Appendix-I

Torque on Small Current Carrying Loop Placed Inside Another Current Carrying Loop

Part A - Magnetic Field at the Center of a Current Carrying Loop: Let us consider a loop of radius R carrying current I in anti-clockwise direction. The loop is taken to be in $\hat{\imath}-\hat{\jmath}$ plane and in accordance with the three unit directions vectors are also shown for reference in Fig. 1..

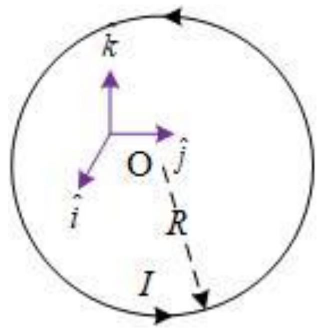


Fig. 1

Biot-Savart's Law stipulated direction and magnitude of magnetic field at a point situated at a distance r from a wire of length $\Delta\vec{l}=\Delta l\hat{l}$ carrying current I along $\hat{l}$. According to the law, as shown in Fig. 2.-

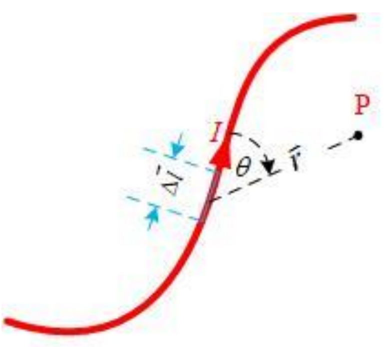


Fig. 2

$$\Delta\vec{B}=\frac{\mu_0 I}{4\pi R^2}\Delta\vec{l}\times\hat{r}\Rightarrow\Delta\vec{B}=\frac{\mu_0 I}{4\pi R^2}\Delta l\hat{l}\ \times\hat{r}$$

$$\Delta\vec{B}=\frac{\mu_0 I\Delta l}{4\pi R^2}\hat{k}\ldots(1).$$

Taking the elemental length Δl is part of a circle of radius R which subtends an angle $\Delta\theta$ at the circle of the circle such that $\Delta l=R\times\Delta\theta\ldots(2)$.

Combining (1) and (2), $\Delta\vec{B}=\frac{\mu_0 IR\Delta\theta}{4\pi R^2}\hat{k}$. Thus, in the instant case, having ascertained direction of the magnetic field at O as $\hat{k}$.its magnitude is $\Delta B=\frac{\mu_0 I\Delta\theta}{4\pi R}\ldots(3)$.

Accordingly, magnitude of the net magnetic field at the center of the circular loop is $B=\int_0^{2\pi}\frac{\mu_0 I}{4\pi R}d\theta\Rightarrow B=\frac{\mu_0 I}{4\pi R}[\theta]_0^{2\pi}\Rightarrow B=\frac{\mu_0 I}{4\pi R}\times 2\pi$. It leads to $\boldsymbol{B}=\frac{\boldsymbol{\mu_0 I}}{\boldsymbol{2R}}\ldots(4)$. This magnetic field at O is in along $\hat{\boldsymbol{k}}$.

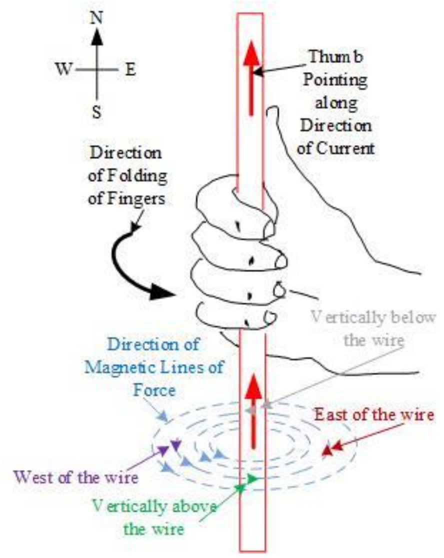


Fig. 3

The direction of magnetic field in this can also ascertained using **Right-Hand-Thumb-Rule** applied to circular loop for convenience as shown in Fig. 3, which when applied to loop or coil is as per Fig.4.

