Typical Problems: Application of Biot-Savart's Law

Problem 1: The wire PQR shown in the figure forms an equilateral triangle. Find the magnetic field B at the center O of the triangle assuming the wire to be uniform.

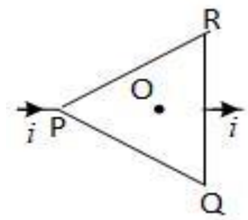
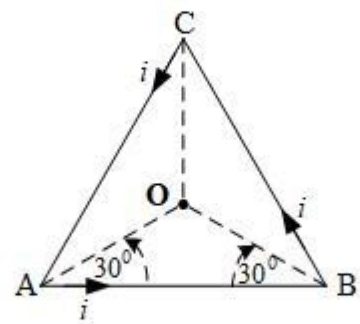


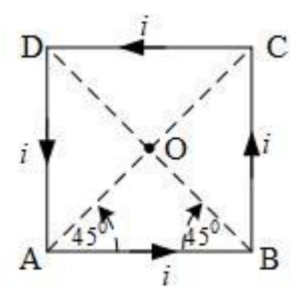
Illustration: Given is a wire of length l, carrying a current i. The problem is in two parts. Given is a wire of length l, carrying a current i. The problem is in two parts.

Part (a): The wire is bent to form an equilateral triangle ABC of side length $a = \frac{l}{3}$, as shown in the figure. The point O, center of the triangle, is symmetrical to its three sides with vertices making an angle $\theta = 30^0$. As discussed in Appendix-I, magnetic field at O due to current in the side AB is $\vec{B}_{\text{AB}} = \frac{\mu_0 i}{2\pi d} \cos\theta\,(\hat{n})$…(1), here $d = \frac{a}{2}\tan\theta$ is distance of point O from the side AB and $\theta = 30^0$ both determined geometrically, and $\hat{n}$ is direction of vector perpendicular to the plane of the triangle ABC. Using (1) and the available data, it leads $\vec{B}_{\text{AB}} = \frac{\mu_0 i}{2\pi\left(\frac{a}{2}\tan\theta\right)} \cos\theta\,(\hat{n}) \Rightarrow \vec{B}_{\text{AB}} = \frac{\mu_0 i \frac{\sqrt{3}}{2}}{\pi \frac{l}{3}\frac{1}{\sqrt{3}}} \Rightarrow \vec{B}_{\text{AB}} = \frac{9\mu_0 i}{2\pi l}(\hat{n})$…(2).



Current in all sides of the triangle is identical and unidirectional, i.e. anti-clockwise, hence each side w.r.t. which point O is symmetrical would produce equal and unidirectional magnetic field and hence at O, $\vec{B} = 3\vec{B}_{\text{AB}}$. Accordingly, magnitude of magnetic field at O is $B = 3 \times \frac{9\mu_0 i}{2\pi l} \Rightarrow B = \boldsymbol{\frac{27\mu_0 i}{2\pi l}}$ **is the answer of part (a).**

Part (b): In this case wire is bent in the form of a square ABCD of side length $a = \frac{l}{4}$, as shown in the figure. The point O, center of the square, is symmetrical to its four sides with vertices making an angle $\theta = 45^0$. As discussed in Appendix-I, magnetic field at O due to current in the side AB is $\vec{B}_{\text{AB}} = \frac{\mu_0 i}{2\pi d} \cos\theta\,(\hat{n})$…(1), here $d = \frac{a}{\sqrt{2}}$ is distance of point O from the side AB and $\theta = 45^0$ both determined geometrically, and $\hat{n}$ is direction of vector perpendicular to the plane of the triangle ABC. Using (1) and the available data, it leads $\vec{B}_{\text{AB}} = \frac{\mu_0 i}{2\pi\left(\frac{a}{2}\tan\theta\right)} \cos\theta\,(\hat{n}) \Rightarrow \vec{B}_{\text{AB}} = \frac{\mu_0 i \frac{1}{\sqrt{2}}}{\pi \frac{l}{4}\times 1} \Rightarrow \vec{B}_{\text{AB}} = \frac{2\sqrt{2}\mu_0 i}{\pi l}(\hat{n})$…(2).



