Complex Numbers

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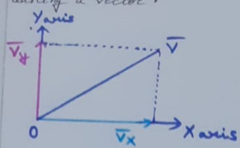
COMPLEX NUMBERS

Taking a vector $\bar{V}$



$\bar{V}_x = V\cos\theta\,\hat{i}$

$\bar{V}_y = V\sin\theta\,\hat{j}$

Here $\hat{i}$ – unit vector along x-axis

$\hat{j}$ – unit vector along y-axis

$V\cos\theta$ is magnitude of $\bar{V}_x$ and represented as $V_x = V\cos\theta$

When V_x is represented with arrow in its top it means it is magnitude

$|\bar{V}_x| = V_x = V\cos\theta$

Likewise, $|\bar{V}_y| = V_y = V\sin\theta$

$$\left.\begin{aligned}\bar{V} &= \bar{V}_x + \bar{V}_y = V_x\hat{i} + V_y\hat{j} \\ &= V\cos\theta\,\hat{i} + V\sin\theta\,\hat{j}\end{aligned}\right\} \text{In 2D}$$

Only magnitude $V = V_x^2 + V_y^2$

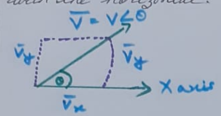
$\tan\theta = \frac{V_y}{V_x} \Rightarrow \theta = \tan^{-1}\left(\frac{V_y}{V_x}\right)$

$\bar{V} \rightleftharpoons \bar{V}_x + \bar{V}_y$

Likewise in 3D $\bar{V} \rightleftharpoons \bar{V}_x + \bar{V}_y + \bar{V}_z$

These components are essentially $\perp^r$ to each other and are called independent components. The process or formule for convertion of $\bar{V}$ into $\bar{V}_x$ and $\bar{V}_y$ is called resolution of vectors.

Now, we will understand application of resolution of vectors in projectile motion. A particle when thrown with some initial velocity making an angle of inclination 'θ' with the horizontal.



Motion of this particle can be independently analysis for $\bar{V}_x$ & $\bar{V}_y$ two components +

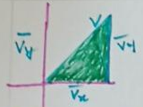
$\bar{V} = \bar{V}_x + \bar{V}_y$

This is called parallelogram of vectors.

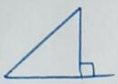
Since each component can be independently analysed therefore equation of motion are applied for each component in scales. These components are $\perp^r$ to each other and arthogenal component.

Now let us see how can we take result of two vectors. This result is called resultant vector.

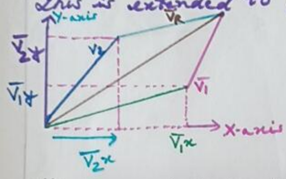
$$\underbrace{\bar{V}_R}_{\text{Resultant}} = \underbrace{\bar{V}_1}_{\text{Vector 1}} + \underbrace{\bar{V}_2}_{\text{Vector 2}}$$



This is Known as triangle



This is extended to polygon vector.



We have resolved V_1 & V_2 into two components as shown in figure

$$\left.\begin{aligned}\bar{V}_x &= \bar{V}_{1x} + \bar{V}_{2x} \\ \bar{V}_y &= \bar{V}_{1y} + \bar{V}_{2y}\end{aligned}\right] \text{Resultant vector \& components}$$

$\bar{V}_R = \bar{V}_x + \bar{V}_y$

$V_R = \sqrt{V_x^2 + V_{2y}^2}$

$\tan\theta_R = \frac{V_y}{V_x} = \tan^{-1}\left(\frac{V_y}{V_x}\right)$

$\bar{V}_x = \bar{V}_{1x} + \bar{V}_{2x} \cdots\cdots \bar{V}_{nx}$

$\bar{V}_x = \sum_{i=1}^{n} \bar{V}_{ix}$

Likewise $\bar{V}_y = \sum_{i=1}^{n} \bar{V}_{iy}$

Determination of magnitude of vectors $\bar{V}_R$ involves use of trigonometric identities therefore the basic identities and their ~~deter~~ derivation are discussed here.