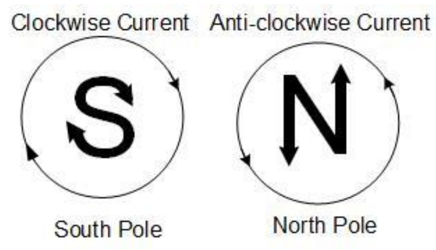


Fig. 4

Area of a loop of radius R is $A=\pi R^2$. This loop if carries ***current i in anticlockwise direction*** then in vector form $\vec{A}=\pi R^2\hat{k}$. Since, current in isolation is treated as a scalar and hence for a current carrying loop a new term is **magnetic dipole moment of a current carrying loop** which is defined as $\vec{\boldsymbol{\mu}}=\boldsymbol{i}\vec{\boldsymbol{A}}$ and in instant case $\vec{\mu}=i\pi R^2\hat{k}\ldots(5)$.

Part B – Interaction of Magnetic Fields of Two Current Carrying Concentric Loops: Two current carrying concentric loops A and B having magnetic dipole moments $\vec{\mu}_A$ and $\vec{\mu}_B$,, respectively will exhibit resultant magnetic dipole moment $\vec{\mu}=\vec{\mu}_A+\vec{\mu}_B\ldots(6)$

Part C – Torque on Current Carrying Loops: Taking two loops in a state of rest, they would experience equal and opposite torques as per Newton's Third Law of Motion. Generally concern is about a small loop of radius r carrying current at the center of an outer loop having much larger radius $R\gg r$. In case of coplanar loop discussions at part B above. In case of coplanar loop resultant magnetic dipole moments are either arithmetic sum or difference of the two dipole moments depending upon direction of their currents as discussed in part A.

But, it becomes an important pair of loops only when they are inclined with respect to each other a torque appears and there. The loops are taken to be concentric and they have radial symmetry about the common axis. There are two possibilities, each of them are analyzed below –

Case 1- Planes of loops are perpendicular to each other:

A circular loop of radius R carrying a current I is placed in $\hat{\imath}-\hat{\jmath}$ plane as shown in the figure. Axis of the loop is along $\hat{k}$. As per (4), in accordance with Biot-Savart's law it will produce magnetic field $\vec{B}=\frac{\mu_0}{2R}\hat{k}$, at the center of the loop. Further, it is given that another circular loop of radius r carrying current i in anti-clockwise direction as seen against $\hat{\imath}$. The small loop, as shown in the Fig. 5, is in $\hat{\jmath}-\hat{k}$ plane.

From the statement of the system shown in the figure on the left, it is observed that –

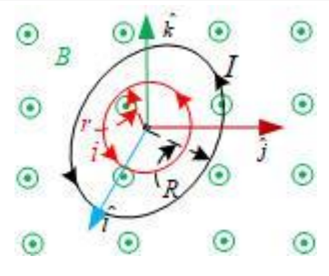


Fig. 5

a) magnetic field at the center of the outer loop of radius R is $\vec{B}$.
b) plane of the smaller loop of radius r is along the magnetic field.
c) given that $r \ll R$, and geometrical symmetry of the loop, force experienced by inner coil as per will produce a torque about diameter of smaller coil. We take for convenience diameter of loop along Y-Y' i.e. $\hat{\jmath}$.

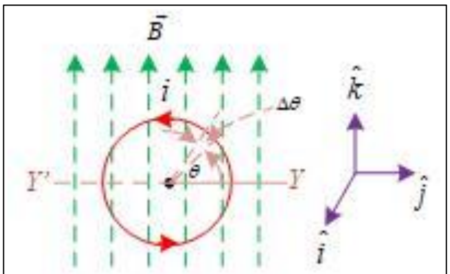


Fig. 6

Taking forward analysis force on a small element of inner loop of length $\Delta\vec{l}=r\Delta\theta\hat{l}$, as per (2), carrying current i,, as per Fig. 6, is as per limited version of Lorentz's Force Law $\Delta\vec{F}=i\vec{l}\times\vec{B}\Rightarrow\Delta\vec{F}=i\big(r\Delta\theta\hat{l}\,\big)\times\vec{B}$…(7). Here, the other part of the law which defines force on free charges, as none is there, is ignored. The force as a result of cross-product is $\Delta\vec{F}=irB\sin\theta\,\Delta\theta(-\hat{\imath})$….(7).

