LET'S DO SOME PROBLEMS-XXXVII

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Some typical problems from the different competitive examinations are being given here to solve and test the knowledge of aspirants preparing for coming Entrance Examinations.

Q1. Find the value of the series:

$$\frac{1}{10^{-9}+1}+\frac{1}{10^{-8}+1}+\cdots+\frac{1}{10^{8}+1}+\frac{1}{10^{9}+1}$$

Hint:

Define,

$$a(k)=\frac{1}{10^{k}+1}$$

$\Rightarrow a(-k) + a(k) = \frac{1}{10^{k}+1}+\frac{1}{10^{-k}+1}$

On simplification, the result is 1

Thus the given expression becomes

$a(-9) + a(-8) + \ldots + a(0) + \ldots + a(8) + a(9)$

$= a(-9) + a(9) + a(-8) + a(8) + \ldots + a(-1) + a(1) + a(0)$

$= 1 + 1 + \ldots + 1 + a(0)$

$= 9 + a(0)$

$= 9 + 1/(1 + 1)$

$= 9.5$

Hence the sum equals to 9.5.

Q2. Evaluate the infinite series:

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}\ldots..$$

Hint:

We know that

$$e^{x}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots.$$

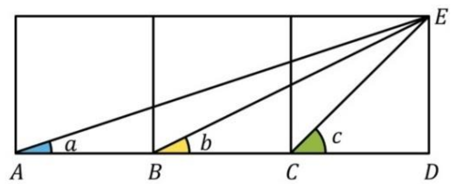
Divide both sides by x

$$\frac{e^{x}}{x}=\frac{1}{x}+\frac{1}{1!}+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots.$$

Differentiate both sides w.r.t. x and put $x=1$

The sum is 1.

Q3. In the following figure comprising of three squares in series, find the sum of the angles *a*, *b*, and *c*.

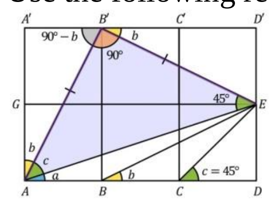


Hint:

Obviously,

c = 45°.

Use the following reconstruction



Triangle *EB'D'* is congruent to triangle *EBD* so angle *EB'D'* is also equal to *b*.

Triangle *B'AA'* is also congruent to triangle *EBD* so angle *B'AA'* is also equal to *b*, and thus *AB'A'* is equal to 90° – *b*.

Because *A'B'D'* is a straight line segment, it has a measure of 180°. Thus we have:

180° = *AB'A'* + *AB'E* + *EB'D'*

180° = (90° – *b*) + *AB'E* + *b*

180° = 90° + *AB'E*

90° = *AB'E*

Triangle *A'B'E* is thus a right triangle.

Furthermore, *AB'* is congruent to *B'E* because they are corresponding hypotenuses of congruent triangles.

Thus *A'B'E* is an isosceles right triangle and its acute angles are each equal to 45° = *c*.

Thus *B'AE* = *c*.

Finally we can solve the problem!

90° = *A'AB*
90° = *A'AB'* + *B'AE* + *EAB*
90° = $b + c + a$
90° = $a + b + c$

Q4. Solve for x in the following equation:

$$\sqrt[3]{x} + \sqrt[3]{(x-16)} = \sqrt[3]{(x-8)}$$

Hint:
Cube both sides of the equation and Use binomial theorem.
$(a + b)^3$
$= a^3 + b^3 + 3a^2b + 3ab^2$
$= a^3 + b^3 + 3ab(a + b)$
The three solutions are
$x = 8$ and $x = 8 \pm (12/7)\sqrt{21}$.

Q5. If a, b, and c are distinct numbers, how many solutions are there to the following equation?

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2\frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2\frac{(x-a)(x-b)}{(c-a)(c-b)} - x^2 = 0$$

Hint:
$x = a$, $x=b$ and $x=c$ satisfies the equation.
Thus the equation has at least 3 solutions $x = a$, $x = b$, and $x = c$.
Each of the first 3 terms is a polynomial of degree at most 2, and the final term is a polynomial x^2 of degree 2. But then the entire left hand side is a polynomial of degree at most 2, which means it can only have at most 2 roots.
We showed the polynomial has at least 3 roots, it must be the degenerate case where the polynomial is a constant identically equal to 0.

This implies the sum of the first three terms is x^2.