Current in all sides of the square is identical and unidirectional, i.e. anti-clockwise, hence each side w.r.t. which point O is symmetrical would produce equal and unidirectional magnetic field and hence at O, $\vec{B} = 4\vec{B}_{\text{AB}}$. Accordingly, magnitude of magnetic field at O is $B = 4 \times \frac{2\sqrt{2}\mu_0 i}{\pi l} \Rightarrow B = \boldsymbol{\frac{8\sqrt{2}\mu_0 i}{\pi l}}$ **is the answer of part (b).**

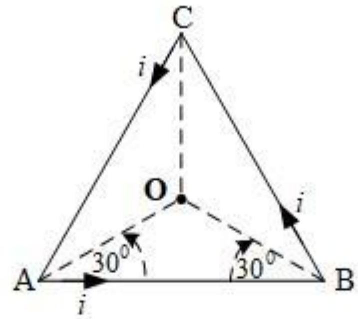
Thus answers are (a) $\boldsymbol{\frac{27\mu_0 i}{2\pi l}}$ **and (b)** $\boldsymbol{\frac{8\sqrt{2}\mu_0 i}{\pi l}}$**.**

Problem 2: A wire of length l is bent in the form of an equilateral triangle and carries an electric current i.

(a) Find the magnetic field B at the center.
(b) If the wire is bent in the form of a square, what would be the value of B at the center?

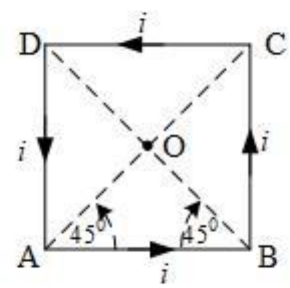
Illustration: Given is a wire of length l, carrying a current i. The problem is in two parts. Given is a wire of length l, carrying a current i. The problem is in two parts.

Part (a): The wire is bent to form an equilateral triangle ABC of side length $a = \frac{l}{3}$, as shown in the figure. The point O, center of the triangle, is symmetrical to its three sides with vertices making an angle $\theta = 30^0$. As discussed in Appendix-I, magnetic field at O due to current in the side AB is $\vec{B}_{\text{AB}} = \frac{\mu_0 i}{2\pi d}\cos\theta\,(\hat{n})$…(1), here $d = \frac{a}{2}\tan\theta$ is distance of point O from the side AB and $\theta = 30^0$ both determined geometrically, and $\hat{n}$ is direction of vector perpendicular to the plane of the triangle ABC. Using (1) and the available data, it leads $\vec{B}_{\text{AB}} = \frac{\mu_0 i}{2\pi\left(\frac{a}{2}\tan\theta\right)}\cos\theta\,(\hat{n}) \Rightarrow \vec{B}_{\text{AB}} = \frac{\mu_0 i\frac{\sqrt{3}}{2}}{\pi\frac{l}{3}\frac{1}{3\sqrt{3}}} \Rightarrow \vec{B}_{\text{AB}} = \frac{9\mu_0 i}{2\pi l}(\hat{n})$…(2).



Current in all sides of the triangle is identical and unidirectional, i.e. anti-clockwise, hence each side w.r.t. which point O is symmetrical would produce equal and unidirectional magnetic field and hence at O, $\vec{B} = 3\vec{B}_{\text{AB}}$. Accordingly, magnitude of magnetic field at O is $B = 3 \times \frac{9\mu_0 i}{2\pi l} \Rightarrow B = \boldsymbol{\frac{27\mu_0 i}{2\pi l}}$ **is the answer of part (a).**

Part (b): In this case wire is bent in the form of a square ABCD of side length $a = \frac{l}{4}$, as shown in the figure. The point O, center of the square, is symmetrical to its four sides with vertices making an angle $\theta = 45^0$. As discussed in Appendix-I, magnetic field at O due to current in the side AB is $\vec{B}_{\text{AB}} = \frac{\mu_0 i}{2\pi d}\cos\theta\,(\hat{n})$…(1), here $d = \frac{a}{\sqrt{2}}$ is distance of point O from the side AB and $\theta = 45^0$ both determined geometrically, and $\hat{n}$ is direction of vector perpendicular to the plane of the triangle ABC. Using (1) and the available data, it leads $\vec{B}_{\text{AB}} = \frac{\mu_0 i}{2\pi\left(\frac{a}{2}\tan\theta\right)}\cos\theta\,(\hat{n}) \Rightarrow \vec{B}_{\text{AB}} = \frac{\mu_0 i\frac{1}{\sqrt{2}}}{\pi\frac{l}{4}\times 1} \Rightarrow \vec{B}_{\text{AB}} = \frac{2\sqrt{2}\mu_0 i}{\pi l}(\hat{n})$…(2).