Therefore, torque experienced by the element of loop about diameter Y-Y' would be $\Delta\vec{\Gamma}=\overrightarrow{QP}\times\Delta\vec{F}$ as per Fig. 7. Combining (7) with the geometry, it leads to $\Delta\vec{\Gamma}=\big(r\sin\theta\,\hat{k}\big)\times\big(irB\sin\theta\,\Delta\theta(-\hat{\imath})\big)$….(8). This expression simplifies into $\Delta\vec{\Gamma}=iBr^2\sin^2\theta\,\Delta\theta(-\hat{\jmath})\Rightarrow\Delta\vec{\Gamma}=\frac{iBr^2}{2}(1-\cos 2\theta)\Delta\theta(-\hat{\jmath})$ ...(9). Thus, net torque would be $\vec{\Gamma}=\left[\int_0^{2\pi}\frac{iBr^2}{2}(1-\cos 2\theta)\Delta\theta\right](-\hat{\jmath})$. The integral simplifies to $\vec{\Gamma}=\frac{iBr^2}{2}\left[\int_0^{2\pi}d\theta-\int_0^{2\pi}\cos 2\theta\ d\theta\right](-\hat{\jmath})\Rightarrow\vec{\Gamma}=\frac{iBr^2}{2}\times 2\pi(-\hat{\jmath})$. Thus, net torque on the inner loop is $\vec{\Gamma}=iB\pi r^2(-\hat{\jmath})$…(10).

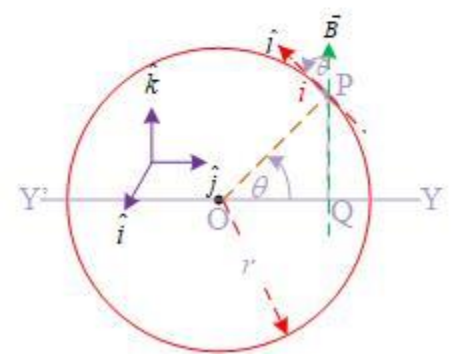


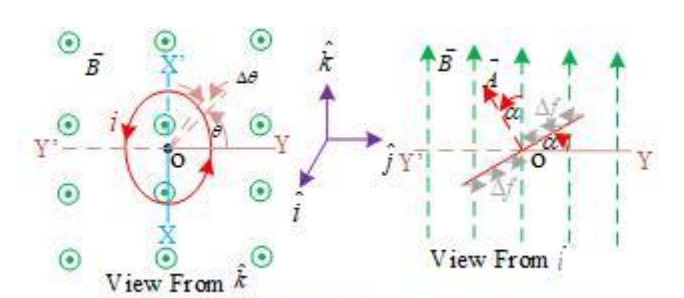
Fig. 7

Combining (4) and (11), magnitude of the torque is $\Gamma=i\left(\frac{\mu_0 I}{2R}\right)\pi r^2\Rightarrow\boldsymbol{\Gamma}=\boldsymbol{\frac{\mu_0\pi i I r^2}{2R}}$**….(11)**

Using (5), expression in (11) can be expressed as $\vec{\Gamma}=i\vec{A}\times\vec{B}\Rightarrow\vec{\Gamma}=\vec{\mu}\times\vec{B}$…(12). Here, magnetic dipole moment of a loop is $\vec{\mu}=i\vec{A}$ and in case of coil $\vec{\mu}=ni\vec{A}$…(13)