In other words:

$$x^2 = a^2\frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2\frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}$$

The problem then translates into how many solutions there are to the equation:
$x^2 - x^2 = 0$

Naturally any value of x satisfies this equation, so number of solutions are infinitely many.

Q6. What is the value of the following series?

$$\frac{1}{(1\times 3)} - \frac{1}{(2\times 4)} + \frac{1}{(3\times 5)} - \frac{1}{(4\times 6)} + \cdots.$$

Hint:
A general term in the series has the form:

$$\frac{1}{(n)\times(n+2)}$$

This can be split into two fractions using partial fractions as:

$$\frac{1}{(n)\times(n+2)} = \frac{1}{(2)\times(n)} - \frac{1}{(2)\times(n+2)}$$

Write each term of the given series using this formula and get the sum assuming k→∞
$S = \frac{1}{4}$

Q7. Suppose a and b are real numbers such that:

$$a\sqrt{a} + b\sqrt{b} = 183$$
$$b\sqrt{a} + a\sqrt{b} = 182$$

Find the value of $\frac{9}{5}(a+b)$.

Hint:
Assume
$x = \sqrt{a}$ and $y = \sqrt{b}$.
Substituting into the equations gives:

$x^3 + y^3 = 183$ …(i)
$xy^2 + x^2y = 182$ …(ii)

use the identity:
$(x + y)^3 = x^3 + y^3 + 3x^2y + 3xy^2$

Thus (i) + 3(ii) gives the cube of the sum of the real numbers x and y, meaning:
$(x + y)^3 = 183 + 3(182)$
$(x + y)^3 = 729 = 9^3$
$x + y = 9$
From (ii) we can then solve for the value of xy as follows:
$xy^2 + x^2y = 182$
$xy(x + y) = 182$
$xy(9) = 182$
$xy = \frac{182}{9}$
Now recall the formula for the square of a sum:
$(x + y)^2 = x^2 + y^2 + 2xy$
Since $(x + y) = 9$, squaring both sides gives $(x + y)^2 = 81$. Thus we have:
$(x + y)^2 = x^2 + y^2 + 2xy$
$81 = x^2 + y^2 + 2(\frac{182}{9})$
$x^2 + y^2 = 81 – 2(\frac{182}{9})$
$x^2 + y^2 = \frac{365}{9}$
$a + b = \frac{365}{9}$
Thus we have:
$\frac{9}{5}\ (a + b) = \frac{9}{5} . \frac{365}{9}$
$= 73$

Q8. Solve for n in the following equation:

$$\lim_{x\to 0} \frac{1}{x} \log_e \frac{e^x + e^{2x} + e^{3x} + \cdots + e^{nx}}{n} = 9$$

Hint:
First we will re-write the limit as follows:

$$\lim_{x\to 0} \frac{\log_e \left(\frac{e^x + e^{2x} + e^{3x} + \cdots + e^{nx}}{n}\right)}{x} = 9$$

Note that the numerator and denominator are both continuous functions. Furthermore, each function approaches 0 as x goes to 0, hence use L'Hospital Rule and differentiate Nr and Dr separately.
Resulting to

$$\frac{n + 1}{2} = 9$$

Giving
n=17

Q9. Solve for m and n to satisfy the equation:

$$5\frac{3}{m} \times n\frac{1}{2} = 19$$

Hint:
$\Rightarrow (5 + \frac{3}{m}) \times (n + \frac{1}{2}) = 19$
$\Rightarrow \frac{(5m+3)}{m} \times \frac{(2n+1)}{2} = 19$
$\Rightarrow (5m + 3)(2n + 1) = 2.m.19$
Since n is a positive whole number, $2n + 1$ is odd. This means $5m + 3$ is even and has 2 as a factor.
Since $\frac{3}{m}$ is a fraction in lowest terms, m is relatively prime to 3 (not a multiple of 3).
$\frac{m+3}{m} = 5 + \frac{3}{m}$
We can conclude m and $5m + 3$ are relatively prime and have no common factors.
Therefore, there are 2 possibilities for $5m + 3$:
it is either equal to 2 or $2 \cdot 19$. Let's consider each case.
$5m + 3 = 2$
$m = -1/5$
(exclude since m should be a positive number as $3/m$ is the fractional part of a mixed number)
$5m + 3 = 2 \cdot 19$
$5m + 3 = 38$
$m = 7$
Then we have:
$(5m + 3)(2n + 1) = 2 \cdot m \cdot 19$
$38(2n + 1) = 38m$
$2n + 1 = m$
$2n + 1 = 7$
$n = 3$