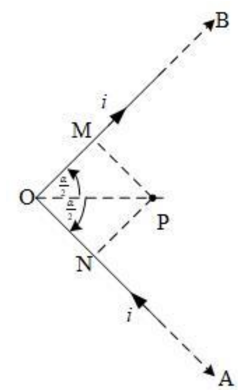


Current in all sides of the square is identical and unidirectional, i.e. anti-clockwise, hence each side w.r.t. which point O is symmetrical would produce equal and unidirectional magnetic field and hence at O, $\vec{B} = 4\vec{B}_{\text{AB}}$. Accordingly, magnitude of magnetic field at O is $B = 4 \times \frac{2\sqrt{2}\mu_0 i}{\pi l} \Rightarrow B = \boldsymbol{\frac{8\sqrt{2}\mu_0 i}{\pi l}}$ **is the answer of part (b).**

Thus answers are (a) $\boldsymbol{\frac{27\mu_0 i}{2\pi l}}$ **and (b)** $\boldsymbol{\frac{8\sqrt{2}\mu_0 i}{\pi l}}$**.**

Problem 3: A long wire carrying current i is bent to form a plane angle α. Find the magnetic field B at a distance x from the vertex.

Illustration: This problem combines asymmetry of point P w.r.t. a long wire AB bent at O at an angle α, such that point P at a distance x from the vertex O, is symmetrical to portions of wire AO and OB, as shown in the figure.



Applying Biot-Savart's Law magnetic field due current i in wire , as illustrated in Appendix-I, is $\overrightarrow{dB} = \frac{\mu_0 i}{4\pi}\frac{\sin\theta}{r^2} dl(\hat{n})\ldots(1)$. Here, dl is length of the element of wire carrying current i, r is the distance of the element from the point P at which magnetic field is to be determined, θ is the angle of the vector $\vec{r}$ w.r.t. $\overrightarrow{dl}$ in which current i is flowing and $\hat{n}$ is the unit direction vector of the magnetic field.

Applying (1) to the portion of wire OA wire the angle θ is clockwise direction and hence it's value is (-)ve and at A $\theta_i \to 0$ while at O $\theta_f = -\left(\pi - \frac{\alpha}{2}\right)$. Therefore, net magnetic field due to wire is $\vec{B}_{AB} = \left[\int_0^{-\left(\pi-\frac{\alpha}{2}\right)} \frac{\mu_0 i}{4\pi}\frac{\sin\theta}{r^2} dl\right]\hat{n}$...(2). This calculation can be simplified decomposing asymmetry of point P with the wire length AO as AO=AN+NO.

Accordingly, field due to wire AN is $\vec{B}_{\text{AN}} = \frac{1}{2}\left(\frac{\mu_0 i}{2\pi b}\right)(-\hat{n})$, here $b = \text{NP} = \text{OP}\sin\frac{\alpha}{2} \Rightarrow b = x\sin\frac{\alpha}{2}$, is half of the field due to a long wire. Thus using the available data $\vec{B}_{\text{AN}} = \frac{\mu_0 i}{4\pi x \sin\frac{\alpha}{2}}(-\hat{n})\ldots(3)$. But, for portion NO it is half of the length of wire symmetrical w.r.t. P along AO. Thus, $\vec{B}_{\text{NO}} = \frac{1}{2}\left(\frac{\mu_0 i}{2\pi b}\cos\frac{\alpha}{2}\right)(-\hat{n})$. Using the available data $\vec{B}_{\text{NO}} = \frac{1}{2}\left(\frac{\mu_0 i}{2\pi\left(x\sin\frac{\alpha}{2}\right)}\cos\frac{\alpha}{2}\right)(-\hat{n}) \Rightarrow \vec{B}_{\text{NO}} = \frac{\mu_0 i}{4\pi x}\cot\frac{\alpha}{2}(-\hat{n})\ldots(4)$.