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

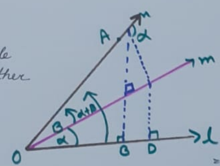
$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

In the diagram EBCD is a rectangle and are $\perp^r$ to each other

$\therefore EC = BD$

& $EB = CD$



In ΔOAB,

$\sin(\alpha+\beta) = \frac{AB}{AO}$

$= \frac{AE+EB}{AO}$

Since $EB = CD$

$= \frac{AE}{AO} \times \frac{AC}{AC} + \frac{CD}{AO} \times \frac{OC}{OC}$

$= \frac{AE}{AC} \times \frac{AC}{AO} + \frac{CD}{OC} \times \frac{OC}{AO}$

$(\Delta AEC)(\Delta AOC)(\Delta OCD)(\Delta AOC)$

$\sin(\alpha+\beta) = \cos\alpha \cdot \sin\beta + \sin\alpha\cos\beta$

Similar for identity $\cos(\alpha+\beta)$ we take ΔOAB

$$\cos(\alpha+\beta) = \frac{OB}{OA} = \frac{OD-BD}{OA} = \frac{OD}{OA} \times \frac{OC}{OC} - \frac{EC}{OA} \times \frac{AC}{AC}$$

Re-arranging

$$\cos(\alpha+\beta) = \frac{OD}{OC} \times \frac{OC}{OA} - \frac{EC}{AC} \times \frac{AC}{OA}$$

$(\Delta OCD)(\Delta OAC)(\Delta EAC)(\Delta ACD)$

Now using above identity we can determine $\cos(\alpha-\beta)$ for which it is to recall.

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$$\cos(\alpha-\beta) = \cos[\alpha+(-\beta)] = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$
$$= \cos\alpha\cos\beta - \sin\alpha(-\sin\beta)$$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Similarly we will determine $\sin(\alpha-\beta)$

$$\sin(\alpha-\beta) = \sin[\alpha+(-\beta)]$$
$$= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$
$$= \sin\alpha\cos\beta + \cos\alpha(-\sin\beta)$$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ — (i)

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ — (ii)

Extending these identities we have $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

$$\sin\beta - \sin A = 2\sin\frac{A+B}{2}\cos\frac{B-A}{2}$$

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha \cdot \sin\beta$$

$$\sin B - \sin A = 2\cos\frac{A+B}{2}\sin\frac{B-A}{2}$$

as $\sin A - \sin B = 2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$A = \alpha-\beta$

$B = \alpha+\beta$

$\frac{A+B}{2} = \alpha$

$\frac{B-A}{2} = \beta$

Similarly

$$\cos(\alpha-\beta) + \cos(\alpha+\beta) = 2\cos\alpha\cos\beta$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{D-C}{2}$$

$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$$
$$= -2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)}$$

$$= \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Likewise

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$

$$\sin 2A = \sin(A+A) = \sin A\cos A + \cos A\sin A$$
$$\sin 2A = 2\sin A\cos A$$

Similarly, we can verify

$$\cos 2A = \cos(A+A) = \cos A\cos A - \sin A\sin A$$

$$\cos 2A = \begin{cases} = \cos^2 A - \sin^2 A \\ = 2\cos^2 A - 1 \\ = 1 - 2\sin^2 A \end{cases} \quad \because [\sin^2\theta + \cos^2\theta = 1]$$

System is represented as function $f(x)$ similar to a polynomial $p(x)$

For example :–

$P(x) = a_0x^n + a_1x^{n-1} \ldots\ldots a_n$

and $f(x) = a_0x^n + a_1x^{n-1} \ldots\ldots a_n$

Eventually $f(x) = p(x)$

Systems way to look at when it is used in Calculas and function is split into its factor while solving

$$f(x) = (x+\alpha_1)(x+\alpha_2)\ldots(x+\alpha_n)$$

Here, $-\alpha_1, -\alpha_2 \ldots -\alpha_n$ are roots of the eq^n polynomial as a function

[To visualize how complex numbers are generated. We will take a simple quadratic equation]

We will take $x^2 + bx + c = 0$; $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\alpha = \frac{-b \pm \sqrt{b^2-4ac}}{2a}; \quad \beta = \frac{-b - \sqrt{b^2-4ac}}{2a}$$

$\alpha = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$ Here $D = b^2-4ac$ and it is called discriminant.

