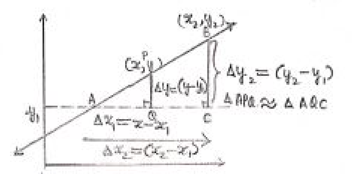
NOTE : five variables can be out of each set of 5 variables given & fifth variable is to be determined. This can be done with the help of (1) & (2).

Equation of a line passing through two point (x_1, y_1) & (x_2, y_2) there are various forms of equations of line.

As Euclid's postulates : Only one line can pass through two points.

* Equation of a line means if x coordinate of a point on the line is known to find y coordinate of the point (or) vice-versa.
* In other words its relation between x, y coordinates of a point on a line.

Figure : 10



Slope of AP $= \frac{\Delta y_1}{\Delta x_1} = m_1$

Slope of AB $= \frac{\Delta y_2}{\Delta x_2} = m_2$

$\because$, APB are co-linear, $m_1 = m_2$

$$\therefore \frac{\Delta y_1}{\Delta x_1} = \frac{\Delta y_2}{\Delta x_2}$$

$$\Rightarrow \frac{(y - y_1)}{(x - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

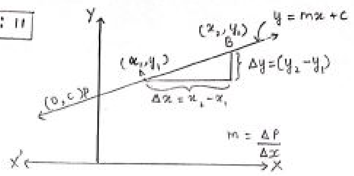
$$\Rightarrow (y - y_1) = \frac{(y_2 - y_1)}{(x_2 - x_1)} \times (x - x_1)$$

There are six variables on this equation.

$\therefore$ only variable of remaining five variable can be found out i.e. by the other variable.

Second form of equation of line

Figure : 11



Q. What is $y = mx + c$?

$$y = mx + c$$

m : Slope of line

c : intercept of line on Y-axis

In this when we put, $x = 0$

$$y = m(0) + c$$
$$y = c$$

First form of equation,

$$(y - y_1) = \frac{(y_2 - y_1)}{(x_2 - x_1)} \times (x - x_1)$$

Second form of equation,

$$y = mx + c$$

$\because$ both equations represent a line having coefficients of variables must be equal to or same.

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x + \left\{ y_1 - \left[\frac{(y_2 - y_1)}{(x_2 - x_1)}\right] x_1 \right\}$$

$$\left.\begin{aligned} A_1 x + B_1 y + C_1 &= 0 \\ A_2 x + B_2 y + C_2 &= 0 \\ \frac{A_1}{A_2} = \frac{B_1}{B_2} &= \frac{C_1}{C_2} \end{aligned}\right\}$$ By the simultaneous solving of linear equation method.

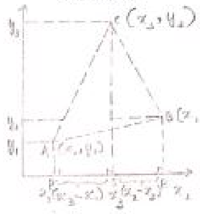
$\boxed{y = mx + c}$

$\therefore$ coefficients of 'y' are equal, coefficients of 'x' must also be equal, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$c = (y_1 - mx_1)$

Area of a triangle by coordinate Geometry

Figure : 12



$PR = x_2 - x_1$

$$ar(\Delta ABC) = ar(APQC) + ar(CQRB) - ar(APRB)$$

$$= \left[\tfrac{1}{2}(x_3 - x_1) \times (y_1 + y_3)\right] + \left[\tfrac{1}{2}(x_2 - x_3) \times (y_2 + y_3)\right] - \left[\tfrac{1}{2}(x_2 - x_1) \times (y_1 + y_2)\right]$$

{$\because$ area of trapezium = 1/2 (distance between parallel side) × (sum of parallel sides) }

$$= \tfrac{1}{2}\left[x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3 - x_3y_3 \quad x_2y_2 - x_3y_2 + x_2y_3 - (x_2y_1 - x_1y_1 + x_2y_2 + x_1y_2)\right]$$

$$= \tfrac{1}{2}\left[x_3y_1 - x_1y_1 - x_1y_3 + x_2y_2 - x_3y_2 + x_2y_3 + x_1y_1 + x_1y_2 - x_2y_2 - x_2y_1\right]$$

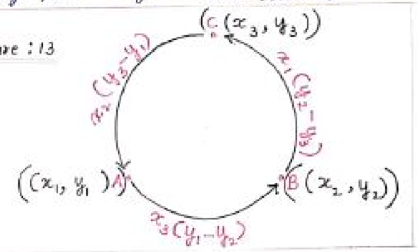
$$= \tfrac{1}{2}\left[(x_1y_2 - x_1y_3) + (x_3y_1 - x_3y_2) + (x_2y_3 - x_2y_1)\right]$$

$$= \boxed{\tfrac{1}{2}\left[x_1(y_2 - y_3)\right] + \left[x_3(y_1 - y_2)\right] + \left[x_2(y_3 - y_1)\right]}$$

Tip to recall, (or easier way to recall),

If we adopt the "Anticlock wise" numbering of vertices of triangle, then using the cyclic property is easier to recall.

Figure : 13



$$area(\Delta ABC) = \tfrac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 + y_2)\right]$$

We can choose either of the two to find whether these given point are colinear or not.

If points are colinear, area of Δ is zero

Slope between any two vertices of a Δ is equal to the slope of another pair of verticies of the triangle.

This student is of Class Xth, at Kendriya Vidyalaya, Dinjan, Assam. She is a regular student of IOMS being held since April'2020. Her hobbies are reading books, story writing, sketching, exploring science mysteries or problems.

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