Thus, net field at P due to wire AO, combining (3) and (4) is $\vec{B}_{\text{AO}} = \vec{B}_{\text{AN}} + \vec{B}_{\text{NO}} = \frac{\mu_0 i}{4\pi x\sin\frac{\alpha}{2}}(-\hat{n}) + \frac{\mu_0 i}{4\pi x}\cot\frac{\alpha}{2}(-\hat{n})$.

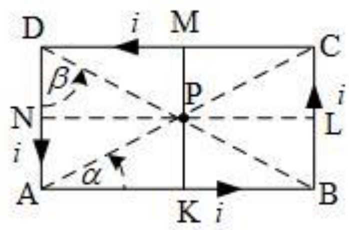
It leads to $\vec{B}_{\text{AO}} = \frac{\mu_0 i}{4\pi x}\frac{\left(1+\cos\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}}(-\hat{n})\ldots(5)$. Here, a little trigonometric manipulation is needed such that $\frac{\left(1+\cos\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}} = \frac{\left(1+\left(2\cos^2\frac{\alpha}{4}-1\right)\right)}{2\sin\frac{\alpha}{4}\cos\frac{\alpha}{4}} = \frac{2\cos^2\frac{\alpha}{4}}{2\sin\frac{\alpha}{4}\cos\frac{\alpha}{4}} = \frac{\cos\frac{\alpha}{4}}{\sin\frac{\alpha}{4}} = \cot\frac{\alpha}{4}\ldots(6)$. Combining, (5) and (6), $\vec{B}_{\text{AO}} = \frac{\mu_0 i}{4\pi x}\cot\frac{\alpha}{4}(-\hat{n})\ldots(7)$.

It is seen from the figure that magnetic field at P due to the same current i in portion OB is symmetrical and additive to the magnetic field due to the current in AO. Thus, magnitude of the net magnetic field at P due to wire AOB is $B = 2B_{AO} = 2 \times \frac{\mu_0 i}{4\pi x}\cot\frac{\alpha}{4} \Rightarrow$ **$B = \frac{\mu_0 i}{2\pi x}\cot\frac{\alpha}{4}$ is the answer.**

N.B.: Complexity in the solution of the problem due to integration involved in application of Biot-Sevart's Law can be avoided by using standard formulation of magnetic field due to a long wire and a section of wire symmetrically placed about a point. It involves systematic observation of the geometry of the problem combining asymmetry into symmetries.

Problem 4: Find the magnetic field B at the center of a rectangular loop of length l and width b, carrying a current i.

Illustration: Given system is shown in the figure in which a wire in shape of rectangular loop of length l and width b is carrying a current i. It is required to determine magnetic field B at point P.



Symmetry in the geometry of the problem reveals that sides AB and CD each of length l are carrying current in anti-clockwise direction with the only difference point P is on the left of the current in the sides AB and CD such that it is in the middle of the both the wires and at a distance $d_1 = \frac{b}{2}$ from them.

Thus magnetic field at P due to both the wires, as per Biot-Savart's Law illustrated in Appendix I, is $\vec{B}_{\text{AB}} = \vec{B}_{\text{CD}} = \frac{\mu_0 i}{2\pi}\frac{\cos\alpha}{d_1}(\hat{n})$…(1), here $\sin\alpha = \frac{\text{AK}}{\text{AP}} = \frac{\frac{l}{2}}{\frac{\sqrt{l^2+b^2}}{2}} \Rightarrow \sin\alpha = \frac{l}{\sqrt{l^2+b^2}}$

Similar geometry in respect of sides BC and CD leads to magnetic field at P is $\vec{B}_{\text{BC}} = \vec{B}_{\text{DA}} = \frac{\mu_0 i}{2\pi}\frac{\sin\beta}{d_2}(\hat{n})$…(1),here $\sin\beta = \frac{\text{PN}}{\text{DP}} = \frac{\frac{b}{2}}{\frac{\sqrt{l^2+b^2}}{2}} \Rightarrow \sin\beta = \frac{b}{\sqrt{l^2+b^2}}$. Here, $d_2 = \frac{l}{2}$

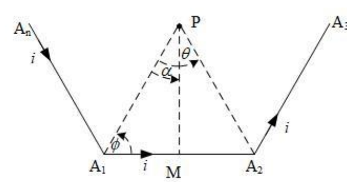
Thus net magnetic field at the center P is $\vec{B} = \vec{B}_{\text{AB}} + \vec{B}_{\text{BC}} + \vec{B}_{\text{CD}} + \vec{B}_{\text{DA}} = 2\frac{\mu_0 i}{2\pi}\frac{\sin\alpha}{d_1} + 2\frac{\mu_0 i}{2\pi}\frac{\sin\beta}{d_2}$.