There are three cases

(a) If $D > 0$ then $\alpha = \frac{-b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}$

$$\beta = \frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$$

or vice versa. Here roots are equal.

(b) If $D = 0$, then $\alpha = \beta = \frac{-b}{2a}$ i.e roots are unequal

(c) If $D < 0$ i.e -ve since square root of +ve number can be determined.

$\therefore D = |-D|$

$$\sqrt{D} = \sqrt{(-1) \times |D|}$$
$$= \sqrt{-1} \times \sqrt{|D|}$$
$$= i\sqrt{|D|}$$

Here $i = \sqrt{-1}$ is an imaginary identity.

Thus, $\alpha = \frac{-b}{2a} + \frac{i\sqrt{|D|}}{2a} = \underbrace{p + iq}_{\text{Complex number}}$

$\beta = \frac{-b}{2a} - \frac{i\sqrt{|D|}}{2a} = \underbrace{p - iq}_{\text{Complex number}}$

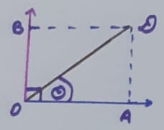
Here p – is real number

q – is imaginary number

And, α and β are called complex conjugate roots.

If we have to find component of P_y along X-axis

$P_{y-x} = P_y \cos 90°$
$= 0$



Let $OP = r$

$OA = P_x = r\cos\theta$

$OB = P_y = r\sin\theta$

P_y has no resolution as effect along X-axis

likewise, $P_{x-y} = P_x \cos 90° = 0$, P_x has no resolution as effect along Y-axis.

Therefore components of a complex number (similar to vector) along $\perp^r$ axes is zero thus can be analysed Inde-pendently.

The logical conclusion is if any complex number like any other vector can be resolved into two components each of them is independent of the other.

Accordingly In complex number $\bar{Z} = p + iq$
real imaginary

We cannot add imaginary part & real part of two complex number. Rather real part are added together and imaginary parts are added together.

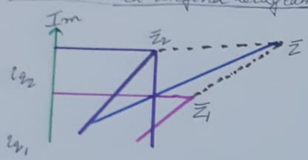
This is explained in the example

$\bar{Z}_1 = p_1 + iq_1$
$\bar{Z}_2 = p_2 + iq_2$

$z = \bar{Z}_1 + \bar{Z}_2 = (p_1 + p_2) + i(q_1 + q_2)$

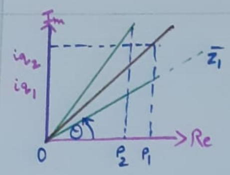
Real part are together — Imaginary part are added together

This is called Argand Diagram



$\therefore \bar{Z} = \bar{Z}_1 + \bar{Z}_2$
$= (p_1 + p_2) + i(q_1 + q_2)$

This is similar to addition of vector and therefore complex numbers is also represented like a vector



In this θ — It is called amplitude of complex number.

Z – It is called argument of complex number.

$\bar{Z} = Z_{Re} + iZ_{im} = Ze^{i\theta} = Z\cos\theta + iz$

$Z_{Re} = Z\cos\theta$
$Z_{im} = Z\sin\theta$

$Ze^{i\theta}$ Exponential form of complex number

$\bar{Z}$ is complex number

Representation of Complex Numbers

Cartesian form — Exponential form

Both forms are equivalent

$\bar{Z} = p + iq \equiv \bar{Z} = Ze^{i\theta} = Z\cos\theta + iz\sin\theta$

($\because p = Z\cos\theta$; $q = Z\sin\theta$, i is coefficient of imaginary term of complex numbers)

For addition:

Cartesian form is used even if complex numbers is available in exponential form, it is converted in cartesian form. But for multiplication polar form of complex number is most suitable by using theory of Indices.

