LET US DO SOME PROBLEMS-XXXIV

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Mathematics Olympiad is a real test of the students' knowledge of the Mathematics across the world. Some questions from the latest test have been selected for the readers to test their standard of preparation. Only answer has been given at the end of each question. No solution is being written. If any reader wants solution, he or she may request the Coordinator's desk for that.

QUESTIONS

Q1. Suppose $r \geq 2$ is an integer, and let m_1, n_1, m_2, n_2,…,m_r, n_r be 2r integers such that $|m_i n_j - m_j n_i| = 1$ for any two integers i and j satisfying $1 \leq i < j \leq r$. Determine the maximum possible value of r.

Ans. 3

Q2. Find all pairs of integers (a, b) so that each of the two cubic polynomials x^3+ax+b and x^3+bx+a has all the roots to be integers.

Ans. The only pair is (0,0)

Q3. Betal marks 2021 points on the plane such that no three are collinear, and draws all possible line segments joining these. He then chooses any 1011 of these line segments, and marks their midpoints. Finally, he chooses a line segment whose midpoint is not marked yet, and challenges Vikram to construct its midpoint using only a straightedge. Can Vikram always complete this challenge?
Note: A straightedge is an infinitely long ruler without markings, which can only be used to draw the line joining any two given distinct points.

Ans. Yes

Q4. A Magician and a Detective play a game. The Magician lays down cards numbered from 1 to 52 face-down on a table. On each move, the Detective can point to two cards and inquire if the numbers on them are consecutive. The Magician replies truthfully. After a finite number of moves the Detective points to two cards. She wins if the numbers on these two cards are consecutive, and loses otherwise.

Prove that the Detective can guarantee a win if and only if she is allowed to ask at least 50 questions.

Q5. In a convex quadrilateral ABCD, $\angle ABD = 30^0$, $\angle BCA = 75^0$, $\angle ACD = 25^0$ and CD = CB. Extend CB to meet the circumcircle of triangle DAC at E. Prove that CE = BD.

Q6. Let ABCD be a trapezium in which AB || CD and AB = 3CD. Let E be the midpoint of the diagonal BD. If [ABCD] = n×[CDE], what is the value of n?

(Here [□] denotes the area of the geometrical figure)

Ans.08

Q7. A number N in base 10, is 503 in base b and 305 in base b + 2. What is the product of the digits of N?

Ans.64

Q8. If $\sum_{k=1}^{N} \frac{2k+1}{(k^2+k)^2} = 0.9999$ then determine the value of N.

Ans. 99

Q9. Let ABCD be a rectangle in which AB + BC + CD = 20 and AE = 9 where E is the mid-point of the side BC. Find the area of the rectangle.

Ans. 19

Q10. Find the number of integer solutions to $\left||x| - 2020\right| < 5$.

Ans. 18

Q11. What is the least positive integer by which 2^5. 3^6. 4^3. 5^3. 6^7 should be multiplied so that, the product is a perfect square ?

Ans. 15

Q12. Let ABC be a triangle with AB = AC. Let D be a point on the segment BC such that BD = $48\frac{1}{61}$ and DC = 61. Let E be a point on AD such that CE is perpendicular to AD and DE = 11. Find AE.

Ans. 25

Q13. A 5-digit number (in base 10) has digits *k, k + 1, k + 2, 3k, k + 3* in that order, from left to right. If this number is m^2 for some natural number m, find the sum of the digits of m.

Ans. 15

Q14. Let ABC be a triangle with AB = 5, AC = 4, BC = 6. The internal angle bisector of C intersects the side AB at D. Points M and N are taken on sides BC and AC, respectively, such that DM || AC and DN || BC. If $(MN)^2 = \frac{p}{q}$, where p and q are relatively prime positive integers then what is the sum of the digits of $|p - q|$?

Ans. 02

Q15. Five students take a test on which any integer score from 0 to 100 inclusive is possible. What is the largest possible difference between the median and the mean of the scores? (The median of a set of scores is the middlemost score when the data is arranged in increasing order. It is exactly the middle score when there are an odd number of scores and it is the average of the two middle scores when there are an even number of scores.)

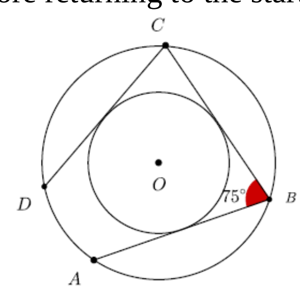
Ans. 40

Q16. Let X = {-5,-4,-3,-2,-1,0,1,2,3,4,5} and S ={$(a, b) \in$ X × X : $x^2 + ax + b$ and $x^3 + bx + a$ have at least a common real zero}.

How many elements are there in S?

Ans. 24

Q17. Given a pair of concentric circles (in the figure below), chords AB, BC, CD, …. of the outer circle are drawn such that they all touch the inner circle. If $\angle$ABC = 75^0, how many chords can be drawn before returning to the starting point?



Ans. 24

Q18. Find the sum of all positive integers n for which $|2^n + 5^n - 65|$ is a perfect square.

Ans. 06

Q19. The product $55\times 60\times 65$ is written as the product of five distinct positive integers. What is the least possible value of the largest of these integers?

Ans. 20

Q20. Three couples sit for a photograph in 2 rows of three people each such that no couple is sitting in the same row next to each other or in the same column one behind the other. How many arrangements are possible?

Ans. 96

Q21. The sides x and y of a scalene triangle satisfy $x + \frac{2\Delta}{x} = y + \frac{2\Delta}{y}$, where Δ is the area of the triangle. If x = 60; y = 63, what is the length of the largest side of the triangle?

Ans. 87

Q22. How many two digit numbers have exactly 4 positive factors? (Here 1 and the number n are also considered as factors of n.)

Ans. 30

Q23. If $\sum_{k=1}^{40}\left(\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}\right) = a + \frac{b}{c}$, where $a, b, c \in$ N, $b < c$, gcd(b, c) = 1, then what is the value of $a + b$?

Ans. 80

Q24. Find the number of pairs (a, b) of natural numbers such that b is a 3-digit number, a + 1 divides b-1 and b divides $a^2 + a + 2$.

Ans. 16

Q25. For a positive integer n, let (n) denote the perfect square integer closest to n. For example, (74) = 81, (18) = 16. If N is the smallest positive integer such that
<91>. <120>. <143>. <180>. <N> = 91. 120. 143. 180. N.

Find the sum of the squares of the digits of N.

Ans. 56

Q26. In triangle ABC, let P and R be the feet of the perpendiculars from A onto the external and internal bisectors of $\angle$ABC, respectively; and let Q and S be the feet of the perpendiculars from A onto the internal and external bisectors of $\angle$ACB, respectively. If PQ = 7, QR = 6 and RS = 8, what is the area of triangle ABC?

Ans. 84