Using the available data $\vec{B} = \left(2\frac{\mu_0 i}{2\pi}\frac{\frac{l}{\sqrt{l^2+b^2}}}{\frac{b}{2}} + 2\frac{\mu_0 i}{2\pi}\frac{\frac{b}{\sqrt{l^2+b^2}}}{\frac{l}{2}}\right)(\hat{n}) = \frac{2\mu_0 i}{\pi\sqrt{l^2+b^2}}\left(\frac{l}{b}+\frac{b}{l}\right)(\hat{n}) \Rightarrow B = \frac{2\mu_0 i}{\pi\sqrt{l^2+b^2}} \times \frac{l^2+b^2}{bl} \Rightarrow$ $B = \boldsymbol{\frac{2\mu_0 i}{\pi l b}\sqrt{l^2+b^2}}$ **is the answer.**

Problem 5: A regular polygon of n sides is formed by bending a wire of total length $2\pi r$ which carries a current i.

(a) Find the magnetic field B at the center of the polygon.
(b) By letting $n \to \infty$, deduce the expression for the magnetic field at the center of a circular current.

Illustration: Given system of a regular polygon having sides n made out by bending a wire of total length $l = 2\pi r$ is shown in the figure. Length of each side is $a = \frac{2\pi r}{n}$ and is carrying current i. Point P, at the center of the polygon is symmetrically placed w.r.t. each side of the polygon at a distance $d = \text{AM}\cos\alpha$. Here, $\text{AM} = \frac{a}{2}$, $\alpha = \frac{\theta}{2}$, where $\theta = \frac{2\pi}{n} \Rightarrow \alpha = \frac{\pi}{n}$. Therefore, $d = \frac{a}{2}\tan\alpha \Rightarrow d = \frac{a}{2}\tan\frac{\pi}{n}$…(1).



Using Biot-Savart's Law, as per Appendix-I, amgnetic field at P due current i in first side A_1A_2 of the polygon is $\vec{B}_1 = \frac{\mu_0 i}{2\pi}\frac{\cos\phi}{d}(\hat{n})$ …(2). Here, again from geometry $\phi = \frac{\pi}{2} - \alpha \Rightarrow \phi = \frac{\pi}{2} - \frac{\pi}{n} \Rightarrow \tan\phi = \tan\left(\frac{\pi}{2} - \frac{\pi}{n}\right) \Rightarrow \tan\phi = \cot\frac{\pi}{n}$ …(3).

Each branch of the polygon would produce equal and unidirectional magnetic field at P such that magnitude of the magnetic field would be $B = nB_1 \Rightarrow B = n \times \frac{\mu_0 i}{2\pi}\frac{\cos\phi}{d}$. Using the available data $B = \frac{n\mu_0 i}{2\pi} \times \frac{\sin\frac{\pi}{n}}{\frac{a}{2}\cot\frac{\pi}{n}} \Rightarrow B = \frac{n\mu_0 i}{\pi\frac{2\pi r}{n}}\sin\frac{\pi}{n}\tan\frac{\pi}{n} \Rightarrow B = \boldsymbol{\frac{\mu_0 i n^2}{2\pi^2 r}\sin\frac{\pi}{n}\tan\frac{\pi}{n}}$ **is the answer of part (a).**

Further, when geometry of wire is circular as $n \to \infty \Rightarrow \frac{\pi}{n} \to 0$, then $\sin\frac{\pi}{n} \to \frac{\pi}{n}$ and $\tan\frac{\pi}{n} \to \frac{\pi}{n}$. Accordingly, $B = \frac{\mu_0 i n^2}{2\pi^2 r} \times \frac{\pi}{n} \times \frac{\pi}{n}$. It simplifies to $B = \boldsymbol{\frac{\mu_0 i}{2r}}$ **is answer of part (b).**

Thus, **answers are (a)** $\boldsymbol{\frac{\mu_0 i n^2}{2\pi^2 r}\sin\frac{\pi}{n}\tan\frac{\pi}{n}}$ **and (b)** $\boldsymbol{\frac{\mu_0 i}{2r}}$**.**