$\bar{Z}_1 = Z_1e^{i\theta_1}$, $\bar{Z}_2 = Z_2e^{i\theta_2}$

$\bar{Z}_1 \times \bar{Z}_2 = (Z_1e^{i\theta_1}) \times (Z_2e^{i\theta_2})$

$\Rightarrow (Z_1 \times Z_2)e^{i\theta_1} \times e^{i\theta_2}$

$\Rightarrow Z_1Z_2e^{i\theta_1 + i\theta_2}$

$= Z_1Z_2e^{i(\theta_1+\theta_2)}$

$\bar{Z}_1 = \bar{Z}_1 \times \bar{Z}_2 = Z_1Z_2e^{i(\theta_1+\theta_2)} = Z_1Z_2\cos\theta(\theta_1 + \theta_2) + iZ_1Z_2\sin(\theta_1+\theta_2)$

Similarly for division exponential form of complex number is used as discussed below

$$\bar{Z}^{-1} = \frac{\bar{Z_1}}{\bar{Z_2}} = \frac{Z_1 e^{i\theta_1}}{Z_2 e^{i\theta_2}} = \left(\frac{Z_1}{Z_2}\right)\left(\frac{e^{i\theta_1}}{e^{i\theta_2}}\right) = \left(\frac{Z_1}{Z_2}\right) e^{i(\theta_1-\theta_2)} \quad \left[\text{Theory of indices } \frac{x^a}{x^b} = x^a \times x^{-b} = x^{a-b}\right]$$

$$= \frac{Z_1}{Z_2}\cos(\theta_1-\theta_2) + i\frac{Z_1}{Z_2}\sin(\theta_1-\theta_2)$$

$$\bar{Z} = \bar{Z_1} + \frac{\bar{Z_2}}{\bar{Z_3}} = \underbrace{Z_1\cos\theta_1}_{\text{Realparts}} + \underbrace{i\sin\theta_1}_{\text{Imaginary}} + \left(\underbrace{\frac{Z_2}{Z_3}\cos(\theta_2-\theta_3)}_{\text{Realpart}} + \underbrace{i\sin(\theta_2-\theta_3)}_{\text{Imaginary}}\right)$$

Add real parts together and like wise imaginary parts

$$\bar{Z} = \left[Z_1\cos\theta_1 + \frac{Z_2}{Z_3}\cos(\theta_2-\theta_3)\right] + i\left[Z_1\sin\theta_1 + \frac{Z_2}{Z_3}\sin(\theta_2-\theta_3)\right]$$

Here, it is essential to understand how exponential form is related to cartesian form i.e $e^{i\theta} = \cos\theta + i\sin\theta$
This require to go back to algebra starting with permutation, combinations and binomial theorem

In algebra $\lfloor\underline{n}$ as $n!$ is read as factorial of n which is mathematically expressed as $\lfloor\underline{n} = n \times (n-1)(n-2)\ldots 2\times 1$
$= n\lfloor\underline{n-1}$

Further conceptually $\lfloor\underline{n}$ is all possible arrangements of n objects, taking all together

$n=1$, A $-\lfloor\underline{1} = 1$

$n=2$; AB, BA $-\lfloor\underline{2} = 2\times 1 = 2$

$n=3$; ABC, ACB, BAC, BCA, CBA, CAB

$2\times 3 = 6$, $\lfloor\underline{3} = 3\times 2\times 1 = 6$

$\lfloor\underline{n} = n\times(n-1)\ldots\ldots 2\times 1$

But when we have nothing i.e $n=0$ there is only one way that nothing is choosen, as there is nothing to choose $\therefore \lfloor\underline{0} = 1$

But when out of n things r things are choosen then it is called permutation and represented mathematically as

$^nP_r = \frac{\lfloor\underline{n}}{\lfloor\underline{n-r}}$ Therefore if out of 3 objects A, B and C two objects than possible arrangements are

$$\left.\begin{array}{l}\text{Choosing BC, BC, CB} - 2\\ \text{choosing CA, CA, AC} - 2\\ \text{Choosing AB, AB, BA} - 2\end{array}\right]\ 2\times 3 = 6$$

This is mathematically represent as $^3P_2 = \frac{\lfloor\underline{3}}{\lfloor\underline{3-2}}$

$$= \frac{\lfloor\underline{3}}{\lfloor\underline{1}}$$

$$= \frac{3\times 2\times 1}{1}$$

$$= 6$$

Combinations

If out of three things two things taken at a time, ignoring arrangement which is called combinations is mathematically expressed as $^nC_r = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\lfloor\underline{n-r}}$

Taking example of ABC again, taking two alphabets at a time

taking B,C → BC & CB – forms one combination
C,A → CA & AC – form one combination
A,B → AB & BA – form one combination

