Magnetic Field at Any Point Inside a Circular Loop Carrying Current

Synopsis

This paper is an outcome of discussions with students of class 9th to 12th on electromagnetism during which it was observed that all texts and references cover derivation of magnetic field at the center of a current carrying circular loop and at any point on axis of the loop, perpendicular to the plane of the loop. An obvious question cropped up 'what could be magnetic field at any point inside the loop lying in its plane?' Interactive Online Mentoring Sessions (IOMS), flagship of Gyan Vigyan Sarita, which focuses on grooming competence to compete among unprivileged children with a sense of Personal Social Responsibility (PSR) in a non-organizational, non-remunerative, non-commercial and non-political manner. As a mentor of the initiative where students are prompted to come out of rote-learning and explore mathematics and science with an out-of-box perspective in their day-today experiences, the obvious question could not be averted. Accordingly, an illustration of the solution to the question has been evolved within the scope of understanding of target students.

Problem Formulation: Consider a point P inside a circular loop of radius R in $\hat{\jmath}-\hat{k}$ plane carrying a current I. The point P is at a distance a from the center of the loop. It is required to determine magnetic field B at P, as shown in the figure.

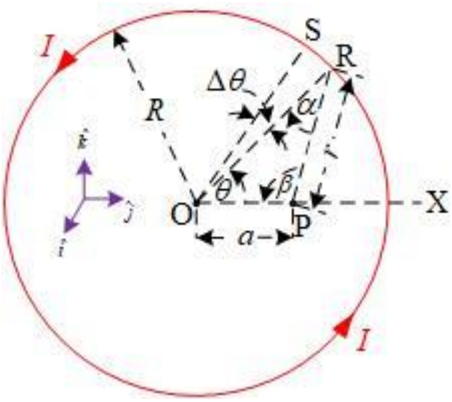


Fig. 1

As per Biot-Savart's Law magnetic field at a point P at a distance $\vec{r}=r\hat{r}$ from a small length of loop $\Delta\vec{l}$ carrying current I is $\Delta\vec{B}=\left(\frac{\mu_0 I}{4\pi}\right)\frac{\Delta\vec{l}\times\hat{r}}{r^2}$….(1). In the system small length of wire is RS such that $\Delta\vec{l}=(R\Delta\theta)\hat{l}$ and point is P at which magnetic flux density is to be determined is displaced by $\vec{r}$. As $\Delta\theta\to 0$ the element $\Delta\vec{l}$ becomes tangential to radial OA and hence (1) can be written as $\Delta\vec{B}=\left(\left(\frac{\mu_0 IR}{4\pi}\right)\frac{\sin\left(\frac{\pi}{2}+\alpha\right)}{r^2}\Delta\theta\right)\hat{\imath}\Rightarrow\ \Delta\vec{B}=\left(\left(\frac{\mu_0 IR}{4\pi}\right)\frac{\cos\alpha}{r^2}\Delta\theta\right)\hat{\imath}$…(2).

Problem Resolution: It is seen that (2) has three variables such that $B=f(r,\alpha,\theta)$ and B at P can be obtained by integrating (2) w.r.t. θ in the interval $[0,2\pi]$ to arrive at net magnetic field at the point due to the loop. Therefore, in function only of θ, by eliminating r and α, with the related parameters R and a which are geometrical constants.

Using properties of triangle in ΔORP, $\frac{OP}{\sin\alpha}=\frac{RP}{\sin\theta}=\frac{OR}{\sin\beta}\Rightarrow\frac{a}{\sin\alpha}=\frac{r}{\sin\theta}=\frac{R}{\sin(\pi-(\alpha+\theta))}\Rightarrow\frac{a}{\sin\alpha}=\frac{r}{\sin\theta}=\frac{R}{\sin(\alpha+\theta)}$.

Accordingly, $r=a\frac{\sin\theta}{\sin\alpha}$ …(3) and $\sin\alpha=\frac{a}{R}\sin(\alpha+\theta)$….(4).

It, further, solves into

$$\sin\alpha=\frac{a}{R}(\sin\alpha\cos\theta+\cos\alpha\sin\theta)\Rightarrow\left(1-\frac{a}{R}\cos\theta\right)\sin\alpha=\frac{a}{R}\sin\theta\cos\alpha.$$