This gives the solution $m = 7$ and $n = 3$. And we can verify this does solve the original problem.

$$5\frac{3}{7} \times 3\frac{1}{2} = 19$$

Q10. Solve for real values of x in the equation: $3^{2x+1} + 4(3^x) - 15 = 0$

Hint:

The equation can actually be converted to a quadratic equation with a few manipulations. Let $u = 3^x$ so that $u^2 = 3^{2x}$.
Then we have:

$3^{2x+1} + 4(3^x) - 15 = 0$

$3^{2x}(3^1) + 4(3^x) - 15 = 0$

$3^{2x}(3) + 4(3^x) - 15 = 0$

$3(3^{2x}) + 4(3^x) - 15 = 0$

$3u^2 + 4u - 15 = 0$

We can then factor the equation:

$(3u - 5)(u + 3) = 0$

$u = 5/3$ or $u = -3$

Since x is a real number, $3^x = u$ is greater than 0, so we can eliminate $u = -3$ as a possibility. We then need to solve for x given that $u = 3^x = 5/3$.

$3^x = 5/3$

$\ln(3^x) = \ln(5/3)$

$x \ln 3 = \ln(5/3)$

$x = \ln(5/3)/(\ln 3) \approx 0.465$

Q11. Solve for x in the equation:

$$x + \frac{x}{1+2} + \frac{x}{1+2+3} + \cdots . + \frac{x}{1+2+3+\cdots+4041} = 4041$$

Hint:

We know that the sum of whole numbers from 1 to n is:

$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

We also know that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

Using this fact, the equation can be written as

$$x\left(1 + \frac{1}{\frac{2.3}{2}} + \frac{1}{\frac{3.4}{2}} + \cdots + \frac{1}{\frac{4041.4042}{2}}\right) = 4041$$

$\Rightarrow x\left(\frac{2}{1.2} + \frac{2}{2.3} + \frac{2}{3.4} + \cdots + \frac{2}{4041.4042}\right) = 4041$

$\Rightarrow 2x\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{4041} - \frac{1}{4042}\right) = 4041$

$\Rightarrow 2x\left(\frac{1}{1} - \frac{1}{4042}\right) = 4041$

$\Rightarrow 2x\left(\frac{4041}{4042}\right) = 4041$

$\Rightarrow x = 2021$

Q12. Solve for real values of x:

$$\left(x - \frac{1}{x}\right)^{\frac{1}{2}} + \left(1 - \frac{1}{x}\right)^{\frac{1}{2}} = x$$

Hint:

Assume:

$$a = \left(x - \frac{1}{x}\right)^{\frac{1}{2}}$$

$$b = \left(1 - \frac{1}{x}\right)^{\frac{1}{2}}$$

We then have:

$a + b = x$

$(a - b)(a + b) = x(a - b)$

$a^2 - b^2 = x(a - b)$

We can simplify the sum of square roots:

$a^2 = x - \frac{1}{x}$

$b^2 = 1 - \frac{1}{x}x$

$a^2 - b^2 = x - 1$

We thus have:

$x - 1 = x(a - b)$

$\Rightarrow \frac{x-1}{x} = a - b$

$\Rightarrow 1 - \frac{1}{x} = a - b$

On eliminating b from these equations, we get

$a = 1$

Going back to the definition of a gives:

$$a = \left(x - \frac{1}{x}\right)^{\frac{1}{2}} = 1$$
$x^2 - x - 1 = 0$

$x = \varphi = \frac{1+\sqrt{5}}{2}$

or

$x = -\frac{1}{\varphi} = \frac{1-\sqrt{5}}{2}$

We finally check each possibility for the original equation:

$(x - 1/x)^{0.5} + (1 - 1/x)^{0.5} = x$

The left hand side is a sum of two square roots so it will be a non-negative number, and that must equal x. Hence we can reject the negative possibility $x = -1/\varphi$. We can then check $x = \varphi$ does satisfy the original equation using the property $1/\varphi = \varphi - 1$:

$(\varphi - 1/\varphi)^{0.5} + (1 - 1/\varphi)^{0.5}$

$= (\varphi - (\varphi - 1))^{0.5} + ((\varphi - 1)/\varphi)^{0.5}$

$= (1)^{0.5} + ((1/\varphi)/\varphi)^{0.5}$

$= 1 + (1/\varphi^2)^{0.5}$

$= 1 + 1/\varphi$

$= 1 + (\varphi - 1)$

$= \varphi$

$= x$

The solution to the original equation is thus the golden ratio φ which is quite a divine answer!