Total no. of combination 3

This is tested mathematically where $n=3$, $r=2$

$$\therefore {}^nC_r = {}^3C_2 = \frac{\lfloor\underline{3}}{\lfloor\underline{2}\lfloor\underline{3-2}} = \frac{\lfloor\underline{3}}{\lfloor\underline{2}\lfloor\underline{1}} = \frac{3\cancel{\lfloor\underline{2}}}{\cancel{\lfloor\underline{2}}\times 1} = 3$$

With this we will proceed further to understand algebra through Binomial Theorem in content of complex number

Binomial Theorem

$(x+a) = x+a$

$(x+a)(x+b) = x^2 + (a+b)x + ab$

Number of combination of two variable at a time

Number of combinations of one variable taken at a time = 2

$(x+a)(x+b)(x+c) = \left[x^2 + (a+b)x + ab\right](x+c)$

$= x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc$

Out of three variables combination of one variable taken at a time = 3.

$$^nC_r = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\lfloor\underline{n-r}}$$

$$^3C_1 = \frac{\lfloor\underline{3}}{\lfloor\underline{1}\lfloor\underline{3-1}}$$

$$= \frac{\lfloor\underline{3}}{\lfloor\underline{2}}$$

$$= \frac{3\cancel{\lfloor\underline{2}}}{\cancel{\lfloor\underline{2}}}$$

$$= 3$$

Out of three variables combination of two variable taken at a time = 3

$$^3C_2 = \frac{\lfloor\underline{3}}{\lfloor\underline{2}\lfloor\underline{3-2}}$$

$$= \frac{\lfloor\underline{3}}{\lfloor\underline{2}\times\lfloor\underline{1}} = \frac{\lfloor\underline{3}}{\lfloor\underline{2}} = \frac{3\cancel{\lfloor\underline{2}}}{\cancel{\lfloor\underline{2}}} = 3$$

Out of 3 variables combination of three variables taken at a time = 1

$$^3C_3 = \frac{\lfloor\underline{3}}{\lfloor\underline{3}\,\lfloor\underline{3-3}}$$

$$= \frac{\cancel{\lfloor\underline{3}}}{\cancel{\lfloor\underline{3}}\,\lfloor\underline{0}}$$

$$= \frac{1}{\lfloor\underline{0}}$$

$$= \frac{1}{1}\left[\because \lfloor\underline{0} = 1\right]$$

$$= 1$$

Taking forward the concept of combination

$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc$$

$$(x+1)(x+1)(x+1) = (x+1)^3 \left[\text{Here } a = 1, b = 1, c = 1\right]$$

$$= x^3 + 3x^2 + 3x + 1$$

$$= x^3 + {}^nC_1 x^2 + {}^nC_2 x + {}^nC_3 1$$

Power of x is decreasing

These identities extended (or extra polated) to $(x+a)^n$

$$(x+1)^n = (1+x)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2$$

$${}^nC_r x^{n-r} a^r \ldots\ldots\ a^n$$

$$1 \le r \le n$$

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 \ldots\ldots\ {}^nC_n x^n$$

This is called binomial theorem or binomial expansion

Taking forward $(1+x)^n$ to $\left(1+\frac{1}{n}\right)^n\Big|_{n\to 0}$ i.e $x = \frac{1}{n}$

$$\left(1+\frac{1}{n}\right)^n\Big|_{n\to\infty} = 1 + {}^nC_1\left(\frac{1}{n}\right) + {}^nC_2\left(\frac{1}{n}\right)^2 \ldots\ldots\ldots$$

$$= 1 + \frac{\lfloor\underline{n}}{\lfloor\underline{1}\,\lfloor\underline{n-1}} \cdot \frac{1}{n} + \frac{n(n-1)\lfloor\underline{n-2}}{\lfloor\underline{2}\,\lfloor\underline{n-2}}\left(\frac{1}{n}\right)^2$$

$$= 1 + \frac{n\lfloor\underline{n-1}}{\lfloor\underline{n-1}} \times \frac{1}{n} + \frac{n(n-1)}{\lfloor\underline{2}}\left(\frac{1}{n}\right)^2$$

$$= 1 + 1 + \frac{n^2}{\lfloor\underline{2}} \cdot \frac{1}{n}\left[\text{because } n-1 = n\big|_{n\to\infty}\right]$$

$$\left(1+\frac{1}{n}\right)^n_{n\to\infty} = 1 + 1 + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \frac{1}{\lfloor\underline{4}} \ldots$$

$$e = \left(1+\frac{1}{n}\right)^n\Big|_{n\to 0} = 1 + \frac{1}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}}$$

$$e \simeq 2.718$$

This e is called Napiers constant or Eucleirs number

This is naturally occuring irrational number and will have lot of utility in physics.