Introducing a normalization parameter $t=\frac{a}{R}$ which defines relative position of point P in the plane of loop w.r.t. its center O we have -

$$\Rightarrow(1-t\cos\theta)\sin\alpha=t\sin\theta\,.\cos\alpha\Rightarrow(1-t\cos\theta)\sin\alpha=t\sin\theta\sqrt{1-\sin^2\alpha}$$

$$(1+t^2\cos^2\theta-2t\cos\theta)\sin^2\alpha=t^2\sin^2\theta\,(1-\sin^2\alpha)$$

$$\Rightarrow(1+t^2(\cos^2\theta+\sin^2\theta)-2t\cos\theta)\sin^2\alpha=t^2\sin^2\theta$$

$$\Rightarrow \sin^2\alpha = \frac{t^2\sin^2\theta}{(1+t^2-2t\cos\theta)}$$

$$\Rightarrow \sin\alpha = \frac{t\sin\theta}{\sqrt{(1+t^2-2t\cos\theta)}} \ldots(5)$$

$$\Rightarrow \cos^2\alpha = 1-\sin^2\alpha = 1-\frac{t^2\sin^2\theta}{(1+t^2-2t\cos\theta)} = \frac{(1+t^2-2t\cos\theta)-t^2\sin^2\theta}{(1+t^2-2t\cos\theta)}$$

$$\Rightarrow \cos^2\alpha = \frac{(1+t^2(1-\sin^2\theta)-2t\cos\theta)}{(1+t^2-2t\cos\theta)} = \frac{(1+t^2\cos^2\theta-2t\cos\theta)}{(1+t^2-2t\cos\theta)} = \frac{(1-t\cos\theta)^2}{(1+t^2-2t\cos\theta)}$$

$$\Rightarrow \cos\alpha = \frac{1-t\cos\theta}{\sqrt{(1+t^2-2t\cos\theta)}} \ldots(6)$$

Combining (3) $\left[r = a\frac{\sin\theta}{\sin\alpha}\right]$ and (5) $\left[\sin\alpha = \frac{t\sin\theta}{\sqrt{(1+t^2-2t\cos\theta)}}\right]$, we have –

$$r = a\frac{\sin\theta}{\frac{t\sin\theta}{\sqrt{(1+t^2-2t\cos\theta)}}} \Rightarrow r = R\sqrt{(1+t^2-2t\cos\theta)} \Rightarrow r^2 = R^2(1+t^2-2t\cos\theta) \ldots(7)$$

Combining (2),(6) and (7) we get –

$$\Delta B_t = \left(\left(\frac{\mu_0 IR}{4\pi}\right)\frac{\cos\alpha}{r^2}\Delta\theta\right) = \left(\frac{\mu_0 IR}{4\pi}\right)\frac{\frac{1-t\cos\theta}{\sqrt{(1+t^2-2t\cos\theta)}}}{R^2(1+t^2-2t\cos\theta)}\Delta\theta \Rightarrow \Delta B_t = \left(\frac{\mu_0 I}{4\pi R}\right)\frac{1-t\cos\theta}{(1+t^2-2t\cos\theta)^{\frac{3}{2}}}\Delta\theta \ldots(8)$$

Therefore, net magnetic field at P is –

$$B_t = \int_0^{2\pi}\left(\frac{\mu_0 I}{4\pi R}\right)\frac{1-t\cos\theta}{(1+t^2-2t\cos\theta)^{\frac{3}{2}}}\Delta\theta \Rightarrow B_t = \left(\frac{\mu_0 I}{4\pi R}\right)\int_0^{2\pi}\frac{1-t\cos\theta}{(1+t^2-2t\cos\theta)^{\frac{3}{2}}}d\theta.$$

Taking the limits outside the integration, for convenience of substitution we get –

$$B_t = \left(\frac{\mu_0 I}{4\pi R}\right)\left[\int\frac{1-t\cos\theta}{(1+t^2-2t\cos\theta)^{\frac{3}{2}}}d\theta\right]_0^{2\pi} \ldots(9)$$

The integration in (9), $F(t) = \left[\int\frac{(1-t\cos\theta)}{(1+t^2-2t\cos\theta)^{\frac{3}{2}}}d\theta\right]_0^{2\pi}$ …(10), at the center of the loop O where $a = 0 \Rightarrow t = \frac{a}{R} = 0$ it reduces to the expression in (10) reduces to $F(0) = [\int d\theta]_0^{2\pi} \Rightarrow F(0) = 2\pi$ …(11). Thus, magnetic field at O, combining (9) and (11) is $B_O = \left(\frac{\mu_0 I}{4\pi R}\right)2\pi \Rightarrow B_O = \frac{\mu_0 I}{2R}$…(12), is in conformity with the known value of B at the center of a current carrying loop.