Q13. Suppose:

$x^2 = 17x + y$

$y^2 = x + 17y$

If $x \neq y$, what is $\sqrt{x^2 + y^2 + 1}$ equal to?

Hint:

Rather than solving for x and y, we can manipulate the two equations. Subtract the second from the first to get:

$x^2 - y^2 = 16x - 16y$

$(x + y)(x - y) = 16(x - y)$

Since $x \neq y$, we know $x - y \neq 0$, so we can divide both sides by $x - y$ to get:

$x + y = 16$

Now sum the two original equations to get:

$x^2 + y^2 = 18x + 18y$

$x^2 + y^2 = 18(x + y)$

Substitute $x + y = 16$ to get:

$x^2 + y^2 = 18(16)$

$x^2 + y^2 = (17 + 1)(17 - 1)$

$x^2 + y^2 = 17^2 - 1$

If we add 1 to both sides and then take the square root of both sides we will solve the problem.

$x^2 + y^2 + 1 = 17^2$

Hence the required answer is 17

Q14. Evaluate the following integral, and use the result to prove that π is less than $\frac{22}{7}$.

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$$

Hint:

Rewrite the integrand after simplifying as

$$\int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}\right)dx$$

$= \frac{22}{7} - \pi$

The integral is equal to $22/7 - \pi$. Since the integrand is positive between 0 and 1, the integral will also be positive. Hence we have:

$0 < 22/7 - \pi$

$\pi < 22/7$

Q15. For all natural numbers n, a sequence a_n satisfies:

$a_{2n} = a_2 a_n + 1$

$a_{2n+1} = a_2 a_n - 2$

If $a_7 = 2$ and $0 < a_1 < 1$, what is the value of a_{25}?

Hint:

For $n = 12$, the sequence satisfies:

$a_{25} = a_2 a_{12} - 2$

For $n = 6$, the sequence satisfies:

$a_{12} = a_2 a_6 + 1$

We are making good progress. If we can solve for a_2 and a_6 then we can calculate a_{12} and consequently a_{25}.
To solve for a_6, we again consider $n = 3$, for which the sequence satisfies:
$a_7 = a_2a_3 – 2$
But since $a_7 = 2$, we have $a_2a_3 = 4$.
Then $n = 3$, the sequence also satisfies:
$a_6 = a_2a_3 + 1$
$a_6 = 4 + 1 = 5$
All that remains is to solve for a_2. We know that $a_2a_3 = 4$, so we can consider a_3 and a_2.
For $n = 1$, the sequence satisfies both conditions:
$a_3 = a_2a_1 – 2$
$a_2 = a_2a_1 + 1$
We can take the product of the two equations to get:
$(a_2a_1 + 1)(a_2a_1 – 2) = a_2(a_3)$
$(a_2a_1 + 1)(a_2a_1 – 2) = 4$
$(a_2a_1)^2 – (a_2a_1) – 2 = 4$
$(a_2a_1)^2 – (a_2a_1) – 6 = 0$
$(a_2a_1 – 3)(a_2a_1 + 2) = 0$
The equation has two solutions: $a_2a_1 = 3$ and $a_2a_1 = -2$.
We can actually eliminate the negative solution. Recall that for $n = 1$ the equation satisfies:
$a_2 = a_2a_1 + 1$
$a_2 – a_2a_1 = 1$
$a_2(1 – a_1) = 1$
$a_2 = \frac{1}{1-a_1}$

Since $0 < a_1 < 1$, we have $\frac{1}{1-a_1}$ is greater than 0, so $1 – a_2 > 0$. Thus both terms in the sequence are positive and $a_2a_2 > 0$. We can eliminate the solution $a_2a_1 = -2$ and it must be that $a_2a_1 = 3$.
Hence we have:
$a_2 = a_2a_1 + 1$
$a_2 = 3 + 1 = 4$
Knowing $a_2 = 4$ and $a_6 = 5$, we can calculate a_{12}:
$a_{12} = a_2a_6 + 1$
$a_{12} = 4(5) + 1 = 21$

Finally we can solve the problem:
$a_{25} = a_2a_{12} – 2$
$a_{25} = 4(21) – 2 = 82$
Hence the answer is $a_{25} = 82$.