Now to determine e^x we go back to binomial theorem

$$e^x = \left(\left(1+\frac{1}{n}\right)^n\right)^x\Big|_{x\to\infty} = \left(1+\frac{1}{n}\right)^{nx}\Big|_{n\to\infty}$$

$$\left(1+\frac{1}{n}\right)^x = 1 + {}^{nx}C_1\left(\frac{1}{n}\right) + {}^{nx}C_2\left(\frac{1}{n}\right)^2 \ldots\ldots\ldots$$

$$= 1 + \frac{\lfloor\underline{nx}}{\lfloor\underline{1}\,\lfloor\underline{nx-1}} \times \frac{1}{n} + \frac{\lfloor\underline{nx}}{\lfloor\underline{2}\,\lfloor\underline{nx-2}} \times \frac{1}{n^2}$$

$$= 1 + \frac{nx\,\cancel{\lfloor\underline{nx-1}}}{\cancel{\lfloor\underline{nx-1}}} \times \frac{1}{x} + \frac{(nx)(nx-1)\cancel{\lfloor\underline{nx-2}}}{\lfloor\underline{2}\,\cancel{\lfloor\underline{nx-2}}} \times \frac{1}{n^2}$$

$$nx - 1 \to nx\big|_{n} = \infty$$

$$e^x = 1 + \frac{x}{\lfloor\underline{1}} + \frac{n^2 x^2}{\lfloor\underline{2}} \times \frac{1}{n^2}$$

$$= 1 + \frac{x}{\lfloor\underline{1}} + \frac{x^2}{\lfloor\underline{2}} + \frac{x^3}{\lfloor\underline{3}} \ldots\ldots$$

Similarly

$$e^{-x} = 1 - \frac{x}{\lfloor\underline{1}} + \frac{x^2}{\lfloor\underline{2}} - \frac{x^3}{\lfloor\underline{3}} \ldots$$

Determining $e^{i\theta}$ and $e^{-i\theta}$

$$e^x = 1 + \frac{x}{\lfloor\underline{1}} + \frac{x^2}{\lfloor\underline{2}} + \frac{x^3}{\lfloor\underline{3}} \ldots\ldots\ldots$$

$$\cancel{i = \sqrt{-1}}\quad i^n = i^{\left|\frac{n}{4}\right| + r}$$

$$\left|\frac{n}{4}\right| \times i^r \Big|_{0<r<4}$$

As per euclid's lemma

$$i = \sqrt{-1},\ i^2 = -1,\ i^3 = -i,\ i^4 = 1$$

for $i^n\big|_{n\in N}$

$$i^5 = i^4 \times i = i$$

$$i^6 = i^4 \times i^2 = -1$$

$$i^7 = i^4 \times i^3 = -i$$

$$i^8 = i^4 \times i^4 = 1$$

$$e^{i\theta} = 1$$

$$e^{i\theta} = 1 + \frac{i\theta}{\lfloor\underline{1}} + \frac{(i\theta)^2}{\lfloor\underline{2}} + \frac{(i\theta)^3}{\lfloor\underline{3}} + \frac{(i\theta)^4}{\lfloor\underline{4}} + \frac{(i\theta)^5}{\lfloor\underline{5}} + \frac{(i\theta)^6}{\lfloor\underline{6}}$$

$$e^{i\theta} = 1 + \frac{i\theta}{\lfloor\underline{1}} - \frac{\theta^2}{\lfloor\underline{2}} - i\frac{\theta^3}{\lfloor\underline{3}} + \frac{\theta^4}{\lfloor\underline{4}} + i\frac{\theta^5}{\lfloor\underline{5}} - \frac{\theta^6}{\lfloor\underline{6}} - \text{(i)}$$

Complex numbers can be refered in cartesian form & exponent-ial form both forms are equivalent.