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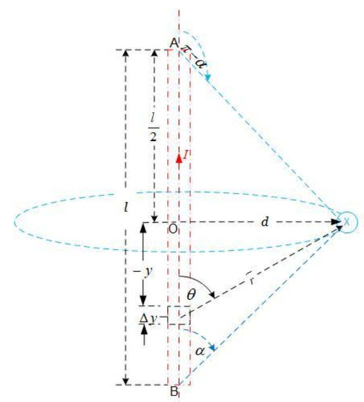
Appendix-I

Magnetic Field due to a Long Current Carrying Wire at a Point P

(Application of Biot-Savart's Law)



Given system is shown in the figure in $\hat{\imath} - \hat{\jmath}$ plane, where a wire AB of length l laid along $\hat{\jmath}$ is carrying a current I from B towards A. It is required to find magnetic field at a point P at a distance d from midpoint O of the wire.

Consider a small element of wire of length $\Delta y \to 0$ at a distance $-y$ from its midpoint. As per Biot-Savart's Law magnetic field at point P due to current I in the element of wire $d\vec{B} = \frac{\mu_0 I}{4\pi}\left(\frac{d\vec{y}\times\hat{r}}{r^2}\right)$.It leads to $d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\sin\theta\, dy\hat{k} \Rightarrow dB = \frac{\mu_0 I}{4\pi}\frac{\sin\theta}{r^2}dy$, in direction entering the plane of the figure as shown therein.

Therefore, magnitude of magnetic field at B due to wire AB is $B = \frac{\mu_0 I}{4\pi}\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{\sin\theta}{d^2+y^2}dy$.

Trigonometrically, $y = d\cot\theta \Rightarrow dy = -d\ \csc^2\theta\, d\theta$. Likewise, limits would change to $-\frac{l}{2} \to \theta$ and $\frac{l}{2} \to (\pi - \theta)$. Using this substitution $B = \frac{\mu_0 I}{4\pi}\int_{\theta}^{\pi-\theta}\frac{\sin\theta}{d^2(1+\cot^2\theta)}(-d\ \csc^2\theta\, d\theta)$. Again using trigonometry $(1 + \cot^2\theta = \csc^2\theta)$, we have $B = \frac{\mu_0 I}{4\pi}\left[\int\frac{\sin\theta}{d^2(\csc^2\theta)}(-d\ \csc^2\theta\, d\theta)\right]_{\theta}^{\pi-\theta}$. This expression solves into $B = \frac{\mu_0 I}{4\pi d}[\int\sin\theta\, d\theta]_{\theta}^{\pi-\theta} \Rightarrow B = -\frac{\mu_0 I}{4\pi d}[\cos\theta]_{\theta}^{\pi-\theta} \Rightarrow B = \frac{\mu_0 I}{4\pi d}[\cos\theta - \cos(\pi-\theta)] \Rightarrow B = \frac{\mu_0 I}{2\pi d}\cos\theta$…(I) Here, $\cos\theta = \frac{\frac{l}{2}}{r} \Rightarrow \cos\theta = \frac{\frac{l}{2}}{\sqrt{d^2+\left(\frac{l}{2}\right)^2}} \Rightarrow \cos\theta = \frac{l}{\sqrt{4d^2+l^2}}$.

The generic expression of magnetic field (I), can be resolved in two specific cases as under -

Case 1: When $d \gg l$. From the figure $\cos\theta \to \frac{l}{2d}$, accordingly $B = \frac{\mu_0 I}{2\pi d} \times \frac{l}{2d} \Rightarrow B = \frac{\mu_0 I l}{4\pi d^2} \Rightarrow \boldsymbol{B \propto \frac{1}{d^2}}$ **proved.**

Case 2: When $d \ll l$. From the figure $\cos\theta \to 1$, accordingly $B = \frac{\mu_0 I}{2\pi d} \Rightarrow \boldsymbol{B \propto \frac{1}{d}}$ **proved.**

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Important Note: You may encounter need of clarification on contents and analysis or an inadvertent typographical error. We would gratefully welcome your prompt feedback on mail ID: subhashjoshi2107@gmail.com. If not inconvenient, please identify yourself to help us reciprocate to you suitably.

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