Q16. Solve:
$$\frac{1}{5\sqrt{4}+4\sqrt{5}}+\frac{1}{6\sqrt{5}+5\sqrt{6}}+\cdots+\frac{1}{11\sqrt{10}+10\sqrt{11}}+\cdots.$$

Hint :
A general term in the series is
$$\frac{1}{n\sqrt{n-1}+(n-1)\sqrt{n}}$$
Multiply the numerator and denominator by the conjugate to simplify.
$$\frac{\sqrt{n-1}}{n-1}-\frac{\sqrt{n}}{n}$$
After substituting for each term, we will get a telescoping sum in which the second and third terms cancel, then the fourth and fifth terms cancel, and so on.
$= \frac{\sqrt{4}}{4}-\frac{\sqrt{5}}{5}+\frac{\sqrt{5}}{5}-\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}-\cdots$
$= \frac{\sqrt{4}}{4}$
= 2/4
= ½

Q17. If $x^5 = 1$, find all possible values of
$$\frac{x}{1+x^2}+\frac{x^2}{1+x^4}+\frac{x^3}{1+x}+\frac{x^4}{1+x^3}$$

Hint:
We will first multiply each fraction by another fraction equal to 1 so that each numerator equals $x^5 = 1$.
$$\frac{x}{1+x^2}.\frac{x^4}{x^4}+\frac{x^2}{1+x^4}.\frac{x^3}{x^3}+\frac{x^3}{1+x}.\frac{x^2}{x^2}+\frac{x^4}{1+x^3}.\frac{x}{x}$$

$=\frac{x^5}{x^4+x^6}+\frac{x^5}{x^3+x^7}+\frac{x^5}{x^2+x^3}+\frac{x^5}{x+x^4}$

Since $x^5 = 1$, we have $x^6 = x$ and $x^7 = x^2$.
Then the expression becomes

$$=\frac{1}{x^4+x}+\frac{1}{x^3+x^2}+\frac{1}{x^2+x^3}+\frac{1}{x+x^4}$$

$= 2$

Thus the sum is always equal to 2.

Q18. Solve the equation for real values of x:

$$2\sqrt[3]{2x+1} = x^3 - 1$$

Hint:

Let $y = \sqrt[3]{2x+1}$

Then we have:

$2x + 1 = y^3$

$2x = y^3 - 1$

$$x = \frac{y^3 - 1}{2}$$

Substituting for y into original equation

$2y = x^3 - 1$

$$y = \frac{x^3 - 1}{2}$$

Define

$$f(t) = \frac{t^3-1}{2}$$

Then we have shown:

$x = (y^3 - 1)/2 = f(y)$

$y = (x^3 - 1)/2 = f(x)$

Since x^3 is a strictly increasing function, f is a strictly increasing function.

Suppose x_0, y_0 is a solution to the pair of equations so that:

$x_0 = f(y_0)$

$y_0 = f(x_0)$

We will show this implies $y_0 = x_0$. It is a proof by contradiction.

For ease of reading, I will omit the subscripts and write $y = y_0$ and $x = x_0$.

First suppose $y < x$. Then we have:

$y < x$

$f(y) < f(x)$

$f(y) < y$

$x < y$

(The first to second step utilizes that f is an increasing function that preserves the order of the inequality).

Thus we have shown the assumption $y < x$ implies $x < y$, a contradiction. Similarly suppose $y > x$. Then we have:

$x < y$

$f(x) < f(y)$

$f(x) < x$

$y < x$

This is a contradiction to the assumption $y > x$.

Since y cannot be greater than x and y cannot be less than x, we must have $y = x$ for any solution to the equations.

The original equation is:

$2y = x^3 - 1$

$2x = x^3 - 1$

$x^3 - 2x - 1 = 0$

We can solve this equation by guessing a solution and then dividing by the linear factor to get a quadratic equation. The usual candidates $x = 1, 0, 2$ do not work, but then we can check $x = -1$ is a solution. Thus $x + 1$ is a linear factor. We then divide by the linear factor using polynomial division:

$$\frac{x^3 - 2x - 1}{x + 1} = x^2 - x - 1$$

We can then solve the second degree equation using Brahmagupta's quadratic formula:

$$x = \frac{1 \pm \sqrt{5}}{2}$$

And behold we have

$\varphi = \frac{1+\sqrt{5}}{2}$ as one of the solutions!

Therefore we have found 3 solutions to the original equation:

$x = -1$

$x = \varphi = \frac{1+\sqrt{5}}{2}$

$x = \frac{1-\sqrt{5}}{2}$