$$\bar{Z} = p + iq = Ze^{i\theta} = Z\cos\theta + iZ\sin\theta$$

Similarly, $e^{-i\theta} = 1 + \frac{(-i\theta)}{\lfloor 1} + \frac{(-i\theta)^2}{\lfloor 2} + \frac{(-i\theta)^3}{\lfloor 3} + \frac{(-i\theta)^4}{\lfloor 4} + \frac{(-i\theta)^5}{\lfloor 5} + \frac{(-i\theta)^6}{\lfloor 6}$

$$= 1 - \frac{i\theta}{\lfloor 1} + \frac{(i\theta)^2}{\lfloor 2} - \frac{(i\theta)^3}{\lfloor 3} + \frac{(i\theta)^4}{\lfloor 4} - \frac{(i\theta)^5}{\lfloor 5} + \frac{(i\theta)^6}{\lfloor 6}$$

$$e^{-i\theta} = 1 - \frac{i\theta}{\lfloor 1} - \frac{\theta^2}{\lfloor 2} + i\frac{\theta^3}{\lfloor 3} + \frac{\theta^4}{\lfloor 4} - \frac{i\theta^5}{\lfloor 5} - \frac{\theta^6}{\lfloor 6} - \text{(ii)}$$

$$e^{i\theta} = 1 + i\frac{\theta}{\lfloor 1} - \frac{\theta^2}{\lfloor 2} - \cancel{i\frac{\theta^3}{\lfloor 3}} + \frac{\theta^4}{\lfloor 4} + \cancel{i\frac{\theta^5}{\lfloor 5}} - \frac{\theta^6}{\lfloor 6}$$

$$e^{-i\theta} = 1 - \frac{i\theta}{\lfloor 1} - \frac{\theta^2}{\lfloor 2} + \cancel{\frac{i\theta^3}{\lfloor 3}} + \frac{\theta^4}{\lfloor 4} - \cancel{\frac{i\theta^5}{\lfloor 5}} - \frac{\theta^6}{\lfloor 6}$$

$$e^{i\theta} + e^{-i\theta} = 2 - \frac{2\theta^2}{\lfloor 2} + \frac{2\theta^4}{\lfloor 4} - \frac{2\theta^6}{\lfloor 6}$$

$$e^{i\theta} + e^{-i\theta} = 2\left[1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} + \frac{\theta^6}{\lfloor 6}\dots\right]$$

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \frac{\theta^6}{\lfloor 6} \dots\dots]$$

$\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$ which is real number.

It is also written $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$. Though constitution of $\cos\theta$, i.e. $e^{i\theta}$ and $e^{-i\theta}$ are complex but $\cos\theta$ is real.

$$e^{i\theta} = 1 + i\frac{\theta}{\lfloor 1} - \frac{\theta^2}{\lfloor 2} - i\frac{\theta^3}{\lfloor 3} + \frac{\theta^4}{\lfloor 4} + i\frac{\theta^5}{\lfloor 5} - \frac{\theta^6}{\lfloor 6}$$

$$-e^{-i\theta} = -\left(1 - \frac{i\theta}{\lfloor 1} - \frac{\theta^2}{\lfloor 2} + \frac{i\theta^3}{\lfloor 3} + \frac{\theta^4}{\lfloor 4} - \frac{i\theta}{\lfloor 5} - \frac{\theta}{\lfloor 6}\right)$$

$$e^{i\theta} - e^{-i\theta} = 1 + \frac{i\theta}{\lfloor 1} - \frac{\theta^2}{\lfloor 2} - \frac{i\theta^3}{\lfloor 3} + \frac{\theta^4}{\lfloor 4} + \frac{i\theta^5}{\lfloor 5} - \frac{\theta^6}{\lfloor 6}$$

$$= 1 + i\theta + \frac{\theta^2}{\lfloor 2} - \frac{i\theta^3}{\lfloor 3} - \frac{\theta^4}{\lfloor 4} + \frac{i\theta^5}{\lfloor 5} + \frac{\theta^6}{\lfloor 6}$$

$$e^{i\theta} - e^{-i\theta} = \frac{2i\theta}{\lfloor 1} - \frac{2i\theta^3}{\lfloor 3} + \frac{2i\theta^5}{\lfloor 5} \dots\dots$$