Likewise, magnetic field at $a = R^- \Rightarrow t = \frac{R-h}{R}\Big|_{h\to 0} = 1-\frac{h}{R}\Big|_{h\to 0}$, we have –

$$F(1) = \left[\int\frac{(1-\cos\theta)}{(1+1-2\cos\theta)^{\frac{3}{2}}}d\theta\right]_0^{2\pi} = \frac{1}{2\sqrt{2}}\left[\int\frac{1}{\sqrt{1-\cos\theta}}d\theta\right]_0^{2\pi} = \frac{1}{2\sqrt{2}}\left[\int\frac{1}{\sqrt{2\sin^2\frac{\theta}{2}}}d\theta\right]_0^{2\pi} \Rightarrow F(1) = \frac{1}{4}\left[\int\frac{1}{\sin\frac{\theta}{2}}d\theta\right]_0^{2\pi}$$

Taking $\frac{\theta}{2} = u \Rightarrow d\theta = 2du$ it leads to-

$$F(1) = \frac{1}{4}\left[\int \operatorname{cosec} u\,(2du)\right]_0^{2\pi} = \frac{1}{2}\left[\int \operatorname{cosec} u\, du\right]_0^{2\pi} = (-)\frac{1}{2}\left[\operatorname{cosec} u \cot u\right]_0^{2\pi} \ldots(13)$$

Making reverse substitution (13) we have -

$$F(1) = (-)\frac{1}{2}\left[\operatorname{cosec}\frac{\theta}{2}\cot\frac{\theta}{2}\right]_0^{2\pi} = \frac{1}{2}\left[\operatorname{cosec} 0 \cot 0 - \operatorname{cosec} \pi \cot \pi\right] = \frac{1}{2}\left[(\infty)\times(\infty) - (\infty)\times(\infty)\right]\ldots(14)$$

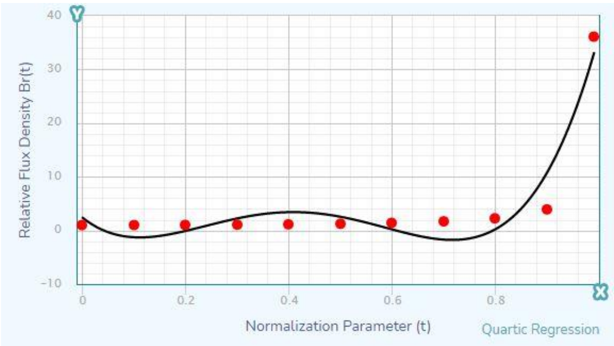
Thus, from (14), $F(1)$ is indeterminate.

Therefore, instead of calculating of determining pattern of $F(t)$, relative flux density $B_a = \frac{B_t}{B_0}$ is, combining (9) and (12) is –

$$B_{r_t} = \frac{\left(\frac{\mu_0 I}{4\pi R}\right)\left[\int \frac{1 - t\cos\theta}{(1+t^2-2t\cos\theta)^{\frac{3}{2}}} d\theta\right]_0^{2\pi}}{\frac{\mu_0 I}{2R}} = \frac{1}{2\pi}\left[\int \frac{1 - t\cos\theta}{(1+t^2-2t\cos\theta)^{\frac{3}{2}}} d\theta\right]_0^{2\pi} \ldots(15).$$

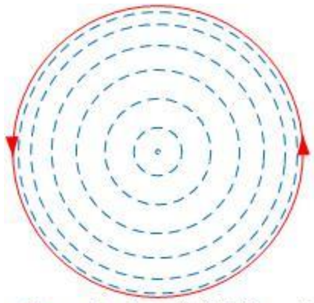
Combining (10) and (15), it leads to $B_{r_t} = \frac{1}{2\pi} \times F(t)\ldots(16)$

The integration $F(t)$ in (10), a part of (16), is not solvable by normal methods and pattern of flux density distribution has been determined using Trapezoidal Rule numerical method, using MS-Excel, in an interval $t = [0, 0.99)$. Results are plotted in Fig. 2 using MyCurveFit, Online Curve Fitting software (https://mycurvefit.com/). As $a \rightarrow R \Rightarrow t \rightarrow 1$ the integration $F(1)$ tends to be indeterminate and hence not plotted. Thus distribution of magnetic field over the cross-section of the loop , which is denser near the perimeter of the loop and rarer at the center, in the form of circular contours of uniform magnetic fields, is shown in Fig. 3.



Variation of Flux Density with Normalization Parameter (t)

Fig. 2



Conceptual graph Of Magnetic Contours in A Loop Carrying Current (Not to the scale)

Fig. 3

Data Calculated Numerically: Data used in Fig. 2, is as under -

$t = \frac{a}{R}$	Br_t	$t = \frac{a}{R}$	Br_t	$t = \frac{a}{R}$	Br_t
0	1	0.4	1.141324	0.8	2.257082
0.1	1.006735	0.5	1.245621	0.9	3.925924
0.2	1.031171	0.6	1.410594	0.99	36.10549
0.3	1.073742	0.7	1.692237	1	Indeterminate

Conclusion: Non-uniform magnetic field in the cross-section of the loop will impact philosophy of design of transformer core which is currently using uniform magnetic material in transformer core. Thus, this paper opens an opportunity to review overall design of transformer specially those used in instrumentation and control where errors due to core losses and magnetizing current are significant at macro level. At micro level it calls for review of magnetic forces that would influence configuration of orbital motion of electrons in atoms. Thus, review of overall spectrum of physics.

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