Let Us Do Some Problems-XXXV

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Gaokao (National Higher Education Entrance Examination) in Mathematics is probably the hardest examination for a student. It is held in China. Every High School Senior takes part in this once-in-a-year event.

Some questions are selected from the different years' examination papers for understanding the standard of the questions and to know what type of teaching is done there. No answers are given. If some student needs the solution he or she may request the Coordinator's desk for that.

QUESTIONS

1. The domain of $f(x)$ is all real numbers and satisfies $f(x + 1) = 2f(x)$.
   For x in (0, 1], $f(x) = x(x – 1)$. For all values of x in $(-\infty, m]$, $f(x) \geq -8/9$.
   What is the maximum value of *m?*

2. Find the domain of the function
   $y = \sqrt{3 - 2x - x^2}$.

3. Throw a fair dice (a cube shaped toy with 1, 2, 3, 4, 5, 6 points on each side) twice. Find the probability that the sum of the numbers on top is less than 10.

4. Given an arithmetic sequence $\{a_n\}$, S_n is the sum of the first n terms. If $a_1+a_2^2=3$, $S_5=10$, then find the value of a_9.

5. Find the number of intersections of the graph of functions $y = sin2x$ and $y = cosx$ on domain $[0, 3\pi]$.

6. Let $f(x)$ be a function on R with period 2. On interval [-1,1],
   $$f(x) = \begin{cases} x + a, -1 \leq x < 0, \\ \left|\frac{2}{5} - x\right|, 0 \leq x < 1, \end{cases}$$
   where $a \in R$. If $f\left(-\frac{5}{2}\right) = f\left(\frac{9}{2}\right)$, then find the value of $f(5a)$.

7. If the real numbers x, y satisfy
   $$\begin{cases} x - 2y + 4 \geq 0, \\ 2x + y - 2 \geq 0, \\ 3x - y - 4 \leq 0, \end{cases}$$
   Then find the range of x^2+y^2.

8. In a triangle ABC, D is the midpoint of BC. Points E, F separate AD into three equal parts. $\overrightarrow{BA}.\overrightarrow{CA} = 4, \overrightarrow{BF}.\overrightarrow{CF} = -1$. Find the value of $\overrightarrow{BE}.\overrightarrow{CE}$.

9. If $sinA = 2\ sinB\ sinC$ in an acute triangle ABC, then find the minimum value of $tanA\ tanB\ tanC$.

10. In a triangle ABC, AC=6, $cosB = \frac{4}{3}$, $C = \frac{\pi}{4}$, then
    (a)What is the length of AB?
    (b)What is the value of $cos\left(A - \frac{\pi}{6}\right)$?

11. Function $f(x) = a^x + b^x$, where $(a > 0, b > 0, a \neq 1, b \neq 1)$

    (1)Suppose $a = 2, b = \frac{1}{2}$
    (a)Solve the equation $f(x)=2$.
    (b)For any $x \in R$, $f(2x) \geq m\ f(x)-6$. What is the maximum value of real number m?

    (2)If $0 < a < 1$, $b > 1$, function $g(x) = f(x)-2$ has only one root. What is the value of ab?

12. Let $U=\{1,2,3,\ldots,100\}$ and $\{a_n\}$ ($n \in N^+$) be a sequence. T is a subset of U. If $T = \phi$, define $S_T = 0$. If $T = \{t_1, t_2, \ldots, t_k\}$, define $S_T = a_{t_1} + a_{t_2} + \cdots + a_{t_k}$.
For example, if $T = \{1, 3, 66\}$, $S_T = a_1 + a_3 + a_{66}$. Now suppose $\{a_n\}(n \in N^+)$ is a geometric sequence with common ratio 3. Also, when $T = \{2, 4\}$, $S_T = 30$.
(1)What is formula for the sequence $\{a_n\}$?
(2)For any positive integer $1 \le k \le 100$, if $T \subset \{1,2,\ldots,k\}$, prove $S_T < a_{k+1}$.
(3)Suppose $C \subset U$, $D \subset U$, $S_C \ge S_D$. Prove that $S_C + S_{C\cap D} \ge 2\, S_D$.

13. In ΔABC, $\angle ABC=90^0$, $BD \perp AC$. E is the midpoint of BC. Prove that $\angle EDC = \angle ABD$.

14. Matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$ and the inverse of matrix B is $B^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 2 \end{pmatrix}$. What is AB?

15. In Cartesian coordinate system xOy, the parametric equation for straight line l is $\begin{cases} x = 1 + \frac{1}{2}t, \\ y = \frac{\sqrt{3}}{2}t \end{cases}$, where t is a parameter.
The parametric equation for ellipse C is $\begin{cases} x = cos\theta, \\ y = 2sin\theta \end{cases}$, where θ is a parameter.
Suppose line l intersects ellipse C at points A, B. How long is line segment AB?

16. Suppose a>0, $|x-1| < \frac{a}{3}$, $|y-2| < \frac{a}{3}$. Prove that $|2x + y - 4| < a$.

17. Suppose $m, n \in N^+$, $n \ge m$. prove that
$(m+1)^m C_m + (m+2)^m C_{m+1} + (m+3)^m C_{m+2} + \ldots + n^m C_{n-1} + (n+1)^m C_n = (m+1)^{n+2} C_{n+2}$