$$\frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{\theta}{\lfloor 1} - \frac{\theta^3}{\lfloor 3} + \frac{i\theta^5}{\lfloor 5}$$

$$= \sin\theta$$

as $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$ [This $\sin\theta$ is also a real number despites its constituents be complex]

$$\overline{Z} = Ze^{i\theta} = Z\cos\theta + i(Z\sin\theta)$$
$$= p + iq$$

$\angle\theta = Z$ Here p is real $= Z\cos\theta$

q is real $= Z\sin\theta$

$$P = Z\cos\theta = Z\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) = \frac{Z}{2}\left[e^{i\theta} + e^{-i\theta}\right]$$

$$q = Z\sin\theta = Z\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) = \frac{Z}{2i}\left[e^{i\theta} - e^{-i\theta}\right]$$

$$\overline{Z} = Ze^{i\theta} = p + iq = \frac{Z}{2}\left[e^{i\theta} + e^{-i\theta}\right] + i\left[\frac{Z}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right]$$

$$= \frac{Z}{2}\left[e^{i\theta} + e^{-i\theta} + e^{i\theta} - e^{-i\theta}\right]$$

$$= \frac{Z}{2} \times 2e^{i\theta} = Ze^{i\theta}$$

This is a proof that how $\overline{Z} = Ze^{i\theta} = Z\cos\theta + iZ\sin\theta$

Observation

(i) Every even term in series is (-)ve and every add term in series is +ve.

(ii) Power of θ and factorial denominator is some number

$\frac{\theta^K}{\lfloor K} \mid K \in \omega$

(iii) All term in expansion are retal.

SOLVED EXAMPLES :—

$$\overline{Z} = Ze^{i\theta}$$

$\overline{\overline{Z}} = Ze^{\overline{i\theta}}$ | Conjugate of $\overline{Z}$

$$= Ze^{-i\theta}$$

$$= \overline{Z} \times \overline{\overline{Z}} = (Ze^{i\theta}) \times (Ze^{-i\theta})$$

$$= Z^2$$

Example 1 :— $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\overline{z_2})$

Sol^n L.H.S

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$$

$$\Rightarrow |z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \left[\because \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}\right]$$

$$\Rightarrow |z_1 + z_2|^2 = z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + z_2\overline{z_1}$$

$$\Rightarrow |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + (\overline{z_1\overline{z_2}})$$

$$\Rightarrow |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\overline{z_2})$$ R.H.S

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Example 2:- $|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$

Sol^n L.H.S

$|z_1+z_2|^2+|z_1-z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) + (z_1-z_2)(\overline{z_1-z_2})$

$= (z_1+z_2)(\bar z_1+\bar z_2) + (z_1-z_2)(\bar z_1-\bar z_2)$ $[\because \overline{z_1+z_2} = \bar z_1+\bar z_2,\ \overline{z_1-z_2} = \bar z_1-\bar z_2]$

$= z_1\bar z_1 + z_2\bar z_2 + z_1\bar z_2 + z_2\bar z_1 + z_1\bar z_1 + z_2\bar z_2 - z_1\bar z_2 - z_2\bar z_1$

$= |z_1|^2 + |z_2|^2 + z_1\bar z_2 + (\overline{z_1\bar z_2}) + |z_1|^2 + |z_2|^2 - z_1\bar z_2 - (\overline{z_1\bar z_2})$

$= |z_1|^2 + |z_2|^2 + \cancel{2Re(z_1\bar z_2)} + |z_1|^2 + |z_2|^2 - \cancel{2Re(z_1\bar z_2)}$

$= 2(|z_1|^2 + |z_2|^2)$ R.H.S

Example 3: $z\bar z = |z|^2$

Sol^n Let $z = a+ib$ And $\bar z = a-ib$

$z\bar z = (a+ib)(a-ib) = a^2 - i^2b^2$

$= a^2 - (-1)b^2$

$= a^2 + b^2$

$= \{\sqrt{a^2+b^2}\}^2$

$= |z|^2$

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The student studies in class XIth (PCM) at Kendriya Vidhyalaya, Dinjan, Dist. Tinsukia, Aassam. She is participating in Interactive Inline Mentoring Session (IOMS).

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