Case 2- Planes of loops are inclined at an angle $\boldsymbol{\alpha}$: Plane of the larger loop creating magnetic field $\vec{B}=B\hat{k}$ is along plane $\hat{\imath}-\hat{\jmath}$. is shown in the Fig 8 (View from $\hat{k}$). Let plane of the smaller loop of radius r is inclined to the plane of larger loop at an angle α, as shown in the Fig. 8 (View from $\hat{\imath}$). Considering radial symmetry the inclination is taken w.r.t, Y-Y' as shown in the figure i.e. $\hat{\jmath}$. In this diameter along X-X' remains aligned to $\hat{\imath}$. Thus, area vector of the loop, with respect to direction current i in it as discussed in part A, is $\vec{A}$ and inclined at an angle α w.r.t.$\vec{B}$.

Each small length of loop $\Delta\vec{l} = r\Delta\theta\hat{l}$, due to current i, would experience force $\Delta\vec{f} = i(r\Delta\theta\hat{l}\,) \times \vec{B}$ as per (7). In this orientation $\Delta\vec{f} = \Delta f(-\hat{j})$ on the semicircular arc on the right of X-X'. Likewise, $\Delta\vec{f} = \Delta f(\hat{j})$ on the semicircular arc on the left of X-X' to current i in the loop.



Conceptual drawing (Not to The scale)

Fig. 8

These distributed forces along the two semicircular arc would form a couple about Y-Y' causing rotation of loop about O along $\hat{\imath}$. In an effort to quantify torque on the inclined loop elemental force $\Delta\vec{f}$ is resolution of QP perpendicular to Y-Y' and it is PR = QP $\sin\alpha$…(13), as shown in the Fig. 9. Accordingly, torque equation in (9) is moderated into-

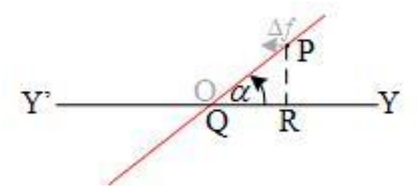


Fig. 9

$$\Delta\vec{\Gamma} = \overrightarrow{PR} \times \left(irB\sin\theta\,\Delta\theta(\hat{l})\right) \Rightarrow \Delta\vec{\Gamma} = (QP\sin\alpha\,\hat{k}) \times \left(irB\sin\theta\,\Delta\theta(\hat{l})\right)$$

$$\Delta\vec{\Gamma} = \left((r\sin\theta)\sin\alpha\,\hat{k}\right) \times \left(irB\sin\theta\,\Delta\theta(\hat{l})\right).$$

On integration it solves into, $\vec{\Gamma} = iB\pi r^2\sin\alpha\,(\hat{\imath}) \Rightarrow \vec{\Gamma} = i\vec{A} \times \vec{B} \Rightarrow \vec{\Gamma} = \vec{\mu} \times \vec{B}$ …(15)

It is seen that torque on a loop carrying current loop/coil produced (by an external magnetic field) derived in (15) and is identical to that derived in (12). In the latter case $\alpha = \frac{\pi}{2}$ when both coils are perpendicular.

Conclusion: In general form torque experienced by a current carrying loop placed in a uniform electric field produced by a large loop carrying current I, as per (12) and (15), would be $\vec{\Gamma} = \vec{\mu} \times \vec{B} = \mu B\sin\alpha\,\hat{\Gamma} \Rightarrow \vec{\Gamma} = \frac{\mu_0\pi i I r^2}{2R}\sin\alpha\,\hat{\Gamma}$…(16). Here, $\mu = i(\pi r^2)$ and $\mu\sin\alpha = i(\pi r^2\sin\alpha)$. Thus, in (16) $\pi r^2\sin\alpha = A\sin\alpha$ is the area of the current carrying turn, resolved perpendicular to the magnetic field perpendicular that it intercepts. Accordingly, $\boldsymbol{\vec{\Gamma}} = \boldsymbol{\vec{\mu}} \times \boldsymbol{\vec{B}}$ is the general expression of torque experienced by a current carrying urn of any shape when placed in a uniform magnetic field and simplifies analysis in complex situations

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